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A TIME-DISCRETE APPROXIMATE SCHEME FOR MULTI-PHASE MEAN CURVATURE FLOW

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1. The initial data for the mean curvature flow

Given a compact smooth hypersurface $\Gamma_0 \subset \mathbb{R}^{n+1}$, one can show the existence of a family of smooth hypersurfaces $\{\Gamma_t\}_{t>0}$ for a while, where the velocity of motion of Γ_t at each point $x \in \Gamma_t$ is given by the mean curvature of Γ_t at that point. This is called the mean curvature flow (abbreviated by MCF). For example, given a sphere as the initial hypersurface, the MCF shrinks the sphere. As the radius gets smaller, the mean curvature gets bigger, and in finite time, the radii of shrinking spheres become zero. In general, this is a typical picture of the MCF. Singular behavior occurs in finite time. If the MCF is written locally as a graph, the PDE which describes this motion is similar to the well-known heat equation, since the velocity corresponds to the time-derivative of the graph and the mean curvature to the Laplacian of the graph (the mean curvature is the sum of principal curvatures) and they are supposed to be equal. It is also well-known that the heat equation has a very strong smoothing effect. Even if the initial data is very rough, the solution gets smoothed out immediately and becomes C^∞. Putting these facts together, it is natural to ask: What is the roughest set with which one can find the MCF starting from it?

There are a few well-known facts about the MCF for such question. First of all, the MCF has the area-decreasing property. To describe this more precisely, let us introduce a few notations. Let $\mathcal{H}^n$ be the n-dimensional Hausdorff measure, which measures the hypersurface area of a given set if the set is a nice C^1 hypersurface in $\mathbb{R}^{n+1}$. Suppose we have a nice family of hypersurfaces $\{\Gamma_t\}$ which is the MCF. Then we can check that

$$\frac{d}{dt}\mathcal{H}^n(\Gamma_t) = -\int_{\Gamma_t} |h_{\Gamma_t}|^2\, d\mathcal{H}^n \le 0 \tag{1.1}$$

and this is what is meant by the area-decreasing property. Here, h_{Γ_t} is the mean curvature vector of Γ_t. For a later use, the following characterization of the mean curvature vector is very important. For a nice C^2 hypersurface Γ, we have the following formula

$$\int_\Gamma \mathrm{div}_\Gamma g\, d\mathcal{H}^n = -\int_\Gamma h_\Gamma \cdot g\, d\mathcal{H}^n \tag{1.2}$$

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for any $g \in C_c^1(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$, which is a set of test vector fields with compact support. $\mathrm{div}_\Gamma g$ is the divergence of g restricted to the tangent space of Γ. The property (1.1) shows that the hypersurface area is like the energy of the problem. For any problem of this kind, it is usually the case that one looks at the "energy class" for the initial data. Thus it is reasonable to restrict the class to the sets Γ_0 with $\mathcal{H}^n(\Gamma_0) < \infty$, or at least locally so, that is, for any $R > 0$, $\mathcal{H}^n(\Gamma_0 \cap \{x : |x| \leq R\}) < \infty$. The analogy with a solution u of the heat equation $u_t = \Delta u$ is

$$\frac{d}{dt}\int |\nabla u|^2 = -\int 2(\Delta u)^2.$$

This is very close in spirit to (1.1), since, if the hypersurface is a graph of u, then $\mathcal{H}^n(\Gamma) = \int \sqrt{1+|\nabla u|^2} \approx \int (1+\frac{1}{2}|\nabla u|^2)$ if $|\nabla u|$ is small. Since we do not need to assume $\nabla u \in L^2$ for the solution of the heat equation, the energy class may still be too restrictive, perhaps. Still, asking $\mathcal{H}^n(\Gamma_0) < \infty$ (or locally finite) is certainly a reasonable restriction. The next point to think about is whether it is reasonable to ask that Γ_0 be closed. In general, it is possible that the closure of Γ_0 is much bigger than Γ_0. All one needs to do is to add countable dense points to Γ_0 to have the closure of Γ_0 being $\mathbb{R}^{n+1}$. There seem to be some points to reflect on, but again it seems reasonable to ask Γ_0 being closed, at least to make the situation simpler. Still, we have the following well-known fact about such set Γ_0 of locally finite measure from geometric measure theory. In general, Γ_0 can be decomposed into two mutually disjoint sets, one is the so-called countably n-rectifiable set, and the other is a purely unrectifiable set. A set is called countably n-rectifiable if is is contained in a countable union of C^1 n-dimensional manifolds except for a null set with respect to $\mathcal{H}^n$. A purely unrectifiable set is the one which does not contain any countably n-rectifiable set of positive $\mathcal{H}^n$ measure, and it is a fractal-like irregular set. It may be possible to define some analogue of mean curvature by certain smoothing procedure for such irregular set, but the L^2 norm of any approximate mean curvature should be very large, and such irregular set will be wiped out immediately. Thus it is reasonable to consider only closed countably n-rectifiable sets as the initial data for MCF. In the following, we present an existence theorem for MCF with such initial data.

2. Definition of Brakke flow

To describe the MCF which starts from closed countably n-rectifiable set, we need the notion of Brakke flow [1]. Here, for brevity of presentation, I describe the Brakke flow without using the notion of varifold. It is a weak solution of MCF and the idea of Brakke flow is that one makes a transition from the family of sets $\{\Gamma_t\}_{t\geq 0}$ to that of Radon measures $\{\mathcal{H}^n\lfloor_{\Gamma_t}\}_{t\geq 0}$.

Definition 2.1. *For $0 < T < \infty$ and open set $U \subset \mathbb{R}^{n+1}$, a family of Radon measures $\{\mu_t\}_{t\in[0,T]}$ is a Brakke flow in U if the following assumptions are all satisfied.*

(1) *For a.e.* $t \in [0,T]$, *there exist an* $\mathcal{H}^n$*-measurable countably* n*-rectifiable set* $\Gamma_t \subset U$ *and an* $\mathcal{H}^n$*-measurable function* $\theta_t : \Gamma_t \to \mathbb{N}$ *such that* $\mu_t = \theta_t \mathcal{H}^n \lfloor_{\Gamma_t}$, *that is,* μ_t *is an* n*-dimensional Hausdorff measure restricted to* Γ_t *and weighted by a natural-number-valued function* θ_t.

(2) $\sup_{t\in[0,T]} \mu_t(K) < \infty$ *for all compact set* $K \subset U$.

(3) *For a.e.* $t \in [0,T]$, μ_t *has a generalized mean curvature vector* h_{μ_t} *such that*

$$\int_0^T \int_K |h_{\mu_t}|^2 \, d\mu_t dt < \infty$$

for any compact set $K \subset U$.

(4) *For any* $0 \le t_1 < t_2 \le T$ *and* $\phi \in C_c^1(U \times [0,T]; \mathbb{R}^+)$, *we have*

$$\int_U \phi \, d\mu_t \Big|_{t=t_1}^{t_2} \le \int_{t_1}^{t_2} \int_U (\nabla \phi - \phi \, h_{\mu_t}) \cdot h_{\mu_t} + \frac{\partial \phi}{\partial t} \, d\mu_t dt.$$

If Γ_t is a smooth MCF, by setting $\mu_t = \mathcal{H}^n \lfloor_{\Gamma_t}$ (so that $\theta_t = 1$) and regarding h_{μ_t} as the usual mean curvature vector of Γ_t, (1)-(4) are all satisfied. In fact, (4) is satisfied with equality in place of inequality. We skip the definition of generalized mean curvature vector h_{μ_t} but it is a notion which coincides with the classical one when Γ_t is smooth. If we assume that θ_t is equal to 1 for a.e. $t \in [0,T]$, so that $\mu_t = \mathcal{H}^n \lfloor_{\Gamma_t}$ for a.e. $t \in [0,T]$, a partial regularity result [1, 2, 4] shows that Γ_t is C^∞ MCF for a.e. $t \in [0,T]$ and $\mathcal{H}^n$ a.e. in U. For the precise statement, see [4]. The function θ_t is called multiplicity and it helps describe the somewhat undesirable "folding" of surface measures due to collisions.

3. Existence theorem of [3]

Assume that $\Gamma_0 \subset \mathbb{R}^{n+1}$ is a closed countably n-rectifiable set with locally finite $\mathcal{H}^n$ measure. Additionally assume that there exists $c_1 \ge 0$ such that $\int_{\Gamma_0} \exp(-c_1|x|) \, d\mathcal{H}^n(x) < \infty$ and that $\mathbb{R}^{n+1} \setminus \Gamma_0$ is not connected. As I mentioned before, Γ_0 can be a very irregular and messy set in $\mathbb{R}^{n+1}$, even with some lower dimensional pieces around, which can grow exponentially near infinity. Next, choose $E_{0,1}, \ldots, E_{0,N}$ which are non-empty, mutually disjoint open sets in $\mathbb{R}^{n+1}$ such that $\mathbb{R}^{n+1} \setminus \Gamma_0 = \cup_{i=1}^N E_{0,i}$, and $N \ge 2$. With the assumption of $\mathbb{R}^{n+1} \setminus \Gamma_0$ being non-connected, we can choose such a set of open sets. Here, $N \ge 2$ is arbitrary, but finite. We may regard this process of assigning a number to each connected component as "labeling". If $\mathbb{R}^{n+1} \setminus \Gamma_0$ has N connected components, it is natural to label them from 1 to N. But it is not necessary to differentiate them if one wishes. If there are countably many connected components, we just label these components from 1 to N, for some N. So the labeling of $\mathbb{R}^{n+1} \setminus \Gamma_0$ is somewhat arbitrary. Note that we always have $\cup_{i=1}^N \partial E_{0,i} = \Gamma_0$, that is, Γ_0 is the topological boundary of these open sets. Depending on how to put labeling, some portion of Γ_0 may be surrounded by open sets with the same labeling. Such portion will vanish instantly at $t = 0$ by the way the MCF is constructed. In fact, the reason for assuming the non-connectedness of $\mathbb{R}^{n+1} \setminus \Gamma_0$ is that, if connected, our method of existence proof should produce trivial solution of instant vanishing at $t = 0$ (in fact, I believe that there is no known method to produce a non-trivial MCF starting from such Γ_0 so far!).

After choosing such open sets, the conclusion of [3] is that there exist $\{E_i(t)\}_{t\geq 0}$ for each $i = 1, \ldots, N$ and a Brakke flow $\{\mu_t\}_{t\geq 0}$ on $\mathbb{R}^{n+1}$ with the following properties.

(1) $E_1(t), \ldots, E_N(t)$ are open and mutually disjoint sets in $\mathbb{R}^{n+1}$ for all $t \geq 0$.

(2) $E_i(0) = E_{0,i}$ for $i = 1, \ldots, N$.

(3) $\mu_0 = \mathcal{H}^n\lfloor_{\Gamma_0}$.

(4) Define $d\mu = d\mu_t\, dt$ which is a Radon measure on $\mathbb{R}^{n+1} \times [0, \infty)$, and define $\Gamma_t = \mathbb{R}^{n+1} \setminus \cup_{i=1}^N E_i(t)$. Then for all $t > 0$, we have $\Gamma_t = \{x \in \mathbb{R}^{n+1} : (x,t) \in \operatorname{spt}\mu\}$.

(5) For each $t \geq 0$, $\int_{\mathbb{R}^{n+1}} \exp(-c_1|x|)\, d\mu_t(x) < \infty$.

(6) Each $E_i(t)$ moves continuously (resp. 1/2-Hölder continuously) with respect to the Lebesgue measure locally in space and time for $t \geq 0$ (resp. $t > 0$).

Here, $\operatorname{spt}\mu$ is the support of μ as a measure on $\mathbb{R}^{n+1} \times [0, \infty)$. These $E_1(t), \ldots, E_N(t)$ may be considered as a set of moving domains starting from $E_{0,1}, \ldots, E_{0,N}$. μ_t is a Brakke flow starting from the initial hypersurface measure $\mathcal{H}^n\lfloor_{\Gamma_0}$, as explained in the previous section (with $U = \mathbb{R}^{n+1}$). Note that Γ_t may be considered as a time-slice of moving boundaries and (4) claims that they coincide with the space-time support of a Brakke flow. Due to (6), the Lebesgue measure of $E_i(t)$ does not jump suddenly in time. Due to (4), Γ_t stays non-empty and μ_t stays non-zero unless all but one $E_i(t)$ becomes empty.

4. Some idea of proof

The proof of the existence theorem involves a construction of time-discrete approximate MCF and certain compactness theorems of varifolds, with many estimates. Here, for the brevity, I sketch some idea of how to construct the time-discrete approximate MCF. Since Γ_0 is assumed to be only countably n-rectifiable, we need to cook up some reasonable approximate quantity which should reduce to the usual mean curvature if Γ_0 is smooth. Let $\Gamma \subset \mathbb{R}^{n+1}$ be a closed countably n-rectifiable set with $\mathcal{H}^n(\Gamma) < \infty$ for simplicity. Let me explain how one can define an approximate mean curvature vector for Γ. Let $\varepsilon > 0$ be small and let $\Phi_\varepsilon(x) = (2\pi\varepsilon^2)^{-\frac{n+1}{2}} \exp(-\frac{|x|^2}{2\varepsilon^2})$. We have $\int_{\mathbb{R}^{n+1}} \Phi_\varepsilon(x)\, dx = 1$ and $\lim_{\varepsilon\to 0+} \Phi_\varepsilon = \delta_0$ (delta function). In [3], we also do a truncation for Φ_ε but I skip it for simplicity. We use (1.2) to motivate the definition of approximate mean curvature vector. Recall that $\operatorname{div}_\Gamma g = \operatorname{tr}(T_x\Gamma \circ \nabla g) = \sum_{i,j=1}^{n+1} (T_x\Gamma)_{ij} \frac{\partial g_i}{\partial x_j}$, where $g = (g_1, \ldots, g_{n+1})$ and $T_x\Gamma : \mathbb{R}^{n+1} \to T_x\Gamma$ is the matrix representing the orthogonal projection from $\mathbb{R}^{n+1}$ to the tangent space $T_x\Gamma$ of

Γ at x. One important fact about $\mathcal{H}^n$ measurable countably n-rectifiable set with locally finite $\mathcal{H}^n$ measure (which is exactly what we are dealing with) is the existence of approximate tangent space. More precisely, for $\mathcal{H}^n$ a.e. $x \in \Gamma$, there exists a unique n-dimensional subspace $T_x\Gamma$ which is a measure theoretic tangent space of Γ, called approximate tangent space. Because of this, $\int_\Gamma \mathrm{div}_\Gamma g \, d\mathcal{H}^n$ for $g \in C_c^1(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ is a well-defined quantity. For $i \in \{1,\dots,n+1\}$ and $y \in \mathbb{R}^{n+1}$ fixed, let g be a smooth vector field whose components are all zero except for the i-th component, and let that component be $\Phi_\varepsilon(x-y)$. To motivate the definition, let us assume that Γ is smooth. Use (1.2) with this g. Writing the i-th component of h_Γ as h_Γ^i, we have

$$\int_\Gamma (T_x\Gamma)_{ij} \frac{\partial \Phi_\varepsilon}{\partial x_j}(x-y)\, d\mathcal{H}^n(x) = -\int_\Gamma h_\Gamma^i(x)\Phi_\varepsilon(x-y)\, d\mathcal{H}^n(x).$$

We then divide both sides by $\varepsilon + \int_\Gamma \Phi_\varepsilon(x-y)\, d\mathcal{H}^n(x)$. When $y \in \Gamma$, due to the property of Φ_ε, the divided right-hand side converges to $-h_\Gamma^i(y)$ as $\varepsilon \to 0+$. When $y \notin \Gamma$, the same quantity converges to 0. Motivated by this observation, one is led to define the approximate mean curvature vector $h_{\varepsilon,\Gamma}$ of Γ as

$$\tilde{h}^i_{\varepsilon,\Gamma}(y) = -\frac{\int_\Gamma (T_x\Gamma)_{ij}\frac{\partial\Phi_\varepsilon}{\partial x_j}(x-y)\, d\mathcal{H}^n(x)}{\varepsilon + \int_\Gamma \Phi_\varepsilon(x-y)\, d\mathcal{H}^n(x)},$$

$$\tilde{h}_{\varepsilon,\Gamma} = (\tilde{h}^1_{\varepsilon,\Gamma},\dots,\tilde{h}^{n+1}_{\varepsilon,\Gamma}),$$

$$h_{\varepsilon,\Gamma}(y) = (\Phi_\varepsilon * \tilde{h}_{\varepsilon,\Gamma})(y) = \int_{\mathbb{R}^{n+1}} \Phi_\varepsilon(x-y)\tilde{h}_{\varepsilon,\Gamma}(y)\, dy.$$

As long as Γ is $\mathcal{H}^n$ measurable and countably n-rectifiable with locally finite $\mathcal{H}^n$ measure, $\tilde{h}_{\varepsilon,\Gamma}$ and $h_{\varepsilon,\Gamma}$ are well-defined and smooth vector fields. One can prove that

$$\sup_{y\in\mathbb{R}^{n+1}} \{\varepsilon^2|h_{\varepsilon,\Gamma}(y)|, \varepsilon^4|\nabla h_{\varepsilon,\Gamma}(y)|\} \le c(n)(1+\varepsilon\mathcal{H}^n(\Gamma)).$$

Define, for $\Delta t \ll \varepsilon^4$, $f_{\varepsilon,\Gamma}(x) = x + \Delta t\, h_{\varepsilon,\Gamma}(x)$. Then $f_{\varepsilon,\Gamma}$ is a diffeomorphizm on $\mathbb{R}^{n+1}$. Then, starting from Γ_0, we may inductively define $\Gamma_{(k+1)\Delta t} = f_{\varepsilon,\Gamma_{k\Delta t}}(\Gamma_{k\Delta t})$ for $k \in \mathbb{N}$. So starting from Γ_0 we move $\Gamma_{k\Delta t}$ by the approximate mean curvature vector $h_{\varepsilon,\Gamma_{k\Delta t}}$ for the time interval of Δt. This seems like a good way to construct an approximate mean curvature flow. However, there are two essential problems with this approach. The first problem is that this does not allow any topological changes for the flow, since $f_{\varepsilon,\Gamma}$ is always diffeomorphism. For example, we may want to split a cross figure into two triple junctions connected by a short line segment. The second problem is that, when we take a limit $\varepsilon \to 0$, we do not have any information on the scale smaller than ε since the approximate mean curvature vector is smoothed out by Φ_ε. For this reason, the actual construction in [3] is different. In each step, before we compute the approximate mean curvature vector, we insert a measure-reducing Lipschitz deformation which has a regularity effect in ε-scale, and also allows desired topological changes. We also work in the framework which guarantee non-triviality of the MCF in the end. Once this is done with suitable set of estimates, the next step is to take a limit. For this, we need certain compactness theorems, analogous to Allard compactness theorem of integral varifolds.

References

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