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Asymptotic analysis of regularly varying solutions of second-order half-linear differential equations

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Abstract. We present the most important results that have arisen from the application of Karamata's theory of regular variation to asymptotic analysis of regularly varying solutions of the second order half-linear differential equation

$$(p(t)|x'|^{\alpha}\text{sgn } x')' + q(t)|x|^{\alpha}\text{sgn } x = 0.$$

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1 Introduction

The second order half-linear differential equation

$$(HL) \qquad (p(t)|x'|^{\alpha}\text{sgn } x')' + q(t)|x|^{\alpha}\text{sgn } x = 0,$$

is considered under the assumption that
(a) $\alpha > 0$ is a constant, and
(b) $p : [a, \infty) \to (0, \infty)$, $q : [a, \infty) \to \mathbb{R}$, $a > 0$, are continuous functions.

In this paper we are concerned with nontrivial solutions of (HL) which exist in an interval of the form $[t_0, \infty)$, $t_0 \geq a$. Such a solution is said to be oscillatory if it has a sequence of zeros clustering at infinity, and nonoscillatory otherwise. If all solutions of (HL) are oscillatory (nonoscillatory) equation (HL) is said to be oscillatory (nonoscillatory). Our attention will be focused on the case where (HL) is nonoscillatory. Since if x satisfies (HL), so does $-x$, it is natural to restrict our consideration to (eventually) positive solutions of (HL).

It is well known that the half-linear equation has many fundamental qualitative properties as the corresponding linear differential equation

$$(L) \qquad (p(t)x')' + q(t)x = 0$$

(see Elbert [3], Došly and Rehak [2]). For example, the classical Sturmian separation and comparison theorems has been extended verbatim to (HL) and basic concepts of the linear oscillation theory have a natural half-linear extensions. Thus, all nontrivial solutions of (HL) are either oscillatory or nonoscillatory. Also, it is shown that (HL) is nonoscillatory if and only if the generalized Riccati differential equation

$$u' + \frac{\alpha}{p(t)^{1/\alpha}}\,|u|^{1+\frac{1}{\alpha}} + q(t) = 0, \tag{1.1}$$

has a solution defined in some neighborhood of infinity.

The systematic study of equations of the form (HL) by means of regularly varying functions (in the sense of Karamata) was initiated by Jaroš, Kusano and Tanigawa [7] and motivated by the monograph of Marić [13], in which a systematic survey was given of the theory of asymptotics of nonoscillatory solutions of second-order linear differential equations, that has been developed by Marić, Tomić and other (see [4, 5, 14, 15]).

In this paper, we present the most important results that have arisen from the application of Karamata's theory of regular variation to asymptotic analysis of regularly varying solutions of the second order linear and half-linear differential equation (L) and (HL).

By a definition, a measurable function $f : (t_0, \infty) \to (0, \infty)$ for some $t_0 > 0$ is said to be *regularly varying (at infinity) of index* $\vartheta \in \mathbb{R}$ if it satisfies

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\vartheta} \quad \text{for all } \lambda > 0. \tag{1.2}$$

Regularly varying function f of index $\vartheta = 0$ is called *slowly varying.* It is known that such a function is characterized by the fact that it has the representation

$$f(t) = c(t)\exp\left\{\int_{t_0}^{t} \frac{\delta(s)}{s} ds\right\}, \quad t \ge t_0, \tag{1.3}$$

for some $t_0 > 0$ and for some measurable functions c, δ such that

$$\lim_{t\to\infty} c(t) = c_0 \in (0,\infty) \quad \textit{and} \quad \lim_{t\to\infty} \delta(t) = \vartheta.$$

If $c(t) \equiv c > 0$ in (1.3), then f is said to be a *normalized regularly varying function of index* ϑ. The totality of regularly varying functions of index ϑ is denoted by $\mathcal{RV}(\vartheta)$ and the totality of normalized regularly varying functions of index ϑ is denoted by $\mathcal{NRV}(\vartheta)$, while the totality of slowly varying function and normalized slowly varying function are denoted by $\mathcal{SV}$ and $\mathcal{NSV}$.

For the detailed presentation of theory of regular variation functions reader is referred to the books [1, 16].

2 Existence of $\mathcal{RV}$−solutions of second-order linear differential equation

The basic questions that are asked concerning asymptotic analysis of positive solutions of second-order linear differential equation in the framework of regular variation are

- QUESTION 1: Is it possible to provide necessary and sufficient conditions for the existence of $\mathcal{RV}$−solutions?
- QUESTION 2: Is it possible to establish the unique explicit asymptotic formula of $\mathcal{RV}$−solutions?

The fundamental results giving answer to the first question were given by Howard, Marić [5], establishing necessary and sufficient conditions for the existence of regularly varying solutions to the linear differential equation

$$(\mathcal{L}) \qquad x'' + q(t)x = 0\,.$$

Theorem 2.1 (Howard, Marić [5]) *Let $q : [a,\infty) \to \mathbb{R}$ be continuous functions, $c \in \left(-\infty, \frac{1}{4}\right)$ any given constant and λ_1, λ_2, $\lambda_1 < \lambda_2$ denote the two real roots of the quadratic equation $\lambda^2 - \lambda + c = 0$. Equation $(\mathcal{L})$ has a fundamental set of solutions $\{x_1(t), x_2(t)\}$ such that*

$$x_1 \in \mathcal{NRV}(\lambda_1), \qquad x_2 \in \mathcal{NRV}(\lambda_2)$$

if and only if

$$\lim_{t\to\infty} t \int_t^\infty q(s)\,ds = c\,.$$

Theorem 2.2 (Howard, Marić [5]) *Let $q : [a,\infty) \to \mathbb{R}$ be continuous functions. Define*

$$\phi(t) = t\int_t^\infty q(s)\,ds - \frac{1}{4}, \quad \psi(t) = \int_t^\infty \frac{|\phi(s)|}{s}\,ds$$

and let the integrals

$$\int^\infty \frac{|\phi(s)|}{s}\,ds \quad \text{and} \quad \int^\infty \frac{\psi(s)}{s}\,ds$$

converge. There exists two linearly independent $\mathcal{RV}$−solutions x_1, x_2 of the linear DE $(\mathcal{L})$ of the form

$$x_1(t) = \sqrt{t}\,\ell_1(t), \qquad x_2(t) = \sqrt{t}\,\ln t\,\ell_2(t), \quad \ell_i \in \mathcal{NSV},\ i = 1,2$$

if and only if

$$\lim_{t\to\infty} t\int_t^\infty q(s)\,ds = 1/4.$$

Moreover, $\ell_i(t) \sim \eta_i > 0$, $i = 1,2$ and $\ell_2(t) \sim \ell_1(t)^{-1}$, $t \to \infty$.

Theorems 2.1, 2.2 for the case $q : [a,\infty) \to (0,\infty)$ were proved by Marić, Tomić [14].

An answer to the second question was given first by Geluk, Marić, Tomić [4] for the case $q : [a,\infty) \to (0,\infty)$ and latter on by Marić [13, Theorem 2.5] for the case q of the arbitrary sign.

Theorem 2.3 *Let $c \in (-\infty, 1/4)$, $c \neq 0$. If $\phi(t) \to 0$ as $t \to \infty$ and*

$$\int^\infty \frac{\phi(s)^2}{s}\,ds < \infty\,,$$

then for two linearly independent $\mathcal{RV}$–solutions of $(\mathcal{L})$

$$x_i \in \mathcal{NRV}(\lambda_i), \quad i = 1,2, \quad x_i(t) = t^{\lambda_i}\ell_i(t), \ l_i \in \mathcal{NSV}$$

there hold

$$x_1(t) \sim t^{\lambda_1} \exp\left\{\int_a^t \left(\frac{\phi(s)}{s} + \frac{2\,\lambda_1}{\rho(s)} \int_s^\infty \rho(\xi)\frac{\phi(\xi)}{\xi^2}\,d\xi\right) ds\right\}, \quad t \to \infty,$$

and

$$x_2(t) \sim t^{\lambda_2}[(1 - 2\lambda_1)\ell_1(t)]^{-1}, \quad t \to \infty.$$

It is natural to expect that these results can ne extended to the more general self-adjoint linear DE (L), assuming that the function $p : [a,\infty) \to (0,\infty)$ and $q : [a,\infty) \to \mathbb{R}$ are continuous (q is allowed to be oscillatory in the sense that it takes both positive and negative values in any neighborhood of infinity). However, the class of $\mathcal{RV}$–functions in the sense of Karamata is not sufficient for such an generalization, since the possible asymptotic behavior of nonoscillatory solutions of (L) is essentially affected by the function p, more precisely, by the integral

$$J_p = \int_a^\infty \frac{dt}{p(t)}.$$

But the generalization can actually be carried out provided the classes of $\mathcal{RV}$–functions, in which the solutions of (L) are sought, are replaced by those of generalized Karamata functions reflecting the essential role played by

$$P(t) = \int_a^t \frac{ds}{p(s)} \quad \text{in the case} \quad J_p = \int_a^\infty \frac{dt}{p(t)} = \infty, \tag{2.1}$$

and

$$\pi(t) = \int_t^\infty \frac{ds}{p(s)} \quad \text{in the case} \quad J_p = \int_a^\infty \frac{dt}{p(t)} < \infty. \tag{2.2}$$

The generalized Karamata functions were introduced by Jaroš and Kusano in [6] by the following definition:

Definition 2.1 (i) A measurable function $g : [t_0,\infty) \to (0,\infty)$ is said to be a *slowly varying with respect to* R, if $g \circ R^{-1}$ is defined for all large t and $g{\circ}R^{-1}(t) = g(R^{-1}(t))$ is slowly varying (in the sense of Karamata), or equivalently, if g is expressed in the form $g(t) = L(R(t))$ for some slowly varying function L. The totality of slowly varying functions with respect to R is denoted by $\mathcal{SV}_R$.

(ii) A measurable function $f : [t_0,\infty) \to (0,\infty)$ is called a *regularly varying function of index* ϑ with respect to R if it is expressed as $f(t) = R(t)^\vartheta g(t)$ for some function g which is slowly varying with respect to R, or as $f(t) = R(t)^\vartheta L(R(t))$ for some slowly varying function L. The set of all regularly varying functions of index ϑ with respect to R is denoted by $\mathcal{RV}_R(\vartheta)$.

As a direct consequence of (1.3) the representation of such generalized regularly varying functions can be obtained. Namely, a function $f \in \mathcal{RV}_R(\vartheta)$ if and only if it is expressed in the form

$$f(t) = c(t)\exp\left\{\int_{t_0}^t \frac{R'(s)}{R(s)}\,\delta(s)ds\right\}, \quad t \geq t_0, \tag{2.3}$$

for some $t_0 > 0$ and for some measurable functions c, δ such that

$$\lim_{t\to\infty} c(t) = c_0 \in (0,\infty) \quad \textit{and} \quad \lim_{t\to\infty} \delta(t) = \vartheta.$$

If the function c in (2.3) is identically a constant on $[t_0,\infty)$, then the function f is called *normalized regularly varying with index* ϑ *with respect to* R. The totality of such functions is denoted by $\mathcal{NRV}_R(\vartheta)$. Use is made of the notation $\mathcal{SV}_R = \mathcal{RV}_R(0)$ or $\mathcal{NSV}_R = \mathcal{NRV}_R(0)$ to justify the use of the terminology slowly varying instead of regularly varying of index 0.

For basic properties of generalized Karamata functions see [6, 8].

Two cases (2.1) and (2.2) were distinguished in [6] and it was shown that the set of generalized Karamata functions $\{\mathcal{RV}_P(\rho) : \rho \in \mathbb{R}\}$ or $\{\mathcal{RV}_{1/\pi}(\rho) : \rho \in \mathbb{R}\}$ formed with the choice of $R(t) = P(t)$ or $R(t) = 1/\pi(t)$ provides a well-suited framework for the asymptotic analysis of eq. (L) with p satisfying $J_p = \infty$ or $J_p < \infty$, respectively. More specifically, sharp conditions for (L) to have a pair of nonoscillatory solutions $\{x_1(t), x_2(t)\}$ belonging to $\{\mathcal{RV}_P(\varrho_1), \mathcal{RV}_P(\varrho_2)\}$ or to $\{\mathcal{RV}_{1/\pi}(\varrho_1), \mathcal{RV}_{1/\pi}(\varrho_2)\}$ for some specified values of ϱ_1 and ϱ_1 were established.

Theorem 2.4 (Jaroš, Kusano [6]) *Assume* $J_p = \infty$. *Let* $c \in (-\infty, \frac{1}{4})$ *and denote by* λ_1, λ_2, $\lambda_1 < \lambda_2$ *the real roots of the quadratic equation* $\lambda^2 - \lambda + c = 0$. *The equation* (L) *is nonoscillatory and has a fundamental set of solutions* $\{x_1(t), x_2(t)\}$ *such that*

$$x_1 \in \mathcal{NRV}_P(\lambda_1), \qquad x_2 \in \mathcal{NRV}_P(\lambda_2)$$

if and only if

$$(C_1) \qquad \lim_{t\to\infty} P(t) \int_t^\infty q(s)\, ds = c\,.$$

Theorem 2.5 (Jaroš, Kusano [6]) *Assume* $J_p = \infty$ *and that*

$$\lim_{t\to\infty} P(t) \int_t^\infty q(s)\, ds = \frac{1}{4}\,.$$

Define

$$\Phi(t) = P(t) \int_t^\infty q(s)\, ds - \frac{1}{4}, \quad \Psi(t) = \int_t^\infty \frac{|\Phi(s)|}{p(s)P(s)}\, ds$$

and suppose that

$$\int^\infty \frac{|\Phi(s)|}{p(s)P(s)}\, ds < \infty \quad \textit{and} \quad \int^\infty \frac{\Psi(s)}{p(s)P(s)}\, ds < \infty\,.$$

Then, the linear DE (L) *possesses a fundamental set of solutions* $\{x_1(t), x_2(t)\}$ *such that*

$$x_i \in \mathcal{NRV}_P\left(\frac{1}{2}\right), \quad i = 1, 2$$

of the form

$$x_1(t) = \sqrt{P(t)}\, \ell_1(t), \qquad x_2(t) = \sqrt{P(t)}\, \ln\big(P(t)\big)\, \ell_2(t), \quad \ell_i \in \mathcal{NSV}_P,\ i = 1, 2\,,$$

and

$$\lim_{t\to\infty} \ell_i(t) = L_i \in (0,\infty), \quad i = 1, 2 \quad \textit{with } L_1 L_2 = 1.$$

Theorem 2.6 (Jaroš, Kusano [6]) *Assume $J_p < \infty$. Let $c \in (-\infty, \frac{1}{4})$ and denote by μ_1, μ_2, $\mu_1 < \mu_2$ the real roots of the quadratic equation $\mu^2 + \mu + c = 0$. The equation (L) is nonoscillatory and has a fundamental set of solutions $\{x_1(t), x_2(t)\}$ such that*

$$x_1 \in \mathcal{NRV}_{1/\pi}(\mu_1), \qquad x_2 \in \mathcal{NRV}_{1/\pi}(\mu_2)$$

if and only if

$$(C_2) \qquad \lim_{t\to\infty} \frac{1}{\pi(t)} \int_t^\infty \pi^2(t) q(s)\, ds = c\,.$$

Theorem 2.7 (Jaroš, Kusano [6]) *Assume $J_p < \infty$ and that*

$$\lim_{t\to\infty} \frac{1}{\pi(t)} \int_t^\infty \pi^2(s) q(s)\, ds = \frac{1}{4}\,.$$

Define

$$\Phi(t) = \frac{1}{\pi(t)} \int_t^\infty \pi^2(s) q(s)\, ds - \frac{1}{4}, \quad \Psi(t) = \int_t^\infty \frac{|\Phi(s)|}{p(s)\pi(s)}\, ds$$

and suppose that

$$\int^\infty \frac{|\Phi(s)|}{p(s)\pi(s)}\, ds < \infty \quad \text{and} \quad \int^\infty \frac{\Psi(s)}{p(s)\pi(s)}\, ds < \infty\,.$$

Then, the linear DE (L) possesses a fundamental set of solutions $\{x_1(t), x_2(t)\}$ such that

$$x_i \in \mathcal{NRV}_{1/\pi}\left(-\frac{1}{2}\right), \quad i = 1, 2$$

of the form

$$x_1(t) = \sqrt{\pi(t)}\, \ell_1(t), \qquad x_2(t) = \sqrt{\pi(t)}\, \ln\left(\frac{1}{\pi(t)}\right) \ell_2(t), \quad \ell_i \in \mathcal{NSV}_P,\ i = 1, 2\,,$$

and

$$\lim_{t\to\infty} \ell_i(t) = L_i \in (0, \infty), \quad i = 1, 2 \quad \text{with } L_1 L_2 = 1.$$

3 Existence of $\mathcal{RV}$–solutions of second-order half-linear differential equation

Considering qualitative resemblance of the linear and the half-linear DE, the main theorems of Howard and Marić (Theorems 2.1, 2.2, 2.3) admits natural generalization to the half-linear equation

$$(\mathcal{HL}) \qquad (|x'|^\alpha \text{sgn}\, x')' + q(t)|x|^\alpha \text{sgn}\, x = 0\,.$$

under the assumption that the function $q : [a, \infty) \to \mathbb{R}$ is conditionally integrable on $[a, \infty)$.

Theorem 3.1 (Jaroš, Kusano, Tanigawa [7]) *Let c be a constant such that*

$$c \in (-\infty, E(\alpha)), \quad \textit{where} \quad E(\alpha) = \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}. \tag{3.1}$$

Let λ_1, λ_2 ($\lambda_1 < \lambda_2$) denote the two real roots of the equation

$$|\lambda|^{1+\frac{1}{\alpha}} - \lambda + c = 0. \tag{3.2}$$

Equation $(\mathcal{HL})$ possesses a pair of regularly varying solutions

$$x_i \in \mathcal{NRV}\Big(\lambda_i^{\frac{1}{\alpha}*}\Big), \quad i = 1, 2, \quad \lambda_i^{\frac{1}{\alpha}*} = |\lambda_i|^{\frac{1}{\alpha}} \operatorname{sgn} \lambda_i \tag{3.3}$$

if and only if

$$\lim_{t\to\infty} t^\alpha \int_t^\infty q(s) ds = c.$$

Theorem 3.2 (Jaroš, Kusano, Tanigawa [7]) *Assume that*

$$\lim_{t\to\infty} t^\alpha \int_t^\infty q(s)\, ds = E(\alpha).$$

Define

$$\Phi(t) = t^\alpha \int_t^\infty q(s)\, ds - E(\alpha), \quad \Psi(t) = \int_t^\infty \frac{|\Phi(s)|}{s}\, ds$$

and suppose that

$$\int^\infty \frac{|\Phi(s)|}{s}\, ds < \infty \quad \textit{and} \quad \int^\infty \frac{\Psi(s)}{s}\, ds < \infty.$$

Then, the half-linear DE (HL) is nonoscillatory and possesses normalized $\mathcal{RV}-$solution of index $\frac{\alpha}{\alpha+1}$ of the form

$$x_1(t) = t^{\frac{\alpha}{\alpha+1}} \ell(t), \quad \textit{with} \quad \ell \in \mathcal{NSV} \quad \textit{and} \quad \lim_{t\to\infty} \ell(t) = L \in (0, \infty).$$

Considering more general half-linear equation (HL), as in the linear case the class of Karamata functions is not sufficient to properly describe asymptotic behavior of nonoscillatory solutions of (HL), which depends on convergence or divergence of the integral

$$I_p = \int_a^\infty \frac{ds}{p(s)^{1/\alpha}}.$$

For these reason, Jaroš, Kusano and Tanigawa in [8] used the generalized regularly varying functions with respect to

$$P(t) = \int_a^t \frac{ds}{p(s)^{1/\alpha}} \quad \text{in the case} \quad I_p = \infty$$

or with respect to

$$\pi(t) = \int_t^\infty \frac{ds}{p(s)^{1/\alpha}} \quad \text{in the case} \quad I_p < \infty,$$

to provide necessary and sufficient condition for (HL) to possesses a pair of generalized regularly varying solutions. They proved the next four theorems, which are generalization of Jaroš and Kusano [6] (Theorems 2.4, 2.5, 2.6, 2.7) for the linear DE:

Theorem 3.3 (Jaroš, Kusano, Tanigawa [8]) *Assume $I_p = \infty$. Let c be a constant satisfying* (3.1) *and λ_1, λ_2 $(\lambda_1 < \lambda_2)$ denote the two real roots of the equation* (3.2). *Equation* (HL) *possesses a pair of solutions*

$$x_i \in \mathcal{NRV}_P\Big(\lambda_i^{\frac{1}{\alpha}*}\Big), \quad i = 1, 2, \tag{3.4}$$

if and only if

$$\lim_{t\to\infty} P(t)^\alpha \int_t^\infty q(s)ds = c.$$

Theorem 3.4 (Jaroš, Kusano, Tanigawa [8]) *Assume $I_p = \infty$ and that*

$$\lim_{t\to\infty} P(t)^\alpha \int_t^\infty q(s)\,ds = E(\alpha).$$

Define

$$\Phi(t) = P(t)^\alpha \int_t^\infty q(s)\,ds - E(\alpha), \quad \Psi(t) = \int_t^\infty \frac{|\Phi(s)|}{p(s)^{1/\alpha}P(s)}\,ds$$

and suppose that

$$\int^\infty \frac{|\Phi(s)|}{p(s)^{1/\alpha}P(s)}\,ds < \infty \quad and \quad \int^\infty \frac{\Psi(s)}{p(s)^{1/\alpha}P(s)}\,ds < \infty.$$

Then, the half-linear DE (HL) has a solution

$$x \in \mathcal{NRV}_P\left(\frac{\alpha}{\alpha+1}\right)$$

such that

$$x_1(t) = P(t)^{\frac{\alpha}{\alpha+1}}\,\ell(t), \quad with \quad \ell \in \mathcal{NSV}_P$$

satisfyng $\lim_{t\to\infty}\ell(t) = L \in (0, \infty)$.

Theorem 3.5 (Jaroš, Kusano, Tanigawa [8]) *Assume $I_p < \infty$. Let c be a constant satisfying* (3.1) *and ρ_1, ρ_2 $(\rho_1 < \rho_2)$ denote the two real roots of the equation*

$$|\rho|^{1+\frac{1}{\alpha}} + \rho + c = 0. \tag{3.5}$$

Equation (HL) *possesses a pair of solutions*

$$x_i \in \mathcal{NRV}_{1/\pi}\Big(\rho_i^{\frac{1}{\alpha}*}\Big), \quad i = 1, 2, \tag{3.6}$$

if and only if

$$\lim_{t\to\infty} \frac{1}{\pi(t)} \int_t^\infty \pi(s)^{\alpha+1} q(s)ds = \alpha\, c.$$

Theorem 3.6 (Jaroš, Kusano, Tanigawa [8]) *Assume $I_p < \infty$ and that*

$$\lim_{t\to\infty} \frac{1}{\pi(t)} \int_t^\infty \pi(s)^{\alpha+1} q(s)\,ds = \alpha\, E(\alpha).$$

Define

$$\Phi(t) = \frac{1}{\pi(t)} \int_t^\infty \pi(s)^{\alpha+1} q(s)\, ds - \alpha\, E(\alpha), \quad \Psi(t) = \int_t^\infty \frac{|\Phi(s)|}{p(s)^{1/\alpha}\pi(s)}\, ds$$

and suppose that

$$\int^\infty \frac{|\Phi(s)|}{p(s)^{1/\alpha}\pi(s)}\, ds < \infty \quad \textit{and} \quad \int^\infty \frac{\Psi(s)}{p(s)^{1/\alpha}\pi(s)}\, ds < \infty\,.$$

Then, the half-linear DE (HL) *has a solution*

$$x \in \mathcal{NRV}_{1/\pi}\left(-\frac{\alpha}{\alpha+1}\right)$$

such that

$$x_1(t) = \pi(t)^{\frac{\alpha}{\alpha+1}}\, \ell(t), \quad \textit{with} \quad \ell \in \mathcal{NSV}_{1/\pi}$$

satisfyng $\lim_{t\to\infty} \ell(t) = L \in (0,\infty)$.

4 Asymptotic behavior of $\mathcal{RV}$−solutions of second-order half-linear differential equation

Since answers to the first question of providing necessary and sufficient conditions for the existence of $\mathcal{RV}$−solutions has been given in Section 2 for the linear DE and in Section 3 for the half-linear DE, a natural question arises about the possibility of acquiring detailed information on the asymptotic behavior at infinity of the solutions whose existence is assured by the above theorems. Thus, in this Section we present results giving answer to the second question about determining the unique explicit asymptotic formula of $\mathcal{RV}$−solutions.

4.1 Asymptotic behavior of regularly varying solutions of $(\mathcal{HL})$

Accurate asymptotic formulas for regularly varying solutions of $(\mathcal{HL})$, whose existence is assured by Theorem 3.1, was established by Kusano and Manojlović in [10].

Let c be a constant satisfying (3.1) and let λ_i, $i = 1,2$, $(\lambda_1 < \lambda_2)$ denote the real roots of the equation (3.2). It is clear that

$$0 < \lambda_1 < \lambda_2 \quad \text{if} \quad c \in (0, E(\alpha)); \qquad \lambda_1 < 0 < \lambda_2 \quad \text{if} \quad c \in (-\infty, 0)$$
$$\text{and} \quad 0 = \lambda_1 < \lambda_2 = 1 \quad \text{if}\; c = 0\,.$$

Note that in the case $c = 0$, Theorem 3.1 ensures existence of a pair of regularly varying solutions of $(\mathcal{HL})$

$$x_1 \in \mathcal{NSV}, \quad x_2 \in \mathcal{NRV}(1) \tag{4.1}$$

if and only if

$$Q(t) := t^\alpha \int_t^\infty q(s) ds \to 0, \quad t \to \infty\,.$$

Thus, taking into account necessity to treat slowly varying functions in a special way, cases $c = 0$ and $c \neq 0$ were examined in [10], separately. So, next two results provide the unique explicit asymptotic formula of $\mathcal{RV}$−solutions (4.1) of $(\mathcal{HL})$.

Theorem 4.1 (Kusano, Manojlović [10]) *Let ϕ_1 be a positive continuous function on $[a,\infty)$ which decreases to 0 as $t\to\infty$ and satisfies*

$$\int_a^\infty \frac{\phi_1(t)^{\frac{1}{\alpha}}}{t}dt=\infty, \qquad \int_a^\infty \frac{\phi_1(t)^{\frac{2}{\alpha}}}{t}dt<\infty.$$

Suppose that the function Q is eventually one-signed and satisfies

$$|Q(t)|=\phi_1(t)+O\Big(\phi_1(t)^{1+\frac{1}{\alpha}}\Big), \quad t\to\infty.$$

Then, equation $(\mathcal{HL})$ possesses a solution $x_1\in\mathcal{NSV}$ such that

$$x_1(t)\sim k_1\exp\Big\{\operatorname{sgn}Q\int_a^t \frac{\phi_1(s)^{\frac{1}{\alpha}}}{s}ds\Big\}, \quad t\to\infty.$$

for some constant $k_1>0$.

Theorem 4.2 (Kusano, Manojlović [10]) *Let ϕ_2 be a continuously differentiable function on $[a,\infty)$ which is slowly varying, decreases to 0 as $t\to\infty$ and satisfies*

$$\int_a^\infty \frac{\phi_2(t)}{t}dt=\infty, \qquad \int_a^\infty \frac{\phi_2(t)^2}{t}dt<\infty.$$

Suppose that the function Q is eventually one-signed and satisfies

$$|Q(t)|=\phi_2(t)+O\big(\phi_2(t)^2\big), \quad t\to\infty.$$

Then, equation $(\mathcal{HL})$ possesses a solution $x_2\in\mathcal{NRV}(1)$ such that

$$x_2(t)\sim k_2\, t\exp\Big\{-\operatorname{sgn}\, Q\int_a^t \frac{\phi_2(s)}{s}ds\Big\}, \quad t\to\infty,$$

for some constant $k_2>0$.

Next two results provide accurate asymptotic formulas for regularly varying solutions (3.3) of $(\mathcal{HL})$, which exist if and only if

$$Q_c(t)=t^\alpha\int_t^\infty q(s)ds-c\to 0, \quad t\to\infty\,.$$

Theorem 4.3 (Kusano, Manojlović [10]) *Let ψ_1 be a positive continuously differentiable function on $[a,\infty)$ which decreases to 0 as $t\to\infty$, has the property that $t|\psi_1'|$ is decreasing and satisfies*

$$\int_a^\infty \frac{\psi_1(t)}{t}dt=\infty, \qquad \int_a^\infty \frac{\psi_1(t)^2}{t}dt<\infty.$$

Suppose that the function $Q_c(t)$ is eventually one-signed and satisfies

$$|Q_c(t)|=\psi_1(t)+O(\psi_1(t)^2), \quad t\to\infty.$$

Then, equation $(\mathcal{HL})$ possesses a solution $x_1\in\mathcal{NRV}\Big(\lambda_1^{\frac{1}{\alpha}*}\Big)$ such that for some constant $k_1>0$

$$x_1(t)\sim k_1\, t^{\lambda_1^{\frac{1}{\alpha}*}}\exp\Big\{\frac{\lambda_1^{\frac{1}{\alpha}*}}{\lambda_1(\alpha-\mu_1)}\operatorname{sgn}Q_c\int_a^t \frac{\psi_1(s)}{s}ds\Big\}, \quad t\to\infty,$$

where $\mu_1=(\alpha+1)\lambda_1^{\frac{1}{\alpha}*}$.

Theorem 4.4 (Kusano, Manojlović [10]) *Let ψ_2 be a positive continuously differentiable slowly varying function on $[a,\infty)$ which decreases to 0 as $t\to\infty$, has the property that $t|\psi_2'|$ is slowly varying and satisfies*

$$\int_a^\infty \frac{\psi_2(t)}{t}dt=\infty, \qquad \int_a^\infty \frac{\psi_2(t)^2}{t}dt<\infty.$$

Suppose that the function $Q_c(t)$ is eventually one-signed and satisfies

$$|Q_c(t)|=\psi_2(t)+O\big(\psi_2(t)^2\big), \quad t\to\infty.$$

Then, equation $(\mathcal{HL})$ possesses a solution $x_2\in\mathcal{NRV}\Big(\lambda_2^{\frac{1}{\alpha}*}\Big)$ such that for some constant $k_2>0$

$$x_2(t)\sim k_2\, t^{\lambda_2^{\frac{1}{\alpha}}}\exp\left\{\frac{\lambda_2^{\frac{1}{\alpha}-1}}{\alpha-\mu_2}\operatorname{sgn} Q_c\int_a^t\frac{\psi_2(s)}{s}ds\right\}, \quad t\to\infty,$$

where $\mu_2=(\alpha+1)\lambda_2^{\frac{1}{\alpha}*}$.

4.2 Asymptotic behavior of generalized regularly varying solutions with respect to P of (HL)

For the half-linear DE (HL) in the case $I_p=\infty$, accurate asymptotic formulas for generalized regularly varying solutions with respect to P, whose existence is assured by Theorem 3.3, was established by Kusano and Manojlović in [11]. As in [10], cases $c=0$ and $c\neq 0$ were treated separately. In the case $c\neq 0$, Theorem 3.3 ensures existence of pair of generalized $\mathcal{RV}-$ solutions (3.4) of $(\mathcal{HL})$ if and only if

$$W(t)=P(t)^\alpha\int_t^\infty q(s)ds-c\to 0, \quad t\to\infty\,,$$

while in the case $c=0$ Theorem 3.3 ensures existence of pair of generalized $\mathcal{RV}-$ solutions of $(\mathcal{HL})$

$$x_1\in\mathcal{NSV}_P \;\text{ and }\; x_2\in\mathcal{NRV}_P(1)$$

if and only if

$$W_0(t)=P(t)^\alpha\int_t^\infty q(s)ds\to 0, \quad t\to\infty\,.$$

Theorem 4.5 (Kusano, Manojlović [11]) *Let $\phi\in\mathcal{SV}_P$ tends to 0 as $t\to\infty$ and satisfies*

$$\int_a^\infty \frac{\phi(t)}{p(t)^{1/\alpha}P(t)}dt=\infty, \qquad \int_a^\infty \frac{\phi(t)^2}{p(t)^{1/\alpha}P(t)}dt<\infty.$$

Suppose that the function W is eventually one-signed and satisfies

$$|W(t)|=\phi(t)+O(\phi(t)^2), \quad t\to\infty.$$

Then, equation (HL) possesses a pair of solutions $x_i\in\mathcal{RV}_P(\lambda_i^{\frac{1}{\alpha}*})$, $i=1,2$ which asymptotic behavior are determined by

$$x_i(t)=\kappa_i\, P(t)^{\lambda_i^{\frac{1}{\alpha}*}}\exp\left\{-\frac{|\lambda_i|^{\frac{1}{\alpha}-1}}{(\alpha+1)\lambda_i^{\frac{1}{\alpha}*}-\alpha}\operatorname{sgn} W\int_a^t(1+o(1))\frac{\phi(s)}{p(s)^{1/\alpha}P(s)}ds\right\},$$

as $t\to\infty$, for some constant $\kappa_i>0$, $i=1,2$.

Theorem 4.6 (Kusano, Manojlović [11]) *Let $\phi \in \mathcal{SV}_P$, $\lim_{t\to\infty}\phi(t) = 0$ and satisfies*

$$\int_a^\infty \frac{\phi(t)}{p(t)^{1/\alpha}P(t)}dt = \infty, \qquad \int_a^\infty \frac{\phi(t)^2}{p(t)^{1/\alpha}P(t)}dt < \infty.$$

Suppose that the function $W_0(t)$ is eventually one-signed and satisfies

$$|W_0(t)| = \phi(t) + O(\phi(t)^2), \quad t \to \infty.$$

Then, equation (HL) possesses a solution $x_2 \in \mathcal{NRV}_P(1)$ which asymptotic behavior is determined by

$$x_2(t) = \kappa_2\, P(t) \exp\left\{-\operatorname{sgn} W_0 \int_a^t (1+o(1))\frac{\phi(s)}{p(s)^{1/\alpha}P(s)}ds\right\}, \quad t \to \infty.$$

for some constant $\kappa_2 > 0$.

Theorem 4.7 (Kusano, Manojlović [11]) *Let $\Phi \in \mathcal{SV}_P$, $\lim_{t\to\infty}\Phi(t) = 0$ and satisfies*

$$\int_a^\infty \frac{\Phi(t)^{\frac{1}{\alpha}}}{p(t)^{1/\alpha}P(t)}dt = \infty, \qquad \int_a^\infty \frac{\Phi(t)^{\frac{2}{\alpha}}}{p(t)^{1/\alpha}P(t)}dt < \infty.$$

Suppose that the function W_0 is eventually one-signed and satisfies

$$|W_0(t)| = \Phi(t) + O\Big(\Phi(t)^{1+\frac{1}{\alpha}}\Big), \quad t \to \infty.$$

Then, equation (HL) possesses a nontrivial slowly varying solution x_1 with respect to P which asymptotic behavior is determined by

$$x_1(t) = \kappa_1 \exp\left\{\operatorname{sgn} W_0 \int_a^t (1+o(1))\frac{\Phi(s)^{\frac{1}{\alpha}}}{p(s)^{1/\alpha}P(s)}ds\right\}, \quad t \to \infty.$$

for some constant $\kappa_1 > 0$.

4.3 Asymptotic behavior of generalized regularly varying solutions with respect to $1/\pi$ of (HL)

For the half-linear DE (HL) in the case $I_p < \infty$, accurate asymptotic formulas for generalized regularly varying solutions with respect to $1/\pi$, whose existence is assured by Theorem 3.5, was also established by Kusano and Manojlović in [11]. As in [10], cases $c = 0$ and $c \neq 0$ were treated separately. In the case $c \neq 0$, Theorem 3.5 ensures existence of pair of generalized $\mathcal{RV}-$ solutions (3.6) of $(\mathcal{HL})$ if and only if

$$\Omega(t) = \frac{1}{\pi(t)}\int_t^\infty \pi(s)^{\alpha+1}q(s)ds - \alpha\, c \to 0, \quad t \to \infty,$$

while in the case $c = 0$ Theorem 3.3 ensures existence of pair of generalized $\mathcal{RV}-$ solutions of $(\mathcal{HL})$

$$x_2 \in \mathcal{NSV}_{1/\pi} \ \text{ and } \ x_1 \in \mathcal{NRV}_{1/\pi}(-1)$$

if and only if

$$\Omega_0(t) = \frac{1}{\pi(t)}\int_t^\infty \pi(s)^{\alpha+1}q(s)ds \to 0, \quad t \to \infty$$

Theorem 4.8 (Kusano, Manojlović [11]) *Let c be a nonzero constant in* $(-\infty, E(\alpha))$. *Let* $\psi \in \mathcal{SV}_{1/\pi}$ *tends to* 0 *as* $t \to \infty$ *and satisfies*

$$\int_a^\infty \frac{\psi(t)}{p(t)^{1/\alpha}\pi(t)}dt = \infty, \qquad \int_a^\infty \frac{\psi(t)^2}{p(t)^{1/\alpha}\pi(t)}dt < \infty.$$

Suppose that the function Ω *is eventually one-signed and satisfies*

$$|\Omega(t)| = \psi(t) + O(\psi(t)^2), \quad t \to \infty.$$

Then, equation (HL) *possesses a pair of solutions* $x_i \in \mathcal{RV}_{1/\pi}(\rho_i^{\frac{1}{\alpha}*})$, $i = 1, 2$ *given by the asymptotic formula*

$$x_i(t) = \kappa_i \left(\frac{1}{\pi(t)}\right)^{\rho_i^{\frac{1}{\alpha}*}} \exp\left\{-\frac{|\rho_i|^{\frac{1}{\alpha}-1}}{\alpha\Big((\alpha+1)\varrho_i^{\frac{1}{\alpha}*}+\alpha\Big)} \operatorname{sgn}\Omega \int_a^t (1+o(1))\frac{\psi(s)}{p(s)^{1/\alpha}\pi(s)}ds\right\}, \quad i = 1, 2,$$

as $t \to \infty$, *for some constants* $\kappa_i > 0$, $i = 1, 2$.

Theorem 4.9 (Kusano, Manojlović [11]) *Let* $\psi \in \mathcal{SV}_{1/\pi}$ *tends to* 0 *as* $t \to \infty$ *and satisfies*

$$\int_a^\infty \frac{\psi(t)}{p(t)^{1/\alpha}\pi(t)}dt = \infty, \qquad \int_a^\infty \frac{\psi(t)^2}{p(t)^{1/\alpha}\pi(t)}dt < \infty.$$

Suppose that the function Ω_0 *is eventually one-signed and satisfies*

$$|\Omega_0(t)| = \psi(t) + O(\psi(t)^2), \quad t \to \infty.$$

Then, equation (HL) *possesses a solution* $x_1 \in \mathcal{NRV}_{1/\pi}(-1)$ *given by the asymptotic formula*

$$x_1(t) = \kappa_1\, \pi(t) \exp\left\{\frac{\operatorname{sgn}\Omega_0}{\alpha}\int_a^t (1+o(1))\frac{\psi(s)}{p(s)^{1/\alpha}\pi(s)}ds\right\}, \quad t \to \infty.$$

for some constant $\kappa_1 > 0$.

Theorem 4.10 (Kusano, Manojlović [11]) *Let* $\Psi \in \mathcal{SV}_{1/\pi}$, $\lim_{t\to\infty}\Psi(t) = 0$ *and satisfies*

$$\int_a^\infty \frac{\Psi(t)^{\frac{1}{\alpha}}}{p(t)^{1/\alpha}\pi(t)}dt = \infty, \qquad \int_a^\infty \frac{\Psi(t)^{\frac{2}{\alpha}}}{p(t)^{1/\alpha}\pi(t)}dt < \infty.$$

Suppose that the function Ω_0 *is eventually one-signed and satisfies*

$$|\Omega_0(t)| = \Psi(t) + O\big(\Psi(t)^{1+\frac{1}{\alpha}}\big), \quad t \to \infty.$$

Then, equation (HL) *possesses a solution* $x_2 \in \mathcal{NSV}_{1/\pi}$ *given by the asymptotic formula*

$$x_2(t) = \kappa_2 \exp\left\{\frac{\operatorname{sgn}\Omega_0}{\alpha^{1/\alpha}}\int_a^t (1+o(1))\frac{\Psi(s)^{\frac{1}{\alpha}}}{p(s)^{1/\alpha}\pi(s)}ds\right\}, \quad t \to \infty$$

for some constant $\kappa_2 > 0$.

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