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On definable amenability of definable groups in simple theories

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1 Abstact

By Stone's representation theorem, We can identify sets of all types (over some set) with Stone spaces having a base of sets of all formulas over the set. So we can naturally extend the topologies to Borel regular probability measures. The measure is called Keisler measure.

Keisler measures have been mainly studied under an assumption of NIP. In NIP, we can consider a notion of Vpnik-Chervonenkis dimension. Therefore, we can approximate Keisler measures as distributions of finitely many types by Vapnik-Chervonenkis theorem.

In this paper, we introduce Keisler measures under an assumption of simple from NIP.

2 Keisler measure

From now, let $\mathcal{M}$ be a monster model of a theory T. That is for some inifinite cardinal $\bar{\kappa}$ such that $2^{\lambda} < \bar{\kappa}$ for all $\lambda < \bar{\kappa}$, $\mathcal{M}$ is $\bar{\kappa}$-saturated strongly $\bar{\kappa}$-homogeneous.

Definition 2.1. Let $A \in [\mathcal{M}]^{<\bar{\kappa}}$ and μ a finitely additive probability measure on $S_x(A)$. Then μ is called *Keisler measure* over A.

We can extend Keisler measures over A to the countably additive probability regular measures on the stone space $S_x(A)$ by using Loeb measure construction.

Every A-type p can be identified with the Dirac measure δ_p at p, so we can consider that $S_x(A)$ is the closed subset of the topological space of all Keisler measures over A.

Example 2.2. Let (G, E) be a random graph. By quantifier elimination, we may consider only measures of Boolean combinations of $x = a, E(x, a)$.

Let $\varphi(x) \in \mathcal{L}(G)$ and $a \in G$. We define μ that $\mu(\varphi(x) \wedge E(x, a)^{\epsilon})$ ($\epsilon \in \{0, 1\}$) when both $\varphi(x) \wedge E(x, a)$, $\varphi(x) \wedge \neg E(x, a)$ are consistent. Then, μ is a Keisler measure.

Definition 2.3.

(1) μ does not *fork* over A if all formulas having positive measure don't fork over A.

(2) $\mathrm{Aut}(\mathcal{M}/A)$-invariant μ is *definable* over A if for every formula a following map is continuous:

$$\begin{array}{ccc} f_\varphi : S_y(A) & \longrightarrow & [0,1] \\ \cup\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\shortmid \\ \mathrm{tp}(b/A) & \mapsto & \mu(\varphi(x,b)). \end{array}$$

Fact 2.4. Let μ be a Keisler measure, $\varphi(x,y) \in \mathcal{L}$ and $\langle b_i \mid i < \omega \rangle$ be an indiscernible sequence.

Assure that there is $\epsilon > 0$ such that $\mu(\varphi(x,b_i)) \geq 0$ for all $i < \omega$. Then $\{\varphi(x,b_i) \mid i < \omega\}$ is consistent.

3 definable group

In this section we are going to define groups acting Keisler measures.

Definition 3.1. $G \subseteq \mathcal{M}$ is *definable group* if the universe and the multiplication of the group G are definable over some parameter.

That is, there is two formulas $G(x), \cdot(x,y,z)$ such that $G = G(\mathcal{M})$ and $a \cdot b = c \iff \models \cdot(a,b,c)$ holds for all $a,b,c \in G$.

When $G(x), \cdot(x,y,z)$ are types, we call G type-definable group.

Example 3.2. Let $T = $ ACF, $\mathcal{M} = \mathbb{C}$, then a general linear group over $\mathbb{C}$ is a definable group over $\emptyset$.

We will consider actions of definable groups on formulas and types.

Definition 3.3.

(1) $\mathcal{L}_G(A) := \{\varphi(x) \wedge G(x) \mid \varphi(x) \in \mathcal{L}_x(A)\}$,

(2) $S_G(A) := \{p \in S_x(A) \mid p(x) \vdash G(x)\}$,

(3) For $g \in G, \varphi(x) \in \mathcal{L}_G(\mathcal{M})$, $g\varphi(x) :\equiv \exists y[\varphi(y) \wedge x = gy]$,

(4) For a (partial) type $\pi(x)$, $g\pi(x) := \{g\varphi(x) \mid \varphi(x) \in \pi(x)\}$.

Note that $\varphi(x) \wedge G(x)$ and $g\varphi(x)$ are not necessarily forlmula.

We can now define actions of definable groups on Keisler measures.

Definition 3.4. Let G be an $\emptyset$-definable group, μ be Keisler measure. For $g \in G$, $\varphi(x) \in \mathcal{L}_G(\mathcal{M})$,

$$g\mu(\varphi(x)) := \mu(g^{-1}\varphi(x)).$$

In a (type-)definable group G, we will often consider subgroups of G having bounded($< \bar{\kappa}$) index in G.

Definition 3.5. Assume that

$$\mathcal{H}_A^G := \{H \leq G \mid H \text{ is type-definable over } A \text{ and } |G : H| < \bar{\kappa}\}.$$

Then, $\mathcal{H}_A^G$ has a least element, that is $G_A^{00} := \bigcap \mathcal{H}_A^G$ is type-definable over A and $G_A^{00} \in \mathcal{H}_A^G$. We call G_A^{00} *connected component of* G *over* A.

Example 3.6. Let $\mathbb{V} = (V; +, \cdot, o, \mathbb{F}, \langle *, * \rangle)$ be an infinite-dimensional vector space over a finite field with some non-degenerate bilinear form.

Let $G = (V, +, o)$, then $G_A^{00} = \{v \in V \mid \text{for all } A\langle v, a \rangle = 0\}$.

Fact 3.7.

(1) For all $A \subseteq \mathcal{M}$, $G_A^{00} = G_\emptyset^{00}$ in NIP .

(2) Let $H \triangleleft G$ be a type-definable normal subgroup having bounded index in G. Then, G/H is hausdorff compact group. So is G/G_A^{00}.

4 definable amenability

For a definable group G, we consider countably additive probability regular measures on $\mathcal{L}_G(\mathcal{M}) := \{\varphi(x) \in \mathcal{L}(\mathcal{M}) \mid \varphi(x) \vdash G(x)\}$. From now, we will call them Keisler measure.

Definition 4.1. G is *definably amenable* if there is a G-action invariant Keisler measure.

Example 4.2.

(1) Amenable group is definably amenable.

(2) Let T be stable, then G is definably amenable.

(3) $SO_3(\mathbb{R})$ is definably amenable.

Fact 4.3. Let T be NIP, G be definably amenable. Then, we may choose a witness μ such that μ is definable over M for some M.

For Keisler measures, the following notion *genericity* is very important.

Definition 4.4.

(1) For a model $M \subseteq \mathcal{M}$, $\varphi(x) \in \mathcal{L}_G(\mathcal{M})$ is *generic* over M if $g\varphi(x)$ does not fork over M for all $g \in G$.

(2) A type $\pi(x)$ is *generic* if there is a model $M \subseteq M$ such that every $\varphi(x) \in \pi(x)$ is generic over M.

Fact 4.5. Assume that T is a theory in which Lascar strong types coincide with KP strong types. Especially, We assume that T is simple or NIP in this paper. Then, for $A \subseteq \mathcal{M}$ and $p \in S_G(A)$, the following are equivalent:

(1) p is generic,

(2) $\mathrm{Stab}(p) = G^{00}_A$,

where $\mathrm{Stab}(p)$ is the subgroup of G generated by $\{g \in G \mid gp \wedge p$ does not fork over $A\}$.

Fact 4.6. In simple theories, for all $A \subseteq \mathcal{M}$ there is a generic type $p \in S_G(A)$. Therefore, there is a global generic type.

However, there is not necessarily a generic type in NIP.

Proposition 4.7. Let T be NIP and G be an $\emptyset$-definable group. Then, G is definably amenable if and only if there is a global generic type.

Proof. ($\Rightarrow$) By the assumption and previous Fact, there is a G-invariant Keisler measure μ which is definable over M for some M, so it does not fork over M. Fix a global type p of an element of the support of μ. For all $\varphi(x) \in p$ $\mu(\varphi(x)) > 0$, hence p does not fork over M.

Fix $g \in G$ and $\varphi(x) \in p$. By G-invariance of μ, $\mu(g\varphi(x)) = \mu(\varphi(x)) > 0$, so $g\varphi(x)$ does not fork over M. Therefore p is generic.

($\Leftarrow$) Let p be a global generic type. By genericity of p, $\mathrm{Stab}(p) = G^{00}_{\emptyset}$. Since $G/G^{00}_{\emptyset}$ is hausdorff compact group, we can consider a Haar measure h on $G/G^{00}_{\emptyset}$. Define $\mathcal{A}_{p,\varphi} := \{g/G^{00}_{\emptyset} \mid g\varphi(x) \in p\}$. As p is definable over M, so is "$y\varphi \in p$". So $\mathcal{A}_{p,\varphi}$ is measurable over $G/G^{00}_{\emptyset}$. Therefore μ is the G-invariant Keisler measure where $\mu(\varphi(x)) := h(\mathcal{A}_{p,\varphi})$ for all $\varphi(x) \in \mathcal{L}_G(\mathcal{M})$. $\square$

In this way, existence of generic types and definable amenability of G are coincide in NIP. On the other hand, there is always a generic type, although G is not necessarily definably amenable.

Problem 4.8. What is an equivalence conditions of definable amenability of definable groups in simple theories.

For this problem, we show the following proposition.

Theorem 4.9. Let T be simple, G be an $\emptyset$-definable group and there is a global generic type p which is stationary (i.e., there is a unique non-forking extension of $p|_A$ for some $A \subseteq \mathcal{M}$). Then, G is definably amenable.

Proof. By the assumption $p|_A$ is generic for all $A \subseteq \mathcal{M}$, so $\mathrm{Stab}(p|_A) = G^{00}_A$. We denote that $\mathrm{Stab}(p|_A) = \mathrm{Stab}(p)$ by stationarity of p. Moreover, stationarity implies definability in simple theories, so $\mathcal{A}_{p,\varphi}$ is measurable over G/G^{00}_A. Therefore μ is the G-invariant Keisler measure where $\mu(\varphi(x)) := h_A(\mathcal{A}_{p,\varphi})$ such that h_A is a Haar measure on G/G^{00}_A. $\square$

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