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EXISTENCE OF A MINIMAL NON-SCATTERING SOLUTIONS TO THE MASS-SUBCRITICAL GENERALIZED KORTEWEG-DE VRIES EQUATION

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1. Introduction

This is a joint work with Satoshi Masaki (Osaka university). We consider the generalized Korteweg-de Vries equation:

$$\text{(gKdV)} \qquad \partial_t u + \partial_x^3 u = \mu \partial_x(|u|^{2\alpha} u), \qquad t, x \in \mathbb{R},$$

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an unknown function, $\alpha > 0$, and $\mu = \pm 1$. The class of equations (gKdV) arises in several fields of physics. For example, equation (gKdV) with $\alpha = 1$ describes a time evolution for the curvature of certain types of helical space curves [15].

We study the long time behavior of solution to (gKdV). Especially, we focus on construction of a non-scattering solution, which is minimal in some sense, to (gKdV). As for (gKdV), the mass-critical case $\alpha = 2$ is most extensively studied in this direction. Killip, Kwon, Shao and Visan [13] constructed a minimal blow-up solution to (gKdV) with the mass critical case in the framework of L^2. Dodson [6] proved the global well-posedness and scattering in L^2 for (gKdV) with the mass critical, defocusing case $\mu = +1$.

We shall show existence of a minimal non-scattering solution of (gKdV) with the mass-subcritical case $\alpha < 2$ by using the concentration compactness argument by Kenig and Merle [10]. As explained in [18], a good well-posedness theory and a decoupling (in)equality play a central role in the concentration compactness argument. However, when $\alpha < 2$, it seems difficult to derive those properties in the usual Sobolev spaces by several reasons. So, we construct a critical element by using a generalized hat-Morrey space which enables us to establish well-posedness theory good enough and to obtain the concentration compactness lemma equipped with a decoupling inequality. We now introduce a generalized hat-Morrey space.

Definition 1.1. *Let $\tau_k^j = [k2^{-j}, (k+1)2^{-j})$ for $j, k \in \mathbb{Z}$. For $1 \leqslant \beta \leqslant \gamma \leqslant \infty$ and $\beta' < \delta \leqslant \infty$, we define a hat-Morrey norm by*

$$\|f\|_{\hat{M}^{\beta}_{\gamma,\delta}} := \left\| |\tau_k^j|^{\frac{1}{\gamma} - \frac{1}{\beta}} \|\hat{f}\|_{L^{\gamma'}(\tau_k^j)} \right\|_{\ell^{\delta}_{j,k}},$$

where $\hat{f}$ stands for Fourier transform of f in x. Banach space $\hat{M}^{\beta}_{\gamma,\delta}$ is defined as set of tempered distributions of which above norm is finite. For $\sigma > 0$, we also define $|\partial_x|^{-\sigma} \hat{M}^{\beta}_{\gamma,\delta} = \{f \in \mathcal{S}'(\mathbb{R}) | |\partial_x|^{\sigma} f \in \hat{M}^{\beta}_{\gamma,\delta}\}$, where $|\partial_x|^{\sigma} = \mathcal{F}^{-1} |\xi|^{\sigma} \mathcal{F}$.

We construct a minimal non-scattering solution of (gKdV) in the framework of the generalized hat-Morrey space $|\partial_x|^{-\sigma}\hat{M}^{\beta}_{2,\delta}$. Before we state our main theorems, we introduce several notation. We introduce a deformations associated with the function space $|\partial_x|^{-\sigma}\hat{M}^{\beta}_{2,\delta}$:

- Translation in physical side: $(T(y)f)(x) := f(x-y),\ y \in \mathbb{R}$.
- Airy flow: $(A(s)f)(x) = (e^{-s\partial_x^3}f)(x),\ s \in \mathbb{R}$.
- Dilation (scaling): $(D(N)f)(x) = N^{\alpha}f(Nx),\ N \in 2^{\mathbb{Z}}$.

Note that $|\partial_x|^{-\sigma}\hat{M}^{\beta}_{2,\delta}$-norm is invariant under the above group actions.

For a solution u on I, take $t_0 \in I$ and set

$$
\begin{aligned}
T_{\max} &:= \sup\{T > t_0 \mid u(t) \text{ can be extended to a solution on } [t_0, T).\},\\
T_{\min} &:= \sup\{T > -t_0 \mid u(t) \text{ can be extended to a solution on } (-T, t_0].\},\\
I_{\max} &= I_{\max}(u) := (-T_{\min}, T_{\max}).
\end{aligned}
$$

Definition 1.2 (Scattering). *We say a solution $u(t)$ to* (gKdV) *scatters forward in time (resp. backward in time) if $T_{\min} = \infty$ (resp. $T_{\max} = \infty$) and if $|\partial_x|^{\sigma}e^{t\partial_x^3}u(t)$ converges in $\hat{M}^{\beta}_{2,\delta}$ as $t \to \infty$ (resp. $t \to -\infty$).*

We first consider the small data scattering for (gKdV).

Assumption 1.3. *Let $5/3 < \alpha \leqslant 20/9$ and $0 < \sigma \leqslant \min(3/5 - 1/\alpha, 1/4 - 2/(5\alpha))$. Define β by $1/\beta = 1/\alpha + \sigma$. Let γ and δ satisfy*

$$
\frac{4}{5\alpha} + 2\sigma \leqslant \frac{1}{\gamma} < \frac{1}{\beta}, \quad \frac{1}{2} - \frac{1}{5\alpha} \leqslant \frac{1}{\delta} < \frac{1}{\beta'}.
$$

Theorem 1.4 (Small data scattering in $|\partial_x|^{-\sigma}\hat{M}^{\beta}_{\gamma,\delta}$). *Suppose α, σ, β, γ. and δ satisfy Assumption 1.3. Then, there exists $\varepsilon_0 > 0$ such that if $|\partial_x|^{\sigma}u_0 \in \hat{M}^{\beta}_{\gamma,\delta}(\mathbb{R})$ satisfies $\||\partial_x|^{\sigma}u_0\|_{\hat{M}^{\beta}_{\gamma,\delta}} \leqslant \varepsilon_0$, then there exists a global solution $u(t)$ to* (gKdV) *satisfying*

$$
u \in C(\mathbb{R}; |\partial_x|^{-\sigma}\hat{M}^{\beta}_{\gamma,\delta}(\mathbb{R})) \cap L_x^{\frac{5\alpha}{2}}(\mathbb{R}; L_t^{5\alpha}(\mathbb{R})) \cap |\partial_x|^{-\frac{1}{3\beta}-\sigma}L^{3\beta}_{t,x}(\mathbb{R}\times\mathbb{R}).
$$

Moreover, u scatters for both time directions.

To seek a critical element, we consider the minimization problem for E_1 defined by

$$
E_1 := \inf\left\{ \inf_{t \in I_{\max}} \||\partial_x|^{\sigma}u(t)\|_{\hat{M}^{\beta}_{2,\delta}} \;\middle|\; \begin{array}{l} u(t) \text{ is a solution to (gKdV) that} \\ \text{does not scatter forward in time.} \end{array} \right\}.
$$

Remark that it holds that

$$
E_1 = \inf\left\{ \||\partial_x|^{\sigma}u(0)\|_{\hat{M}^{\beta}_{2,\delta}} \;\middle|\; \begin{array}{l} u(t) \text{ is a solution to (gKdV) that does} \\ \text{not scatter forward in time, } 0 \in I_{\max}(u). \end{array} \right\}.
$$

by the time translation symmetry. By Theorem 1.4, we see that $E_1 > 0$. Furthermore, for the focusing case $\mu = -1$, we have $E_1 \leqslant \||\partial_x|^{\sigma}Q\|_{\hat{M}^{\beta}_{2,\delta}}$, where Q is a (unique) positive even solution of $-Q'' + Q = Q^{2\alpha+1}$.

The goal is to determine the explicit value of E_1. In what follows, we consider the focusing case $\mu = -1$ only. However, the focusing assumption is used only for assuring E_1 is finite. Our analysis work also in the defocusing case $\mu = +1$ if we assume E_1 is finite.

Assumption 1.5. *We suppose that* $5/3 < \alpha < 12/5$ *and* $\max(0, 1/2 - 1/\alpha) < \sigma < \min(3/5 - 1/\alpha, 1/4 - 2/(5\alpha))$. *Define* $\beta \in (5/3, 2)$ *by* $1/\beta = 1/\alpha + \sigma$ *and let* $1/\delta \in (1/2 - 1/(5\alpha), 1/\beta')$.

Theorem 1.6 (Analysis of E_1). *Suppose that Assumption 1.5 is satisfied. Then, there exists a minimizer* $u_1(t)$ *to* E_1 *in the following sense:* $u_1(t)$ *is a solution to* (gKdV) *with maximal interval* $I_{\max}(u_1) \ni 0$ *and*

(i) $u_1(t)$ *does not scatter forward in time;*

(ii) $u_1(t)$ *attains* E_1 *in such a sense that either one of the following two properties holds;*

(a) $\| |\partial_x|^\sigma u_1(0) \|_{\hat{M}^\beta_{2,\delta}} = E_1$;

(b) $u_1(t)$ *scatters backward in time and* $u_{1,-} := \lim_{t\to-\infty} e^{t\partial_x^3} u_1(t)$ *satisfies* $\| |\partial_x|^\sigma u_{1,-} \|_{\hat{M}^\beta_{2,\delta}} = E_1$.

So far, we do not have any additional property, such as precompactness of the flow, of the critical solution u_1 constructed in Theorem 1.6. It is not necessarily by a technical reason. Indeed, a similar minimization problem is considered for energy critical nonlinear Schrödinger equation in [16], and a minimizer satisfying properties (i) and (ii)-(b) is given[1]. Remark that the minimizer satisfying (ii)-(b) does not possess precompactness of the flow for negative time direction.

If we consider the minimization problem for E_2 defined by

$$E_2 \quad := \quad \inf \left\{ \varlimsup_{t\uparrow T_{\max}} \| |\partial_x|^\sigma u(t) \|_{\hat{M}^\beta_{2,\delta}} \;\middle|\; \begin{array}{l} u(t) \text{ is a solution to (gKdV) that} \\ \text{does not scatter forward in time.} \end{array} \right\},$$

we obtain a compactness of the critical element. Indeed, we have the following result.

Theorem 1.7 (Analysis of E_2). *Suppose that Assumption 1.5 is satisfied. Then, there exists a minimizer* $u_2(t)$ *to* E_2 *in the following sense:* $u_2(t)$ *is a solution to* (gKdV) *with maximal interval* $I_{\max}(u_2) \ni 0$ *and*

(i) $u_2(t)$ *does not scatter forward and backward in time;*

(ii) *Three quantities*

$$\sup_{t\in\mathbb{R}} \| |\partial_x|^\sigma u_2(t) \|_{\hat{M}^\beta_{2,\delta}}, \quad \varlimsup_{t\uparrow T_{\max}} \| |\partial_x|^\sigma u_2(t) \|_{\hat{M}^\beta_{2,\delta}}, \quad \varlimsup_{t\downarrow T_{\min}} \| |\partial_x|^\sigma u_2(t) \|_{\hat{M}^\beta_{2,\delta}}$$

are equal to E_2.

(iii) $u_2(t)$ *is precompact modulo symmetries, i.e., there exist a scale function* $N(t) : I_{\max} \to \mathbb{R}_+$ *and a space center* $y(t) : I_{\max} \to \mathbb{R}$ *such that the set* $\{(D(N(t))T(y(t)))^{-1} u_2(t) \mid t \in I_{\max}\} \subset |\partial_x|^{-\sigma} \hat{M}^\beta_{2,\delta}$ *is precompact.*

Note by definition, we see $E_1 \leqslant E_2$.

The rest of the article is organized as follows. In Section 2, we give an outline of the proof of the small data scattering for (gKdV) (Theorem 1.4). In Section 3, we shall mention about how to construct a minimal non-scattering solution to (gKdV) (Theorems 1.6 and 1.7) by using the concentration compactness.

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[1] Furthermore, in this case there is no minimizer which attains minimum value at finite time as in (ii)-(a). See [16]

2. Small Data Scattering

In this section we prove small data scattering for (gKdV) (Theorem 1.4). To this end, we consider integral form of (gKdV):

$$u(t) = e^{-(t-t_0)\partial_x^3}u_0 + \mu \int_{t_0}^{t} e^{-(t-s)\partial_x^3}\partial_x(|u|^{2\alpha}u)(s)ds. \tag{2.1}$$

For an interval $I \subset \mathbb{R}$, we introduce function spaces $L(I)$, $M(I)$, and $S(I)$as follows:

$$\begin{aligned} L(I) &:= \left\{ u \in \mathcal{S}'(I \times \mathbb{R}) \,\middle|\, \|u\|_{L(I)} := \left\| |\partial_x|^{\frac{1}{\alpha}} u \right\|_{L_x^{5\alpha}(\mathbb{R};L_t^{\frac{5\alpha}{3}}(I))} < \infty \right\}, \\ M(I) &:= \left\{ u \in \mathcal{S}'(I \times \mathbb{R}) \,\middle|\, \|u\|_{M(I)} := \left\| |\partial_x|^{\frac{1}{2\alpha}} u \right\|_{L_x^{\frac{10\alpha}{3}}(\mathbb{R};L_t^{\frac{5\alpha}{2}}(I))} < \infty \right\}, \\ S(I) &:= \left\{ u \in \mathcal{S}'(I \times \mathbb{R}) \,\middle|\, \|u\|_{S(I)} := \|u\|_{L_x^{\frac{5\alpha}{2}}(\mathbb{R};L_t^{5\alpha}(I))} < \infty \right\}. \end{aligned}$$

For an interval $I \subset \mathbb{R}$, we say a function $u \in M(I) \cap S(I)$ is a solution to (gKdV) on I if u satisfies (2.1) in the $M(I) \cap S(I)$ sense. Modifying a well-posedness result in [17], we have

Lemma 2.1. *Let $5/3 < \alpha \leqslant 20/9$. Denote by $Z(I)$ either $L(I)$ or $M(I)$. Let $t_0 \in \mathbb{R}$ and I be an interval with $t_0 \in I$. Then, there exists a universal constant $\delta > 0$ such that if a tempered distribution u_0 and an interval $I \ni t_0$ satisfy*

$$\eta_0 = \eta_0(I; u_0, t_0) := \left\| e^{-(t-t_0)\partial_x^3}u_0 \right\|_{S(I)} + \left\| e^{-(t-t_0)\partial_x^3}u_0 \right\|_{Z(I)} \leqslant \delta,$$

then there exists a unique solution $u(t)$ on I to (gKdV) satisfying

$$\|u\|_{S(I)} + \|u\|_{Z(I)} \leqslant 2\eta_0.$$

Moreover, the solution satisfies $u(t) - e^{-(t-t_0)\partial_x^3}u_0 \in C(I; \hat{L}^\alpha)$.

Furthermore we obtain an existence result.

Proposition 2.2. *Let α, σ, β, γ and δ satisfy the assumption of Theorem 1.4. Then, for any $u_0 \in |\partial_x|^{-\sigma}\hat{M}^{\beta}_{\delta,\gamma}$ and $t_0 \in \mathbb{R}$ there exists an interval $I \subset \mathbb{R}$, $I \ni t_0$ such that there exists a unique solution $u(t)$ on I to (gKdV). The solution belongs to $C(I; |\partial_x|^{-\sigma}\hat{M}^{\beta}_{\gamma,\delta} + \hat{L}^\alpha)$.*

The key point in the proof of Proposition 2.2 is the following refined Strichartz' estimate for the Airy equation which is due to [19, Theorem 1.3].

Lemma 2.3. *Let $\sigma \in (0, 1/4)$. Let (p, q) satisfy*

$$0 \leqslant \frac{1}{p} \leqslant \frac{1}{4} - \sigma, \qquad \frac{1}{q} \leqslant \frac{1}{2} - \frac{1}{p} - \sigma.$$

Define α and s by

$$\frac{2}{p} + \frac{1}{q} = \frac{1}{\alpha}, \qquad s = -\frac{1}{p} + \frac{2}{q}.$$

Further, we define β, γ, and δ by

$$\frac{1}{\beta}=\frac{1}{\alpha}+\sigma,\qquad \frac{1}{\gamma}=\begin{cases}\frac{1}{\beta}-\frac{1}{p} & \text{if } \frac{1}{q}\geqslant\frac{1}{p}+\sigma,\\ \frac{1}{\beta}-\frac{1}{q}+\sigma & \text{if } \frac{1}{q}<\frac{1}{p}+\sigma,\end{cases}\qquad \frac{1}{\delta}=\frac{1}{2}-\frac{1}{\max(p,q)}.$$

Then, there exists a positive constant C depending on p, q, σ such that the inequality

$$\left\| |\partial_x|^s e^{-t\partial_x^3} f \right\|_{L_x^p(\mathbb{R};L_t^q(\mathbb{R}))} \leqslant C \| |\partial_x|^\sigma f \|_{\hat{M}^\beta_{\gamma,\delta}} \tag{2.2}$$

holds for any $f \in |\partial_x|^{-\sigma}\hat{M}^\beta_{\gamma,\delta}$.

Proof of Proposition 2.2. One sees from Lemma 2.3 that if $\alpha > 8/5$ and $0 < \sigma \leqslant 1/4 - 2/(5\alpha)$ then

$$\left\| e^{-(t-t_0)\partial_x^3} u_0 \right\|_{L(\mathbb{R})} + \left\| e^{-(t-t_0)\partial_x^3} u_0 \right\|_{S(\mathbb{R})} \leqslant C \, \| |\partial_x|^\sigma u_0 \|_{\hat{M}^\beta_{\gamma,\delta}} < \infty. \tag{2.3}$$

Hence, there exists an open neighborhood $I \subset \mathbb{R}$ of t_0 such that $\eta_0(I; u_0, t_0) \leqslant \delta$, where δ and η_0 are defined in Lemma 2.1. Since $u(t) - e^{-(t-t_0)\partial_x^3}u_0 \in C(I; \hat{L}^\alpha)$ and $|\partial_x|^\sigma e^{-(t-t_0)\partial_x^3}u_0 \in C(I; \hat{M}^\beta_{\gamma,\delta})$, we obtain the result. □

Proof of Theorem 1.4. To prove the theorem, it suffices to show that $u(t) - e^{-(t-t_0)\partial_x^3}u_0 \in C(I, |\partial_x|^{-\sigma}\hat{M}^\beta_{\gamma,\delta})$. This is obtained by mimicing the argument in [17, 18]. See [19, Proof of Theorem 1.4]. □

As a byproduct of the above arguments, we obtain the scattering criterion for (gKdV).

Theorem 2.4 (scattering criterion). *Suppose α, σ, β, γ, and δ satisfy Assumption 1.3. Let $u_0 \in |\partial_x|^{-\sigma}\hat{M}^\beta_{\gamma,\delta}$ and let $u(t)$ be a solution to* (gKdV) *with maximal lifespan $I_{\max} \ni 0$. The following three statements are equivalent*

- *$u(t)$ scatters forward in time in the sense of Definition 1.2;*
- *$\|u\|_{L([0,T_{\max}))} < \infty$;*
- *$\|u\|_{S([0,T_{\max}))} < \infty$;*

Further, if either one of the above (hence all of the above) holds then $e^{t\partial_x^3}u(t)$ converges as $t \to \infty$ in $\hat{L}^\alpha \cap |\partial_x|^{-\sigma}\hat{L}^\beta$.

3. Minimizing problem

3.1. **Linear profile decomposition.** In this subsection, we establish the linear profile decomposition in $|\partial_x|^{-\sigma}\hat{M}^\beta_{2,\delta}$. The linear profile decomposition essentially consists of two parts. The first part is concentration compactness and the second part is the inductive procedure to obtain a decomposition.

Let us begin with the concentration compactness part. The hat-Morrey space $\hat{M}^\alpha_{\beta,\gamma}$ is realized as a dual of a Banach space [16, Theorem 2.17]. Therefore, a bounded set of the hat-Morrey space is compact in the weak-* topology.

Theorem 3.1 (Concentration compactness in $|\partial_x|^{-\sigma}\hat{M}^{\beta}_{2,\delta}$). *Suppose that* $\alpha > 8/5$ *and* $0 < \sigma < 1/4 - 2/(5\alpha)$. *Let* β, γ, δ *satisfy* $1/\beta = 1/\alpha + \sigma$,

$$\frac{4}{5\alpha} + 2\sigma < \frac{1}{\gamma} < \frac{1}{\beta}, \quad \text{and} \quad \frac{1}{2} - \frac{1}{5\alpha} < \frac{1}{\delta} < \frac{1}{\beta'}.$$

Let $\{u_n\}_n \subset |\partial_x|^{-\sigma}\hat{M}^{\beta}_{\gamma,\delta}$ *a bounded sequence;*

$$\||\partial_x|^{\sigma}u_n\|_{\hat{M}^{\beta}_{\gamma,\delta}} \leqslant M \tag{3.1}$$

for some $M > 0$. *If the sequence further satisfies*

$$\left\|e^{-t\partial_x^3}u_n\right\|_{L(\mathbb{R})\cap S(\mathbb{R})} \geqslant m \tag{3.2}$$

for some $m > 0$ *then there exist such that*

$$|\partial_x|^{\sigma}(T(y_n)^{-1}A(s_n)^{-1}D(N_n)^{-1}u_n) \rightharpoonup |\partial_x|^{\sigma}\psi$$

as $n \to \infty$ *weakly-** *in* $\hat{M}^{\beta}_{\gamma,\delta}$ *with* $\|\psi\|_{\hat{M}^{\beta}_{\gamma,\delta}} \geqslant C(M, m) > 0$.

Proof of Theorem 3.1. See [19, Theorem 4.1]. □

We next move to the main issue of this subsection, linear profile decomposition. Let us define a set of deformations as follows

$$G := \{D(N)A(s)T(y) \mid \Gamma = (N, s, y) \in 2^{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R}\}. \tag{3.3}$$

We often identify $\mathcal{G} \in G$ with a corresponding parameter $\Gamma \in 2^{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R}$ if there is no fear of confusion. Let us now introduce a notion of orthogonality between two families of deformations.

Definition 3.2. *We say two families of deformations* $\{\mathcal{G}_n\} \subset G$ *and* $\{\widetilde{\mathcal{G}}_n\} \subset G$ *are* orthogonal *if corresponding parameters* $\Gamma_n, \widetilde{\Gamma}_n \in 2^{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R}$ *satisfies*

$$\lim_{n\to\infty}\left(\left|\log\frac{N_n}{\widetilde{N}_n}\right| + \left|s_n - \left(\frac{N_n}{\widetilde{N}_n}\right)^3\widetilde{s}_n\right| + \left|y_n - \frac{N_n}{\widetilde{N}_n}\widetilde{y}_n\right|\right) = +\infty. \tag{3.4}$$

Theorem 3.3 (Linear profile decomposition in $|\partial_x|^{-\sigma}\hat{M}^{\beta}_{2,\delta}$). *Suppose that* α, σ, β, γ, *and* δ *satisfy Assumption 1.5. Let* $\{u_n\}_n$ *be a bounded sequence in* $|\partial_x|^{-\sigma}\hat{M}^{\beta}_{2,\delta}$. *Then, there exist* $\psi^j \in |\partial_x|^{-\sigma}\hat{M}^{\beta}_{2,\delta}$, $r^j_n \in |\partial_x|^{-\sigma}\hat{M}^{\beta}_{2,\delta}$, *and pairwise orthogonal families of deformations* $\{\mathcal{G}^j_n\}_n \subset G$ $(j = 1, 2, \dots)$ *parametrized by* $\{\Gamma^j_n = (h^j_n, s^j_n, y^j_n)\}_n$ *such that, extracting a subsequence in* n,

$$u_n = \sum_{j=1}^{J}\mathcal{G}^j_n\psi^j + r^J_n \tag{3.5}$$

for all $n, J \geqslant 1$ *and*

$$\lim_{J\to\infty}\varlimsup_{n\to\infty}\left(\left\||\partial_x|^{\frac{1}{3\alpha}}e^{-t\partial_x^3}r^J_n\right\|_{L^{3\alpha}_{t,x}(\mathbb{R}\times\mathbb{R})} + \left\|e^{-t\partial_x^3}r^J_n\right\|_{L_t^{\frac{5\alpha}{2}}L_x^{5\alpha}(\mathbb{R}\times\mathbb{R})}\right) = 0. \tag{3.6}$$

Moreover, a decoupling inequality

$$\varlimsup_{n\to\infty}\||\partial_x|^{\sigma}u_n\|^{\delta}_{\hat{M}^{\beta}_{2,\delta}} \geqslant \sum_{j=1}^{J}\||\partial_x|^{\sigma}\psi^j\|^{\delta}_{\hat{M}^{\beta}_{2,\delta}} + \varlimsup_{n\to\infty}\|r^J_n\|^{\delta}_{\hat{M}^{\beta}_{2,\delta}} \tag{3.7}$$

holds for all $J \geqslant 1$. Furthermore, if u_n is real-valued then so are ψ^j and r_n^J.

Proof of Theorem 3.3. See [19, Theorem 4.3] □

3.2. **Outline of Proof of Theorem 1.6.** Let us begin with the analysis of E_1. We first take a minimizing sequence $\{u_n(t), t_n\}_n \subset |\partial_x|^{-\sigma}\hat{M}^{\beta}_{2,\delta} \times \mathbb{R}$ as follows; $t_n \in I_{\max}(u_n)$ and

$$(3.8) \qquad \|u_n\|_{S([t_n,T_{\max}))} = \infty, \quad \||\partial_x|^{\sigma} u_n(t_n)\|_{\hat{M}^{\beta}_{2.\delta}} \leqslant E_1 + \frac{1}{n}.$$

By time translation symmetry, we may suppose that $t_n \equiv 0$. We apply the linear profile decomposition theorem (Theorem 3.3) to the sequence $\{u_n(0)\}_n$. Then, up to subsequence, we obtain a decomposition

$$u_n(0) = \sum_{j=1}^{J} \mathcal{G}_n^j \psi^j + r_n^J$$

for $n, J \geqslant 1$ with the properties (3.6), (3.7), and pairwise orthogonality of $\{\mathcal{G}_n^j\}_n \subset G$. By extracting subsequence and changing notations if necessary, we may assume that for each j and $\{x_n^j\}_{n,j} = \{\log N_n^j\}_{n,j}, \{s_n^j\}_{n,j}, \{y_n^j\}_{n,j}$, either $x_n^j \equiv 0$, $x_n^j \to \infty$ as $n \to \infty$, or $x_n^j \to -\infty$ as $n \to \infty$ holds. Let us define a nonlinear profile $\Psi^j(t)$ associated with (ψ^j, s_n^j) as follows: For each j, we let

- if $s_n^j \equiv 0$ then $\Psi^j(t)$ is a solution to (gKdV) with $\Psi^j(0) = \psi^j$;
- if $s_n^j \to \infty$ as $n \to \infty$ then $\Psi^j(t)$ is a solution to (gKdV) that scatters forward in time to $e^{-t\partial_x^3}$ psi^j;
- if $s_n^j \to -\infty$ as $n \to \infty$ then $\Psi^j(t)$ is a solution to (gKdV) that scatters backward in time to $e^{-t\partial_x^3}\psi^j$;

Let

$$(3.9) \qquad V_n^j(t) := D(N_n^j)T(y_n^j)\Psi^j((N_n^j)^3 t + s_n^j).$$

Here, we define an approximate solution

$$(3.10) \qquad \tilde{u}_n^J(t,x) = \sum_{j=1}^{J} V_n^j(t,x) + e^{-t\partial_x^3} r_n^J.$$

The main step is to show that there exists Ψ^j that does not scatter forward in time. Suppose not. Then, all Ψ^j scatters forward in time and so $\||\partial_x|^{\sigma}\psi^j\|_{\hat{M}^{\beta}_{2.\delta}} < E_1$ for all j. Then, we shall observe that $\tilde{u}_n^J$ is an approximately solves (gKdV) and that is close to u_n. Furthermore, by the stability estimate [19, Theorem 3.6], we have $\|u_n\|_{S(\mathbb{R}_+)} < \infty$ for sufficiently large n. This contradicts with the definition of $\{u_n\}_n$. Thus, we see that there exists j_0 such that Ψ^{j_0} does not scatter. Then, $\||\partial_x|^{\sigma}\psi^{j_0}\|_{\hat{M}^{\beta}_{2.\delta}} \geqslant E_1$ by definition of E_1. One also sees from (3.7) that $\||\partial_x|^{\sigma}\psi^{j_0}\|_{\hat{M}^{\beta}_{2.\delta}} \leqslant E_1$. Hence, $\||\partial_x|^{\sigma}\psi^{j_0}\|_{\hat{M}^{\beta}_{2.\delta}} = E_1$.

Let us show that $u_1 := \Psi^{j_0}$ attains E_1. The case $s_n^{j_0} \to \infty$ as $n \to \infty$ is excluded since this implies $u_c(t)$ scatters forward in time. If $s_n^{j_0} \equiv 0$ then

$\Psi^{\jmath_0}(0) = \psi^{\jmath_0}$ and so $\||\partial_x|^\sigma u_1(0)\|_{\hat{M}^\beta_{2,\delta}} = E_1$. Finally, if $s_n^{\jmath_0} \to -\infty$ as $n \to \infty$ then $\lim_{t\to-\infty} e^{t\partial_x^3}\Psi^{\jmath_0}(t) = \psi^{\jmath_0}$. Hence, $u_{1,-} := \lim_{t\to-\infty} e^{t\partial_x^3}\Psi^{\jmath_0}(t)$ satisfies $\||\partial_x|^\sigma u_{1,-}\|_{\hat{M}^\beta_{2,\delta}} = E_1$.

3.3. **Outline of Proof of Theorem 1.7.** We finally consider analysis of E_2. By definition of E_2, it is possible to choose a minimizing sequence of solutions $\{u_n(t)\}_n$ so that all $u_n(t)$ does not scatter forward in time and

$$E_2 \leqslant \varlimsup_{t\uparrow T_{\max}(u_n)} \||\partial_x|^\sigma u_n(t)\|_{\hat{M}^\beta_{2,\delta}} \leqslant E_2 + \frac{1}{n}.$$

Hence, there exists $t_n, t_n' \in I_{\max}(u_n)$, $t_n < t_n'$, so that

$$\|u_n\|_{S([t_n,t_n'])} \geqslant n, \qquad \sup_{t\in[t_n,T_{\max})} \||\partial_x|^\sigma u_n(t)\|_{\hat{M}^\beta_{2,\delta}} \in \left[E_2, E_2 + \frac{2}{n}\right].$$

Indeed, we first choose t_n so that the second property holds. Then, since $\|u_n\|_{S([t_n,T_{\max}))} = \infty$, we can choose t_n' so that the first property is true.

By time translation symmetry, we may suppose that $t_n' \equiv 0$. We now apply linear profile decomposition to $u_n(0)$ to get the decomposition

$$u_n(0) = \sum_{\jmath=1}^{J} \mathcal{G}_n^\jmath \psi^\jmath + r_n^J$$

for $n, J \geqslant 1$ with the properties (3.6), (3.7), and pairwise orthogonality of $\{\mathcal{G}_n^\jmath\}_n \subset G$. By extracting subsequence and changing notations if necessary, we may assume that for each j and $\{x_n^\jmath\}_{n,\jmath} = \{\log N_n^\jmath\}_{n,j}, \{s_n^\jmath\}_{n,\jmath}, \{y_n^\jmath\}_{n,\jmath}$, we have either $x_n^\jmath \equiv 0$, $x_n^\jmath \to \infty$ as $n \to \infty$, or $x_n^\jmath \to -\infty$ as $n \to \infty$. Let us define nonlinear profile $\Psi^\jmath$ associated with $(\psi^\jmath, s_n^\jmath)$ in the same way as in the proof of Theorem 1.6. We also define $V_n^\jmath$ and $\tilde{u}_n^J$ by (3.9) and (3.10), respectively.

Then, mimicking the proof of Theorem 1.6, one sees that at least one $\Psi^\jmath$ does not scatter forward in time. We further see from decoupling inequality (3.7) and small data scattering that the number of the profiles that do not scatter is finite. Renumbering, we may suppose that $\Psi^\jmath(t)$ do not scatter forward in time if and only if $j \in [1, J_1]$. Here, $1 \leqslant J_1 < \infty$. Arguing as in [16], we see that $J_1 = 1$, $\varlimsup_{t\uparrow T_{\max}(\Psi^1)} \||\partial_x|^\sigma \Psi^1(t)\|_{\hat{M}^\beta_{2,\delta}} = E_2$, $\psi^\jmath \equiv 0$ for $j \geqslant 2$, and $r_n^1 \to 0$ as $n \to \infty$ in $|\partial_x|^{-\sigma}\hat{M}^\beta_{2,\delta}$. As a result,

$$u_n(0) = \mathcal{G}_n^1\psi^1 + o_n(1) \quad \text{in } |\partial_x|^{-\sigma}\hat{M}^\beta_{2,\delta}. \tag{3.11}$$

If $s_n^1 \to \infty$ as $n \to \infty$ then $\Psi^1(t)$ scatters forward in time, a contradiction. Because of $\|u_n\|_{S([t_n,0])} \geqslant n$, the same argument works for negative time direction. We see that $\Psi^1(t)$ does not scatter backward in time and that the case $s_n^1 \to -\infty$ as $n \to \infty$ is excluded. Moreover, together with $\sup_{t\in[t_n,T_{\max})} \||\partial_x|^\sigma u_n(t)\|_{\hat{M}^\beta_{2,\delta}} \in [E_2, E_2 + \frac{2}{n}]$, we have

$$\varlimsup_{t\downarrow T_{\min}(\Psi^1)} \||\partial_x|^\sigma \Psi^1(t)\|_{\hat{M}^\beta_{2,\delta}} = \sup_{t\in I_{\max}(\Psi^1)} \||\partial_x|^\sigma \Psi^1(t)\|_{\hat{M}^\beta_{2,\delta}} = E_2.$$

So far, we have proven that Ψ^1 satisfies the first two properties of Theorem 1.7. Let us finally prove the precompactness modulo symmetry. Take an arbitrary sequence $\{\tau_n\} \subset I_{\max}(\Psi^1)$. Then, we can choose $t_n \in (T_{\min}(\Psi^1), \tau_n)$ so that $u_n(t) := \Psi$, $t'_n = \tau_n$, and this t_n satisfies the same assumption as above. The decomposition (3.11) reads as existence of $\psi \in |\partial_x|^{-\sigma}\hat{M}^{\beta}_{2,\delta}$, $\{N_n\}_n \subset \mathbb{R}_+$, and $\{y_n\}_n \subset \mathbb{R}$ such that

$$\Psi^1(\tau_n) = D(N_n)T(y_n)\phi + o_n(1) \quad \text{in } |\partial_x|^{-\sigma}\hat{M}^{\beta}_{2,\delta}.$$

This is nothing but a sequential version of precompactness. A standard argument then upgrades this property to the continuous one.

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