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A smoothness problem for differential equations with state-dependent delay

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For a fixed $h>0$ let U be a subset of functions $\phi:[-h,0]\rightarrow \mathbb{R}^{n}$, and let a map $f:U\rightarrow \mathbb{R}^{n}$ be given. The delay differential equation of the form

$$x'(t)=f(x_{t}) \tag{1}$$

can represent classical equations with constant delay, and equations with state-dependent delay as well. For given $\phi\in U$, a solution of (1) with initial condition

$$x_{0}=\phi, \tag{2}$$

is a map $x:[-h, t_{\phi})\rightarrow \mathbb{R}^{n}$ such that the segments x_{t}, defined by $x_{t}(s)=x(t+s)$ for $s\in[-h, 0]$, are in U for all $t\in[0, t_{\phi})$, the initial condition (2) is satisfied, moreover the restriction of x to $(0, t_{\phi})$ is differentiable, and (1) holds for all $t\in(0, t_{\phi})$.

If X^{0} denotes the Banach space $C([-h, 0], \mathbb{R}^{n})$ with the maximum norm $|\phi|_{0}=\max_{-h\leq s\leq 0}|\phi(s)|$, $U\subset X^{0}$ is open, and $f:U\rightarrow \mathbb{R}^{n}$ is p-times continuously differentiable, $p\geq 1$ is an integer, then problem (1)–(2) has a unique solution $x=x^{\phi}$, which is maximal in the sense that any other solution is a restriction of x^{ϕ}. Then $F(t, \phi)=x_{t}^{\phi}$, $\phi\in U$, $0\leq t<t_{\phi}$, defines a continuous semiflow so that the solution operators $F(t, \cdot):U\rightarrow U$ are C^{p}-smooth on nonempty domains. See [1, 2]. This is the classical framework for equations with constant delays.

For differential equations with state-dependent delay (SDDEs) the above does not work. H.-O. Walther [6] developed a suitable theory to handle SDDEs: If U is an open subset of $X^{1}=C^{1}([-h, 0], \mathbb{R}^{n})$, equipped with the norm $|\phi|_{1}=|\phi|+|\phi'|$, $f:U\rightarrow \mathbb{R}^{n}$ is C^{1}-smooth, and each $Df(\phi)\in L_{c}(X^{1}, \mathbb{R}^{n})$ has an extension $D_{e}f(\phi)\in L_{c}(X^{0}, \mathbb{R}^{n})$ with the continuity property

$$U\times X\ni(\phi, \chi)\mapsto D_{e}f(\phi)\chi\in \mathbb{R}^{n} \text{ is continuous,}$$

then the set (in case it is nonempty)

$$X_{f}^{1}=\{\phi\in U:\phi'(0)=f(\phi)\}$$

is a C^{1}-submanifold of X^{1} with codimension n. The manifold X_{f}^{1} can serve as a phase space for SDDEs: Each $\phi\in X_{f}^{1}$ uniquely determines a maximal solution x^{ϕ} of (1)–(2), the map $(t, \phi)\mapsto x_{t}^{\phi}$ defines a continuous semiflow on X_{f}^{1} with C^{1}-smooth solution operators $X_{f}^{1}\ni\phi\mapsto x_{t}^{\phi}\in X_{f}^{1}$.

It is an open problem whether, for SDDEs, better than C^{1}-smoothness (C^{p}-smoothness with an integer $p>1$) can be obtained for the solution operators, see [3]. In a recent paper [5] we show that, in general, for a C^{p}-map $f:U\rightarrow \mathbb{R}^{n}$ on an open subset U of X^{1} with the above extension property of Df, the solution manifold $X_{f}^{1}\subset X^{1}$ is only C^{1}-smooth, not twice continuously differentiable, no matter how large p is.

The higher smoothness (C^{p} with $p>1$) of the solution operators is a key technical property in the qualitative theory of semiflows. For example, local bifurcation results at stationary points often require smooth local center manifolds, and the smoothness of solution operators play an essential role in the proofs for smooth center manifolds.

A C^p-smooth local unstable manifold is obtained in [4] without knowing higher smoothness of the solution operators. It is expected that such a result works for local center manifolds at stationary points as well.

The aim of this note is to present a new idea from [5] to the smoothness problem, and to mention problems, possible new directions of research related to this approach.

The approach of [5] will be shown on the simple example

$$x'(t) = g(x(t - r(x(t)))), \tag{3}$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$ and $r : X^0 \to (0, h)$ are C^p-smooth, $p \geq 1$. Equation (3) is a particular case of (1) with

$$f = g \circ \mathrm{ev} \circ (\mathrm{id} \times (-r))$$

where the evaluation map $\mathrm{ev} : X^0 \times [-h, 0] \to \mathbb{R}^n$ is defined by

$$\mathrm{ev}(\phi, t) = \phi(t).$$

Then the restriction of f to X^1 is C^1-smooth, and Df has the above mentioned extension property. The framework of Walther [6] can be applied to get C^1-smooth solution operators on the solution manifold. C^p-smoothness for $p > 1$ is not known.

The lack of smoothness in SDDEs, and in particular in (3), comes from the lack of smoothness of the map ev. However, ev has interesting smoothness properties.

Let $X^k = C^k([-h, 0], \mathbb{R}^n)$ denote the Banach spaces of the k-times continuously differentiable functions $\phi : [-h, 0] \to \mathbb{R}^n$ equipped with the usual norm $|\phi|_k = \sum_{j=0}^{k} |\phi^{(j)}|_0$. Now define the map ev_k as the restriction of ev to $X^k \times [-h, 0]$. It is shown in [5] that ev_k is C^k-smooth, and it is not C^{k+1}-smooth.

We will use the well known millification technique. If $\eta : \mathbb{R} \to \mathbb{R}$ is a C^∞-smooth function with $\mathrm{supp}\,\eta \subset [-1, 1]$, and $\int_{\mathbb{R}} \eta(s)\,ds = 1$, then for $\epsilon > 0$ define $\eta_\epsilon(t) = (1/\epsilon)\eta(t/\epsilon)$, $t \in \mathbb{R}$. Clearly, $\mathrm{supp}\,\eta_\epsilon \subset [-\epsilon, \epsilon]$ and $\int_{\mathbb{R}} \eta_\epsilon(s)\,ds = 1$. For $\phi \in X^0$ define $\hat{\phi} : \mathbb{R} \to \mathbb{R}^n$ as the extension of ϕ so that $\hat{\phi}(t) = \phi(-h)$ for $t < -h$, and $\hat{\phi}(t) = \phi(0)$ for $t > 0$. Let

$$\hat{\phi} * \eta_\epsilon(t) = \int_{\mathbb{R}} \hat{\phi}(t - s)\eta_\epsilon(s)\,ds = \int_{\mathbb{R}} \hat{\phi}(s)\eta_\epsilon(t - s)\,ds, \quad t \in \mathbb{R},$$

and define the mollification $m_\epsilon(\phi)$ of ϕ as the restriction of $\hat{\phi} * \eta_\epsilon$ to the interval $[-h, 0]$. The linear map $m_\epsilon : X^0 \to X^0$ is called a mollifier.

It is easy to see that, for each $\phi \in X^0$, the function $m_\epsilon(\phi) : [-h, 0] \to \mathbb{R}^n$ is C^∞-smooth, and thus $m_\epsilon(X^0) \subset X^k$ for all integers $k \geq 0$. In addition, the linear maps

$$m_{\epsilon,k} : X \ni \phi \mapsto m_\epsilon(\phi) \in X^k$$

are continuous for all integers $k \geq 0$.

Assume that there exists $\delta > 0$ with $r(X^0) \subset (\delta, h - \delta)$ Choose $\epsilon \in (0, \delta)$, and consider the map

$$f_\epsilon = g \circ \mathrm{ev} \circ (m_\epsilon \times (-r))$$

and the equation

$$x'(t) = f_\epsilon(x_t). \tag{4}$$

Observe that

$$f_\epsilon(\phi) = g \circ \mathrm{ev}_p \circ (m_{\epsilon,p} \times (-r))(\phi)$$

for all $\phi \in X^0$. Combining that the map $m_{\epsilon,p} : X^0 \to X^p$ is linear continuous, the maps $g : \mathbb{R}^n \to \mathbb{R}^n$, $r : X^0 \to (0,h)$ and $\mathrm{ev}_p : X^p \times [-h,0] \to \mathbb{R}^n$ are C^p-smooth, it follows that $f_\epsilon : X^0 \to \mathbb{R}^n$ is C^p-smooth. Therefore, for equation (4) with initial condition $x_0 = \phi \in X^0$, the classical theory, developed for equations with constant delay, can be applied to get C^p-smooth solution operators on the phase space X^0.

By the definition of m_ϵ, equation (4), the so called mollified version of equation (3), can be written as

$$x'(t) = g\left(-\int_{-r(x_t)-\epsilon}^{-r(x_t)+\epsilon} x(t+u)\eta_\epsilon(-r(x_t)-u)\,du\right). \tag{5}$$

This form of the mollified equation shows that equation (5) is obtained from (3) so that the term $x(t - r(x_t))$ with a discrete delay is changed by the the term with distributed delay

$$-\int_{-r(x_t)-\epsilon}^{-r(x_t)+\epsilon} x(t+u)\eta_\epsilon(-r(x_t)-u)\,du.$$

The above idea works in a more general context, see [5]. In [5] several other examples can be found.

The lack of smoothness disappears for the mollified equation. However, we changed the original equation. Thus, several new problems arise. What is the relation between the dynamics of equations (3) and (4)? It is expected that, by letting $\epsilon \to 0+$, equation (4) will give useful information on the dynamics of equation (3). For equation (4) Hopf bifurcation results are available to get information on the direction of the bifurcation and on the stability of the obtained periodic orbits. What properties can be preserved from equation (4) to equation (3) as $\epsilon \to 0+$? It is also an interesting problem, how the results depend on the smooth function η used to define the mollifier m_ϵ.

In several applications, the discrete delay, like in equation (3), is considered only for the sake of simplicity. A term with distributed delay is often more realistic.

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