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Behavior of solutions to a chemotaxis system with general sensitivity functions

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1 Introduction

This manuscript is based on the joint work with Kentarou Fujie (Tokyo University of Science).

We consider the following system.

$$(PP)\begin{cases} \tau u_t = \nabla\cdot(\nabla u - u\nabla\chi(v)) & \text{in } \Omega\times(0,T), \\ \eta v_t = \Delta v - v + u & \text{in } \Omega\times(0,T), \\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = 0 & \text{on } \partial\Omega\times(0,T), \\ u(\cdot,0) = u_0, v(\cdot,0) = v_0 & \text{in } \Omega. \end{cases}$$

Here, $\tau > 0$, $\eta > 0$, $\Omega \subset \mathbf{R}^n$ ($n \geq 2$) is a bounded and convex domain with smooth boundary $\partial\Omega$, χ is smooth on $(0,\infty)$ satisfying $\chi'(v) > 0$ ($v > 0$), $\nu = \nu(x)$ is the outer normal unite vector at $x \in \partial\Omega$ and initial conditions u_0 and v_0 are smooth and positive.

The system (PP) is introduced to describe the aggregation of cellular slime molds. When the environmental situation worsens, they aggregate to a single milt-cellular body. During this aggregation process, a chemical signal is secreted by cells to guide the collective movements. We refer to this property as chomotaxis. Functions u and v represent the density of cells and the chemical concentration, respectively.

This system has the conservation of mass:

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx \quad \text{for } t \geq 0. \tag{1}$$

The function $\chi(v)$ represents the relation between the movement of cells and the chemical concentration. The term $u\chi'(v)\nabla v = u\nabla\chi(v)$ stands for chemotaxis. The positivity of $\chi'(v)$ means that this chemical substance is an attractant. This function χ is called sensitivity function.

In this manuscript, we mainly treat sensitivity functions satisfying that

$$\lim_{v\to\infty}\chi'(v) = 0. \tag{2}$$

This assumption represents slowdown of cells' response to strong stimulus. Some researchers treat the system (PP) with $\chi(v) = \chi_0 \log v$, where χ_0 is a positive constant. This type sensitivity function χ satisfies the assumption (2).

2 Linear sensitivity case

In this manuscript, we consider properties of solutions to (PP) under the assumption (1). On the other hand, there are many researches on solutions to (PP) with $\chi(v) = \chi_1 v$, where χ_1 is a positive constant. This type sensitivity function is called linear sensitivity function. The system (PP) with a linear sensitivity function has a Lyapunov function. Let $\eta = \tau = 1$ for simplicity. Let (u, v) be a solution to (PP) with $\chi(v) = \chi_1 v$. Putting

$$F(u,v) = \int_\Omega (u \log u - \chi_1 uv)dx + \frac{\chi_1}{2}\int_\Omega (|\nabla v|^2 + v^2)dx,$$

we have that

$$\frac{d}{dt}F(u,v) + \int_\Omega (v_t)^2 dx + \int_\Omega u|\nabla(\log u - \chi_1 v)|^2 dx = 0.$$

This Lyapunov function is very useful, when we investigate properties of solutions. In fact, if $n = 2$ and $\int_\Omega u_0 dx < \frac{4\pi}{\chi_1}$, it follows from the Lyapunov function and the Tringer-Moser inequality that

$$\delta \int_\Omega uvdx \le F(u,v)$$

with some positive constant $\delta > 0$ and that

$$\int_\Omega u \log u dx < \infty.$$

This and the standard bootstrap argument lead us to the boundedness of solutions (u, v)(see [10]). This means that the boundedness of solutions follows from the Lyapunov function.

Moreover, the blowup of solutions comes from the Lyapunov function. We denote the set of positive integers by $\mathcal{N}$. Let $n = 2$, $\chi(v) = \chi_1 v$ and let $\lambda \in (0,\infty) \setminus \{(4\pi/\chi_1)\mathcal{N}\} = \{\lambda > 0 : \lambda \ne (4\pi/\chi_1)j \text{ for } j = 1, 2, 3, \cdots\}$. Then,

$$\mathcal{F}_\lambda = \inf\left\{F(u,v) : (u,v) \text{ is a stationary solution satisfying } \int_\Omega udx = \lambda\right\} > -\infty.$$

For $\lambda > 4\pi/\chi_1$ with $\lambda \notin (4\pi/\chi_1)\mathcal{N}$, there are a pair of positive continuous functions (u_0, v_0) satisfying $F(u_0, v_0) < \mathcal{F}_\lambda$ and $\int_\Omega u_0 dx = \lambda$. The Lyapunov function guarantees that the solution blows up (see [7]). And, we have that the solutions blows up in finite time by using the differential inequality on the Lyapunov function (see [16]).

Here, if $\limsup_{t \to T_{max}}(\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} = \infty$ with some $T_{max} \in (0, \infty]$, we say that the solution (u, v) blows up at the time T_{max} and that T_{max} is blowup time or maximal existence time.

When $\chi(v)$ is not a linear function, any Lyapunov functions are not found yet. Then, in that case, the arguments mentioned in this section do not work.

3 Nonlinear sensitivity case

In the nonlinear sensitivity case, there are the following researches on the boundedness of classical solutions.

If $\Omega \subset \mathbf{R}^n$ $(n \geq 2)$ and $\chi'(v) \leq \dfrac{a}{(b+v)^p}$ $(a > 0,\ b \geq 0,\ p > 1)$, then solutions to (PP) exist globally in time and are bounded ([14, 5]).

If $\Omega \subset \mathbf{R}^n$ $(n \geq 2)$ and $\chi(v) = \chi_0 \log v$ $(\chi_0 < \sqrt{2/n})$, then solutions to (PP) exist globally in time and are bounded ([15, 1]).

The following research is the one on the time-global existence of weak solutions.

If $\Omega \subset \mathbf{R}^n$ $(n \geq 2)$ and $\chi(v) = \chi_0 \log v$ $(\chi_0 < \sqrt{n/(n-2)})$, a weak solution satisfying u^p, $v^p \in L^1_{loc}(\Omega \times (0, \infty))$ $(0 < p < 1)$ exists globally in time ([13]).

In the nonlinear sensitivity case, any Lyapunov functions are not found yet. Then, the arguments mentioned in the previous section do not work for solutions to (PP) with nonlinear sensitivity functions. Considering this situation, we must consider simple systems. Then, we consider the limiting system of (PP) as τ or $\eta = 0$. Moreover, considering the research on solutions to the limiting system, we think that the above conditions for the boundedness of solutions are not critical.

First, we consider the limiting system of (PP) as $\eta = 0$.

$$(PE)\begin{cases} \tau u_t = \nabla \cdot (\nabla u - u \nabla \chi(v)) & \text{in } \Omega \times (0,T), \\ 0 = \Delta v - v + u & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(\cdot, 0) = u_0 & \text{in } \Omega. \end{cases}$$

The following hold for solutions to (PE) ([11, 3]).

If Ω is a bounded domain in $\mathbf{R}^2$ and $\lim_{v\to\infty}\chi'(v) = 0$, then solutions to (PE) exist globally in time and are bounded.

If Ω is a bounded ball in $\mathbf{R}^n$ $(n \geq 3)$, u_0 is radial and $\chi(v) = \chi_0 \log v$ $(\chi_0 \in (0, 2/(n-2)\ )$, then solutions to (PE) exist globally in time and are bounded.

If Ω is a bounded ball in $\mathbf{R}^n$ $(n \geq 3)$, u_0 is radial and $\chi(v) = \chi_0 \log v$ $(\chi_0 > 2n/(n-2))$, then there exist blowup solutions to (PE).

Next, we consider the limiting system of (PP) as $\tau = 0$.

$$(EP)\begin{cases} 0 = \nabla\cdot(\nabla u - u\nabla\chi(v)) & \text{in } \Omega\times(0,T), \\ \eta v_t = \Delta v - v + u & \text{in } \Omega\times(0,T), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega\times(0,T), \\ v(\cdot,0) = v_0 & \text{in } \Omega, \\ \displaystyle\int_\Omega u(x,t)dx = \lambda & \text{in } (0,T). \end{cases}$$

Here, λ is a positive constant.

We impose the last condition for solutions to (EP), since solutions to (PP) satisfy (1). This last condition and the first equation of the system (EP) guarantee that

$$u = \frac{\lambda \exp(\chi(v))}{\int_\Omega \exp(\chi(v))dx}.$$

Then, the system (EP) can be transformed into the following system.

$$(NLP)\begin{cases} \eta v_t = \Delta v - v + \frac{\lambda\exp(\chi(v))}{\int_\Omega \exp(\chi(v))dx} & \text{in } \Omega\times(0,T), \\ u = \dfrac{\lambda\exp(\chi(v))}{\int_\Omega \exp(\chi(v))dx} & \text{in } \Omega\times(0,T), \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega\times(0,T), \\ v(\cdot,0) = v_0 & \text{in } \Omega. \end{cases}$$

Classical solutions to (NLP) satisfy the following properties ([12]).

If Ω is a bounded domain in $\mathbf{R}^2$ and $\lim_{v\to\infty}\chi'(v) = 0$, then solutions to (NLP) exist globally in time and are bounded.

If Ω is a bounded domain in $\mathbf{R}^n$ $(n \geq 3)$, $\chi(v) = \chi_0 \log v$ and $\chi_0 \in (0, n/(n-2))$, then solutions to (NLP) exist globally in time and are bounded.

If Ω is a bounded ball in $\mathbf{R}^n$ ($n \geq 3$), $\chi(v) = \chi_0 \log v$ and $\chi_0 > n/(n-2)$, then there exist blowup solutions to (EP).

Remark 3.1 *If Ω is a bounded domain in $\mathbf{R}^2$ and $\chi(v) = \chi_1 v$ ($\chi_1 > 0$), there exist blowup solutions to (EP).*

Considering results on solutions to the limiting systems of (PP) as τ or $\eta = 0$, we consider the system (PP) in the case where τ or η is sufficiently small and get the following results.

Theorem 3.2 ([4]) *Suppose that $n \geq 3$, Ω is a bounded and convex domain in $\mathbf{R}^n$ and that $\limsup_{v\to\infty} v\chi'(v) < n/(n-2)$. Then, solutions to (PP) exist globally in time and are bounded if τ is sufficiently small.*

This property of solutions is different from the one in the case where the sensitivity function is linear.

Remark 3.3 *In the case where $\chi(v) = \chi_0 \log v$, the function χ satisfies that $\lim_{v\to\infty} \chi'(v) = 0$ and that $\limsup_{v\to\infty} v\chi'(v) = \chi_0$.*

The following results are on solutions to (PP) with a linear sensitivity function.

Theorem 3.4 ([16]) *If $n \geq 3$ and Ω is a bounded ball in $\mathbf{R}^n$, there exist radial blowup solutions.*

4 Sketch of proof of Theorem 3.2

Finally, we describe a sketch of proof of Theorem 3.2. For simplicity, we assume that $\eta = 1$. .

The following two lemmas say estimates of solutions independent of the time constant τ. The first lemma is shown by the standard energy argument and the second lemma comes from the properties of the heat kernel.

Lemma 4.1 *Suppose that (u, v) is a solution to (PP). There are positive constants T_{min} and $\tilde{L}$ satisfying*

$$\|(u,v)\|_{L^\infty(\Omega\times(0,T_{min}))} \leq \tilde{L} \quad \text{for } \tau \in (0,1].$$

Lemma 4.2 *Suppose that (u, v) is a solution to (PP). There is a positive constant $v_* > 0$ such that*

$$v(x,t) \geq v_* \quad \text{for } (x,t) \in \Omega \times (0, T_{max}(\tau)) \text{ and } \tau \in (0,1].$$

Here, $T_{max}(\tau)$ is the maximal existence time of the classical solution (u, v) to (PP) with the time constant τ.

In order to investigate solutions to (PP), we consider the following functions.

$$z(x,t) = \frac{e^{\chi(v(x,t))}}{\int_\Omega e^{\chi(v(y,t))}dy}, \quad w(x,t) = \frac{u(x,t)}{z(x,t)} \quad \text{for } x \in \overline{\Omega} \text{ and } t \in (0, T_{max}(\tau)).$$

Those functions satisfy the following system.

$$(TPP) \begin{cases} v_t = \Delta v - v + \dfrac{w\exp(\chi(v))}{\int_\Omega \exp(\chi(v))dx} & \text{in } \Omega \times (0,\infty), \\ \tau w_t = \dfrac{1}{z}\nabla\cdot(z\nabla w) - \left(\dfrac{\tau}{z}z_t\right) w & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ v(\cdot,0) = v_0, \ w(\cdot,0) = \frac{u_0}{e^{\chi(v_0)}}\int_\Omega e^{\chi(v_0)}dx & \text{on } \Omega. \end{cases}$$

Lemma 4.1 entails the following estimate.

$$\|w\|_{L^\infty(\Omega\times(0,T_{min}))} \le L = \frac{\tilde{L}|\Omega|e^{\chi(\tilde{L})}}{e^{\chi(v_*)}}. \tag{3}$$

Putting $H = 2\max\left\{\|u_0\|_{L^\infty(\Omega)}, \|w(0)\|_{L^\infty(\Omega)}, L\right\}$ and

$$S(\tau) = \sup\{t > 0 : \sup_{0<s<t} \|w(s)\|_{L^\infty(\Omega)} \le H\} \quad \text{for } \tau \in (0,1],$$

we have that $S(\tau) \in (T_{min}, T_{max}(\tau))$ for $\tau \in (0,1]$.

Since we expect that

$$w(t) \to \lambda \quad \text{as } \tau \to 0 \quad \text{for } t > T_{min},$$

we shall show that

$$|\nabla w(t)| \to 0 \quad \text{as } \tau \to 0 \quad \text{for } t > T_{min}.$$

The following inequality follows from properties of the semi-group.

$$\begin{aligned} \|v(t)\|_{L^\infty(\Omega)} &\le \|v_0\|_{L^\infty(\Omega)} + \int_0^t \|e^{(t-s)(\Delta-1)}w(s)\frac{e^{\chi(v(s))}}{\int_\Omega e^{\chi(v(s))}dx}\|_{L^\infty(\Omega)} \\ &\le \|v_0\|_{L^\infty(\Omega)} + C\int_0^t \frac{e^{s-t}}{(t-s)^\beta}\|w(s)\|_{L^\infty(\Omega)}\|\frac{e^{\chi(v(s))}}{\int_\Omega e^{\chi(v(s))}dx}\|_{L^q(\Omega)}. \end{aligned}$$

Here $q > n/2$, $n/q < 2\beta < 2$. Put $\chi_* = \limsup_{v\to\infty} v\chi'(v)$. We see that

$$\frac{\|e^{\chi(v)}\|_{L^q(\Omega)}}{\int_\Omega e^{\chi(v)}dx} \le \frac{\|e^{\chi(v)}\|_{L^1(\Omega)}^{1/q}\cdot\|e^{\chi(v)}\|_{L^\infty(\Omega)}^{(q-1)/q}}{\|e^{\chi(v)}\|_{L^1(\Omega)}} \le C\frac{\|(v+1)^{\chi_*}\|_{L^\infty(\Omega)}^{(q-1)/q}}{|\Omega|^{(q-1)/q}e^{(q-1)\chi(v_*)/q}}.$$

Then, the following inequality holds.

$$\|v(t)\|_{L^\infty(\Omega)} \le \|v_0\|_{L^\infty(\Omega)} + C(H)\int_0^t \frac{e^{s-t}}{(t-s)^\beta}\left(\|v(s)\|_{L^\infty(\Omega)}^{\chi_*\frac{(q-1)}{q}} + 1\right).$$

Here and henceforth, $C(H)$ is a positive constant depending on the constant H.

Since constants q and χ_* satisfy that $\frac{(q-1)}{q}\chi_* < \frac{(q-1)}{q}\frac{n}{n-2} < 1$, the above inequality leads us to

$$\|v(t)\|_{L^\infty(\Omega)} \le C(H) \quad \text{for} \quad t \in (0, S(\tau)).$$

This means that $u = zw$ is bounded in $\Omega \times (0, S(\tau))$. Combining this with semi-group properties, we imply that for $\alpha, \beta \in (0,1)$ with $1+\alpha < 2\beta$

$$\|(\Delta-1)^\beta v(t)\|_{L^\infty(\Omega)} \le C\|(\Delta-1)v_0\|_{L^\infty(\Omega)} + C(H) \quad \text{for } t \in (0, S(\tau)).$$

Then, the parabolic regularity leads us to

$$\|\nabla(v(t)-v(s))\|_{L^\infty(\Omega)} \le C(H)|t-s|^{\alpha/2} \quad \text{for} \quad t,s \in (0, S(\tau)). \tag{4}$$

For each $t_0 \in (0, S(\tau))$, put

$$A(t_0) = \frac{1}{z(t_0)}\nabla\cdot(z(t_0)\nabla\cdot) \ \text{ in } \Omega, \quad \frac{\partial\,\cdot}{\partial\nu} = 0 \ \text{ on } \partial\Omega.$$

Let $G(x,y,t) = G(x,y,t;t_0)$ be the heat kernel of $\partial_t - A(t_0)$ in Ω with the homogeneous Neumann boundary condition. The following estimate of the heat kernel $G(x,y,t)$ comes from the estimate (4) ([9]).

$$\left|\frac{\partial^j}{\partial t^j}\frac{\partial^\mu}{\partial x^\mu}G(x,y,t)\right| \le \frac{C_1}{t^{(n+2j+|\mu|)/2}}\exp\left(-C_2\frac{|x-y|^2}{t}\right) \quad (0 \le 2j+|\mu| \le 2). \tag{5}$$

Here C_i $(i=1,2)$ depends on H and $\|z(t_0)\|_{C^{1+\alpha}(\overline{\Omega})}$ $(0<\alpha<1/2)$.

We get the following estimates by (5) and Lemma 4.2.

Lemma 4.3 *For $t_0 \in (0, S(\tau))$ and $\tau \in (0,1]$, there exists a positive constant $\Lambda = \Lambda(\min_{\overline{\Omega}} z(x,t_0), \|z(t_0)\|_{L^\infty(\Omega)}, \Omega)$ such that*

$$\|\nabla e^{tA(t_0)}w_0\|_{L^q(\Omega)} \le C\left(1+\frac{1}{\sqrt{t}}\right)e^{-\Lambda t}\|w_0\|_{L^q(\Omega)} \quad \textit{for } t>0,\ q\in(1,\infty)$$

and that

$$\|\nabla e^{tA(t_0)}w_0\|_{L^q(\Omega)} \le Ce^{-\Lambda t}\|\nabla w_0\|_{L^q(\Omega)} \quad \textit{for } t>0,\ q\in\{2\}\cup(n,\infty).$$

Let $T \in (0, T_{min}/2)$. We take an integer J satisfying $J-1 < S(\tau)/T \le J$. Put $T_j = jT$ $(j = 1, 2, \cdots, J-1)$ and $T_J = S(\tau)$.

For $t \in (T_j, T_{j+1})$. Put $\zeta = (t - T_j)/\tau$ and $W(\zeta) = w(t)$. The function W satisfies that

$$W_\zeta = A(T_j)W + \nabla(\log P) \cdot \nabla W - \tau Q W \quad \text{in } \Omega \times (0, T/\tau)$$

and that

$$W(\zeta) = e^{\zeta A(T_j)}W(0) + \int_0^\zeta e^{(\zeta-\xi)A(T_j)}\{\nabla P(\xi) \cdot \nabla W(\xi) - \tau Q(\xi)W(\xi)\}d\xi$$

for $\zeta \in (0, T/\tau)$, where

$$P(\zeta) = \log \frac{z(t)}{z(T_j)} \quad \text{and} \quad Q(\zeta) = \frac{z_t(t)}{z(t)}.$$

For α, $\beta \in (0,1)$ with $1 + \alpha < 2\beta$, it follows from (4), Lemma 4.3 and the maximal regularity that

$$\|\nabla \log \frac{z(t)}{z(T_j)}\|_{L^\infty(\Omega)} \le C(H, v_0, v_*)T^{\alpha/2} \quad \text{for } t \in (T_j, T_{j+1}),$$

$$\tau \int_0^\zeta \|\nabla e^{(\zeta-\xi)A(T_j)}Q(\xi)W(\xi)\|_{n+1} d\xi \le C\tau^{n/(n+1)} \quad \text{for } \zeta \in (0, T/\tau)$$

and that

$$\|\nabla e^{\zeta A(T_j)}W(0)\|_{L^{n+1}(\Omega)} \le Ce^{-\zeta\Lambda}\|\nabla W(0)\|_{L^{n+1}(\Omega)} \quad \text{for } \zeta \in (0, T/\tau).$$

From those, we imply that

$$\|\nabla w(t)\|_{L^{n+1}(\Omega)} \le C\tau^{1/6} \quad \text{for } t \in [T/2, S(\tau)],$$

if $0 < \tau \ll T \ll 1$, Then, if $0 < \tau \ll 1$, we have that

$$\|w(t)\|_\infty \le \|w_0\|_\infty + C\tau^{1/6} \le \frac{3}{2}L = \frac{3}{4}H \quad \text{for } t \in [T/2, S(\tau)].$$

This means that $S(\tau) = \infty$, if $\tau \ll 1$. Then, we obtain that

$$\sup_{t \ge 0} \left(\|w(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)}\right) < \infty$$

and that

$$\sup_{t \ge 0} \|u(t)\|_{L^\infty(\Omega)} < \infty.$$

It follows from this and the standard bootstrap argument that

$$\|(u, v)\|_{C^{2+2\theta.1+\theta}(\Omega \times (0,\infty))} < \infty$$

with some $\theta \in (0, 1/2)$. Thus, we have Theorem 3.2.

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