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On the number of independent orders

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Abstract

In this note, we present and prove some lemmas that are useful when studying the number of independent orders. We can show $\kappa_{srd}^{m}(T)=\infty\Rightarrow\kappa_{srd}^{1}(T)=\infty$, using these lemmas. Its proof will be given in a forthcoming paper. (The details are not given in this note.)

We fix a complete theory T, and we work in a very saturated model of T. Letters $x, y, \ldots$ are used to denote finite tuples of variables. X is a set of x-tuples and Y is a set of y-tuples. In many cases, they have the form

$$X=(x_{\eta})_{\eta\in\omega^{n}} \text{ and } Y=(y_{\nu})_{\nu\in n\times\omega},$$

where $n\in\omega$. For sets Z, W of finite tuples of variables and a set $\Gamma(Z, W)$ of formulas, the set of all formulas $\exists z_{0}\ldots\exists z_{m-1}(\gamma_{0}(z, w)\wedge\ldots\gamma_{m-1}(z, w))$, where $m\in\omega, \gamma_{i}(z_{i}, w_{i})\in\Gamma$, $z_{i}\subset Z$, $w_{i}\subset W$, is denoted by $\exists Z\Gamma(Z, W)$.

Definition 1. Let $n\in\omega$.

1. Let $X=(x_{\eta}:\eta\in\omega^{n})$ be a set of variables. Let $\Delta(X)$ be a set of formulas whose free variables are in X. We say that Δ has the subarray property if there is a set $A=(a_{i_{0},\ldots,i_{n-1}}:\langle i_{0},\ldots,i_{n-1}\rangle\in\omega^{n})$ such that for any strictly increasing functions $f_{i}:\omega\rightarrow\omega$ $(i<n)$, $A_{f_{0},\ldots,f_{n-1}}=(a_{f_{0}(i_{0}),\ldots,f_{n-1}(i_{n-1})}:\langle i_{0},\ldots,i_{n-1}\rangle\in\omega^{n})$ realizes Δ.

2. Let $Y=(y_{\nu})_{\nu\in n\times\omega}$. Let $\mathcal{E}(Y)$ be a set of formulas whose free variables are in Y. We say that $\mathcal{E}$ has the (n-dimensional) subsequence property if there is a set $B=(b_{i,j})_{\langle i,j\rangle\in n\times\omega}$ such that for any strictly increasing functions $f_{i}:\omega\rightarrow\omega$ $(i<n)$, $B_{f_{0},\ldots,f_{n-1}}=(b_{i,f_{i}(j)})_{\langle i,j\rangle\in n\times\omega}$ realizes $\mathcal{E}(Y)$.

Lemma 2. *Suppose that $\Delta(X)$, where $X = (x_\eta : \eta \in \omega^n)$, has the sub-array property. Then a realization $A = (a_\eta : \eta \in \omega^n)$ of Δ can be chosen as an indiscernible array in the following sense:*

(*) For finite subsets F, F' of ω^n, if F and F' are isomorphic as $\{\leq_0, \dots, \leq_{n-1}\}$-structures then a_F and $a_{F'}$ have the same L-type.

Proof. For simplicity, we assume $n = 2$. We write X as $X = (X_0, X_1, \dots)$, where $X_i = (x_{i,j})_{j\in\omega}$. For each i, let $X_i = (x_{ij})_{j\in\omega}$ be the i-th row vector of X. Then

$$\Delta = \Delta((X_i)_{i\in\omega}) = \Delta(X_0, X_1, \dots)$$

has the subsequence property. So, for $A = (A_i)_{i\in\omega}$ realizing Δ, we can assume the A_i's form an indiscernible sequence. Similarly, we can also assume $(A'_j)_{j\in\omega}$, where $A'_j = (a_{i,j})_{i\in\omega}$, is an indiscernible sequence. So A is an indiscernible array. □

For $A = (a_\eta)_{\eta\in\omega^n}$ and a subset F of ω^2, a_F will denote the set $(a_\eta)_{\eta\in F}$.

Lemma 3. *Suppose that $\Delta(X)$ is realized by an indiscernible array $A = (a_\eta : \eta \in \omega^n)$. Let $X^* = (x_\eta)_{\eta\in I^n}$, where I is an arbitrary ordered set. We define $\Delta^*(X^*)$ by: For all φ and $F^* \subset_{fin} I^n$,*

$$\varphi(x_{F^*}) \in \Delta^* \iff \varphi(x_F) \in \Delta, \text{ for some } F \subset \omega^n \text{ with } F \cong_{\leq_0,\dots,\leq_{n-1}} F^*.$$

Then Δ^* is consistent and is realized by an indiscernible array.

Proof. It is sufficient to show the consistency, since the indiscernibility condition can be added to Δ^*. Let $\varphi_i(x_{F_i^*}) \in \Delta^*$ $(i < m)$. Choose $F_i \subset \omega^n$ $(i < m)$ witnessing the definition of Δ^*. Then $\varphi_i(a_{F_i})$ holds for all $i < m$. We can also choose $F'_i \subset \omega^n$ such that $F_0^* \dots F_{n-1}^* \cong F'_0 \dots F'_{n-1}$. By the indiscernibility, $\varphi_i(a_{F'_i})$ holds for all $i < m$. This shows that $\bigwedge \varphi_i(x_{F_i^*})$ is satisfiable. □

Lemma 4. *Suppose that $\mathcal{E}(Y)$, where $Y = (y_{\langle i,j\rangle} : \langle i,j\rangle \in n \times \omega)$, has the n-dimensional subsequence property. Then $\mathcal{E}(Y)$ is realized by $B = (b_{\langle i,j\rangle} : \langle i,j\rangle \in n \times \omega\}$ with the following property:*

(**) By letting $B_i = (b_{i,j})_{j\in\omega}$ $(i < n)$, B_i is an indiscernible sequence over $\bigcup_{k\neq i} B_k$.

Proof. Easy. □

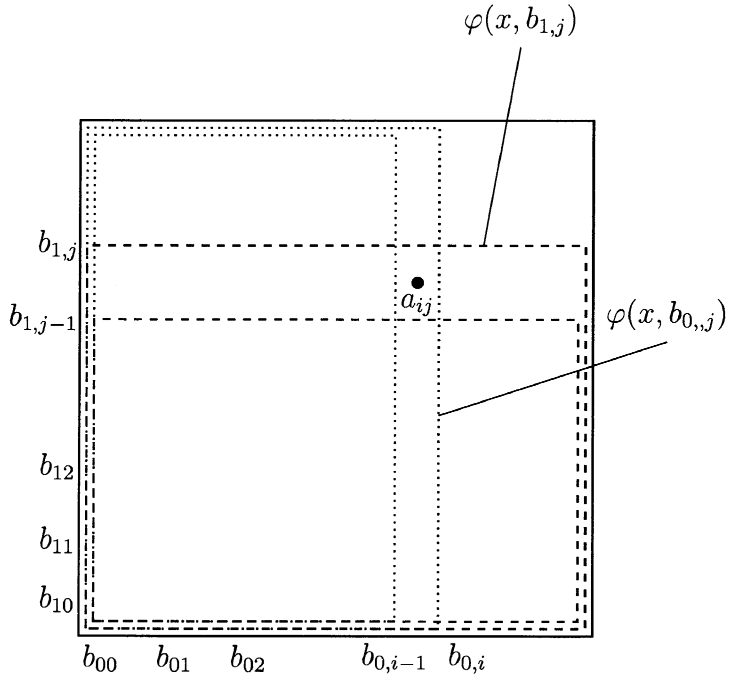
Example 5. Let $\varphi(x, y)$ be a formula. We say that T has n independent orders uniformly defined by φ if there are $A = (a_\eta : \eta \in \omega^n)$ and $B = (b_{i,j})_{\langle i,j\rangle \in n\times\omega}$ such that, for all $\eta \in \omega^n$ and $\langle i, j\rangle \in n \times \omega$,

$$\varphi(a_\eta, b_{ij}) \text{ holds iff } j \geq \eta(i).$$

Let

$$\Gamma(X, Y) := \{\varphi(x_\eta, y_{i,j})^{\text{ if } j \geq \eta(i)} : \eta \in \omega^n, \langle i, j\rangle \in n \times \omega\}.$$

Then T has n independent orders iff $\Gamma(X, Y)$ is consistent (with T). The set $\Delta(X) := \exists Y\Gamma(X, Y)$ has the subarray property and $\mathcal{E}(Y) := \exists X\Gamma(X, Y)$ has the n-dimensional subsequence property. (Notice that Δ and $\mathcal{E}$ are sets of first-order formulas.)



2-dimensional case

From now on, $\Gamma_{\varphi,n,\omega}(X, Y)$ denotes the set described by the above example. By Lemma 3 (or by a direct argument), $\Gamma_{\varphi,n,\mathbb{Q}}$ is naturally defined. In particular, if T has n independent orders defined by φ, then $\Gamma_{\varphi,n,\mathbb{Q}}(X, Y)$ is consistent, and $\Delta(X) := \exists Y\Gamma_{n,\varphi,\mathbb{Q}}(X, Y)$ has the subarray property. We simply write $\Gamma_{\varphi,n}$ if we are not interested in the ordered set (ω or $\mathbb{Q}$).

Definition 6 (The Number of Independent Orders)**.** Let $m, n \in \omega$. We write

1. $\kappa^m_{ird}(T) \geq n$ if $\Gamma_{\varphi(x,y),n}$ is consistent for some $\varphi(x,y)$ with $|x| = m$.
2. $\kappa^m_{ird}(T) = n$ if $\kappa^m_{ird}(T) \geq n$ and $\kappa^m_{ird}(T) \not\geq n+1$.
3. $\kappa^m_{ird}(T) = \infty$ if $\kappa^m_{ird}(T) \geq n$ $(\forall n)$.

Definition 7 (The Number of Independent Strict Orders). Let $\Gamma^s_{\varphi(x,y),n}(X,Y)$ be the set:

$$\Gamma_{\varphi(x,y),n}(X,Y) \cup \bigcup_{j<n} \{\forall x(\varphi(x, y_{i,j}) \to \varphi(x, y_{i+1,j})) : i \in \omega\}.$$

We write

1. $\kappa^m_{srd}(T) \geq n$ if $\Gamma^s_{\varphi(x,y),n}$ is consistent for some $\varphi(x,y)$ with $|x| = m$.
2. $\kappa^m_{srd}(T) = n$ if $\kappa^m_{srd}(T) \geq n$ and $\kappa^m_{srd}(T) \not\geq n+1$.
3. $\kappa^m_{srd}(T) = \infty$ if $\kappa^m_{srd}(T) \geq n$ $(\forall n)$.

The definition of above invariants are due to Shelah, but with a slight modification.

Remark 8. 1. Suppose that T has the independence property. Then $\kappa^1_{ird}(T) = \infty$: Since T has the independence property, there is a formula $\varphi(x,y)$ with $|x| = 1$ and $I = (b_i)_{i\in\omega}$ such that $\{\varphi(x,b_i)^{\text{ if } i \in F} : i \in \omega\}$ is consistent for any $F \subset \omega$. Choose an indiscernible sequence $I^* = (b_i)_{i\in\omega^2}$ extending I. Then I^* realizes $\exists X \Delta_{\varphi,\omega}(X,Y)$. By compactness, this shows $\kappa^1_{ird}(T) = \infty$.

2. Let T_{rg} be the theory of random graphs. Then $\kappa^1_{ird}(T_{rg}) = \infty$ and $\kappa^m_{srd}(T) = 1$.

3. If T has the order property, then $\kappa^m_{ird}(T) \geq m+1$. If T has the strict order property, then $\kappa^m_{srd}(T) \geq m+1$: Both can be proven similarly. For the case of strict order property, choose $\psi(x,y)$ with $|x| = 1$ and $I = (b_i)$ witnessing the property. For $u = u_0, \ldots, u_{m-1}$, let $\varphi_i(u,y) := \psi(u_i, y)$ $(i < m)$. Then $\{\varphi_i(u, b_j)^{\text{if } j \geq \eta(i)} : i < m,\ j \in \omega\}$ is consistent, for any $\eta \in \omega^m$. This shows $\kappa^m_{srd}(T) \geq m+1$, since there is a formula with additional variables such that each φ_i is a specialization of the formula.

References

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