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Laplacian growth on a branched Riemann surface

Björn Gustafsson[1]

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Abstract

Laplacian growth refers to domain evolution driven by harmonic gradients, for example the gradient of the Green's function with a fixed pole. It makes sense on Riemannian manifolds of arbitrary dimension, and there is a notion of weak solution which allows for changes of topology of the domain during the evolution. However, here we discuss the possibility, in the case of two dimensions, of avoiding changes of topology at the price of allowing the evolution to go up on a branched Riemann covering surface of the original surface. If the initial domain is simply connected one can then describe the evolution by means of conformal mappings from the unit disk, a kind of Loewner evolution.

There appear difficulties which are not yet completely solved. Preliminary results can be found in joint work arXiv:1411.1909 with Yu-Lin Lin, which presently is under further progress with also Joakim Roos as an author.

Keywords: Laplacian growth, Hele-Shaw flow, Polubarinova-Galin equation, Loewner-Kufarev equation, string equation, partial balayage, branched Riemann surface.

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[1]Department of Mathematics, KTH, 100 44, Stockholm, Sweden.
Email: gbjorn@kth.se

1 Introduction

This is a summary of a talk given at RIMS, Kyoto, on June 6, 2017. It represents work in progress [8] with Yu-Lin Lin and Joakim Roos. A partial preprint is available as [7], which in its turn is a continuation of a previous paper [6]. For general background and references on Laplacian growth, or Hele-Shaw flow moving boundary problems, see [11].

2 Laplacian growth, general description

There are many variants, but the traditional description of **Laplacian growth** (**LG**) is given in terms of the following data and definitions.

- $\mathcal{M}$ is a Riemannian manifold, $a \in \mathcal{M}$ a fixed point.
- Ω is any subdomain of $\mathcal{M}$ with $a \in \Omega$.
- $G_\Omega(\cdot, a)$ denotes the Dirichlet Green's function of Ω with pole at a.
- $V_n = -\frac{\partial G_\Omega(\cdot,a)}{\partial n}$ is the speed by which $\partial\Omega$ is imposed to move, in the outward normal direction.

Then: given an initial domain $\Omega(0)$ one asks for the evolution $\Omega(t)$, for t in some interval containing $t = 0$.

LG originally arose in a fluid problem discussed by **Henry Selby Hele-Shaw** [15]. Special features of this moving boundary problem for Hele-Shaw flow, i.e. **LG**, are

- The dynamical law is **nonlocal** ("motion by harmonic measure").
- **Weak solution** makes **LG** decouple into a series of elliptic problems.
- There is an ordinary PDE description as **highly degenerate parabolic** problem (of Stefan type).
- **LG** is extremely **well-posed** as $t \nearrow$ (injection, swelling domains).
- **LG** is extremely **ill-posed** as $t \searrow$ (suction, shrinking domains).

3 History

The subject has switched several times between Great Britain and Russia, but since around 1980 it has been truly international.

- **Henry Selby Hele-Shaw** (1854-1941), British engineer and inventor. Made experiments which are described in *The flow of water*, Nature **58** (1898) [15]. See also the historical review [29].
- Russian school 1945-...(**Galin, Polubarinova, Kochina, Kufarev, Vinogradov,...**); formulation of basic equations, first existence proof.
- British school 1958-... (**Saffman, Taylor, Richardson, Ockendon, Elliott, Howison, Lacey, King,...**); discovery of fingering instability in the suction version, many explicit examples, etc.
- Contributions from many countries 1980-... (**Sakai, Friedman, DiBenedetto, Reissig, Wolfersdorff, Tanveer, Escher, Simonett, Carleson, Makarov, Hedenmalm, Shimorin, Tian, Lin, Onodera,...**); proofs of existence of various kinds of solutions, geometric properties, etc.
- New Russian school 1990-... (**Mineev-Weinstein, Wiegmann, Zabrodin, Krichever, Marshakov,...**); connections to integrable systems and other branches of modern mathematical physics.
- Books by **Etingof, Varchenko, Vasil'ev, Teodorescu, Gustafsson**: [28], [12], [11].

Examples of physical processes in general which are governed by **LG** dynamical laws are:

- Viscous fluid in a Hele-Shaw cell.
- Ground water movement (porous medium flow by Darcy's law).
- Electrochemical depositing.
- 2D quantum gravity.
- Coulomb gas ensembles.
- Quantum Hall regimes.

- (Internal) diffusion limited aggregation **(I)DLA**.
- Random matrix ensembles.
- Dispersionless limit of Toda integrable hierarchy.

4 String equation

In two dimensions, with $\mathcal{M} = \mathbb{C}$ to start with, the **Green's function** is

$$G_\Omega(x, a) = -\log|x - a| + \text{harmonic},$$

vanishing on $\partial\Omega$. A convenient description of **LG** is that it is a smooth evolution $\Omega(t) \subset \mathcal{M}$ such that

$$\frac{d}{dt}\int_{\Omega(t)} \varphi\, dxdy = \int_{\partial\Omega(t)} \varphi\, d\theta \quad \forall\varphi \in C^\infty(\mathcal{M}), \tag{4.1}$$

holds, where (with $*$ denoting the **Hodge star** operator)

$$d\theta = - * dG(\cdot, a) = -\frac{\partial G_{\Omega(t)}(\cdot, a)}{\partial n}\, ds \quad \text{on } \partial\Omega.$$

This one-form $d\theta$ can also be identified with the **harmonic measure** of $\partial\Omega$ with respect to the point a. In the sequel we choose $a = 0$.

The governing law (4.1) can reformulated in several ways, for example as:

1) **String equation** and corresponding Hamiltonian formulation.

2) **Polubarinova-Galin** equation (for conformal map).

3) **Loewner-Kufarev** equation (for conformal map).

4) Variational inequality **weak solution**.

Our main result concerns a combination of 3) and 4). On integrating (4.1) with respect to t over an interval $0 \le t \le T$ one gets

$$\int_{\Omega(T)} \varphi\, dx \wedge dy = \int_0^T \int_{\partial\Omega(t)} \varphi\, dt \wedge d\theta + \int_{\Omega(0)} \varphi\, dx \wedge dy \quad \forall\varphi \in C^\infty(M).$$

Locally, (t, θ) can be used as coordinates, in place of (x, y). Indeed,

$$\begin{cases} t = t(x,y) = \text{the time when } \partial\Omega(t) \text{ reaches } (x,y), \\ \theta = \theta(x,y) = \text{an angular variable along each } \partial\Omega(t). \end{cases}$$

This gives the **string equation**:

$$dt \wedge d\theta = dx \wedge dy, \quad \text{equivalently} \quad \frac{\partial(t,\theta)}{\partial(x,y)} = 1. \tag{4.2}$$

The string equation is not far from being on Hamiltonian form. Consider $H = \theta$ as a **Hamiltonian** function and set

$$\omega = y\,dx - H\,dt.$$

Then (4.2) says that $d\omega = 0$, with $t = t(x, y)$ as above. Now relax t to be a free and independent variable. Then ω can be interpreted as the **action 1-form** in (x, y, t)-space, the **action** itself, along a curve γ, being

$$S = \int_\gamma \omega.$$

In Hamiltonian mechanics one asks for curves γ for which the action becomes stationary. The criterion for this is that

$$i(\xi)d\omega = 0$$

for any tangent vector $\xi = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{t}\frac{\partial}{\partial t}$ along the curve. Here dot denotes derivative with respect to an evolution parameter, which we may take to be t itself, and $i(\xi)$ denotes interior multiplication by ξ. Spelling this out gives the traditional **Hamilton equations**, expressing stationarity of action as

$$\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}. \tag{4.3}$$

From the family of solutions of (4.3) one can recover (4.2).

It should be remarked that the trajectories for (4.3), i.e. the curves $H(x, y) = \text{constant}$, are not the same as the trajectories for the fluid particles in the Hele-Shaw problem.

5 Simply connected case

In the simply connected case, there is a more explicit description in terms of conformal maps $f(\cdot, t) : \mathbb{D} \to \Omega(t)$ from the unit disk (normalization $f(0) = 0$, $f'(0) > 0$ assumed). Indeed, writing $z = x + \mathrm{i}y = f(e^{\mathrm{i}\theta}, t)$ and assuming that f is actually univalent in a neighborhood of $\overline{\mathbb{D}}$ we have the **Polubarinova-Galin** equation

$$\operatorname{Re}\left[\dot{f}(\zeta, t)\, \overline{\zeta f'(\zeta, t)}\right] = 1, \quad \zeta \in \partial\mathbb{D}.$$

This can be identified with (4.2), and also with the **string equation** in the version of **Mineev-Weinstein, Wiegmann, Zabrodin** [31], [19]: for any normalized univalent f in a neighborhood of $\overline{\mathbb{D}}$ there holds

$$\{f, f^*\} = 1.$$

Here the Poisson bracket is defined by

$$\{f, g\} := \zeta \frac{\partial f}{\partial \zeta} \frac{\partial g}{\partial M_0} - \zeta \frac{\partial g}{\partial \zeta} \frac{\partial f}{\partial M_0},$$

in terms of $f^*(\zeta) = \overline{f(1/\bar{\zeta})}$ and the **harmonic moments** $\{M_0, M_1, M_2, \dots\}$ of $\Omega = f(\mathbb{D})$:

$$M_k := \frac{1}{\pi} \int_\Omega z^k \, dxdy = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega} z^k \bar{z} dz.$$

Thus we think of f as a function of ζ and the moments: $f = f(\zeta; M_0, M_1, \dots)$. It can be shown that simply connected domains are locally determined by their moments, and Laplacian growth for such domains is characterized by

$$\begin{cases} M_k = \text{constant}, & k \geq 1, \\ M_0 = 2t + \text{constant}. \end{cases}$$

Therefore, the derivative $\partial/\partial M_0$, which is taken with the other moments $M_1, M_2, \dots$ kept fixed, effectively agrees with the time derivative in the Hele-Shaw problem.

In terms of the Taylor expansion

$$f(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^{k+1} \quad (a_0 > 0)$$

the moments are given by

$$M_k = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} f(\zeta)^k f^*(\zeta) f'(\zeta)d\zeta = \sum (j_0+1)a_{j_0}\cdots a_{j_k}\bar{a}_{j_0+\ldots+j_k+k},$$

where the last expression is known as **Richardson's formula** [21]. This is a highly nonlinear relationship, and even when f is a polynomial of low degree it is virtually impossible to invert it, to obtain $a_k = a_k(M_0, M_1, \ldots)$. Note that such an inversion would give explicit solutions to the Laplacian growth problem. However, there are in the polynomial case at least explicit expressions for the (nonzero) Jacobi determinant for the change, see [17], [27].

Remark 5.1. The moments M_k make sense for arbitrary analytic functions f (not necessarily univalent) on $\overline{\mathbb{D}}$, and for arbitrary $k \in \mathbb{Z}$. However, when f is not locally univalent the moments $M_0, M_1, M_2, \ldots$ do not determine f, even on the infinitesimal level.

6 Loewner-Kufarev equation

The **Polubarinova-Galin** equation (**PG**) [3], [20] can be solved for $\dot{f} = \partial f/\partial t$. This gives an equation of **Loewner-Kufarev** type (**LK**) [18], [16], [30] namely

$$\dot{f} = \nabla(0)f, \tag{6.1}$$

where

$$\nabla(0)f(\zeta,t) := \frac{\zeta f'(\zeta,t)}{2\pi}\int_0^{2\pi}\frac{1}{|f'(e^{i\theta},t)|^2}\frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta}\,d\theta \tag{6.2}$$

can be thought of as a directional derivative, representing a tangent vector in the infinite dimensional space of univalent functions. The equations (6.1), (6.2) make sense also if f' has zeros in $\mathbb{D}$, even though zeros on $\partial\mathbb{D}$ cause some troubles. When there are zeros in $\mathbb{D}$, **LK** is stronger than **PG**.

Goal: We set out to solve (6.1) for $0 \le t < \infty$, given f at $t = 0$. This requires relaxation to weak solutions, otherwise it is not always possible.

In the test function description (4.1) of **LG** it is enough to use functions which are harmonic in $\Omega(t)$. This gives the characterization

$$\frac{d}{dt}\int_{\Omega(t)} h\,dxdy = 2\pi h(0) \quad \forall h \in \mathrm{Harm}(\overline{\Omega(t)}),$$

and after integration

$$\int_{\Omega(t)} h\,dxdy = \int_{\Omega(0)} h\,dxdy + 2\pi t h(0) \quad \forall h \in \mathrm{Harm}(\overline{\Omega(t)}). \tag{6.3}$$

For subharmonic functions h one has the same, but with inequality $\geq$ instead. The above expresses that

$$\chi_{\Omega(t)} \cong \chi_{\Omega(0)} + 2\pi t\delta_0,$$

where $\cong$ denotes **graviequivalence**. In the direction $\chi_{\Omega(0)} + 2\pi t\delta_0 \mapsto \chi_{\Omega(t)}$ it is a form of balayage, **partial balayage** (see [9]). We then write

$$\mathrm{Bal}(2\pi t\delta_0 + \chi_{\Omega(0)}, 1) = \chi_{\Omega(t)}. \tag{6.4}$$

7 Weak solutions and Balayage

Laplacian growth makes sense on Riemannian manifolds, and in the well-posed time direction $t \nearrow$ there is a **good notion of weak solution**, which is global (allows $t \to \infty$).

- However, the domains $\Omega(t)$ are then **not always simply connected**, and hence not always on the form $f(\mathbb{D}, t)$. If we insist on having a solution on the form $\Omega(t) = f(\mathbb{D}, t)$ we must allow $\Omega(t)$ to spread on a Riemann surface above $\mathbb{C}$.
- The problem is that the Riemann surface we would need is **not given in advance**, it has to be created along with the solution. Whenever a zero of f' approaches $\partial\mathbb{D}$ one has to add a branch point to make sure that the solution can spread on a covering surface.

Definition 7.1. An evolution $\Omega(t)$ for $t \geq 0$ is called a **weak solution** of **LG** if, for each $t > 0$, $\Omega(0) \subset \Omega(t)$ and

$$\int_{\Omega(t)} h\,dxdy \geq \int_{\Omega(0)} h\,dxdy + 2\pi t h(0). \tag{7.1}$$

for every $h \in L^1(\Omega(t))$ which is **subharmonic** in $\Omega(t)$.

The above inequality (7.1), saying that $\Omega(t)$ is a kind of **quadrature domain** for subharmonic functions, is equivalent to the balayage statement (6.4). The theory of quadrature domains for subharmonic functions was developed by M. Sakai [23], [24], [25], and construction of such quadrature domains were later developed into a notion of partial balayage tacitly used in [5], and fully elaborated in [10], [9]. Partial balayage is also closely related to weighted equilibrium distributions [22].

Theorem 7.1 ([24], [4]). *Given any bounded open set $\Omega(0)$ there exists a unique weak solution $\Omega(t) \subset \mathbb{C}$, $0 \leq t < \infty$, in the above sense.*

- The uniqueness statement actually requires some additional precision concerning nullsets.
- The theorem generalizes to much more general settings, and to Riemannian manifolds of any dimension.

The weak solution can be constructed as the solution of an **obstacle problem**: For any $t > 0$, let u be the smallest function satisfying

$$\begin{cases} u \geq 0, \\ \Delta u \leq 1 - \mu, \end{cases}$$

where $\mu = \chi_{\Omega(0)} + 2\pi t$, and define $\Omega(t)$ by

$$\Omega(t) = \{u > 0\}.$$

Thus $u = 0$ outside $\Omega(t)$ and one has

$$\chi_{\Omega(t)} = \mu + \Delta u.$$

This can be seen to be equivalent to (7.1) and hence provides a proof for existence and uniqueness of weak solution

Comments:

- Weak solutions are made up of just bounded open sets $\Omega(t)$, and these are allowed to change topology during the evolution.
- Solutions within the framework of conformal mappings break down when changes of topology occurs.

Now, our project (not yet finished) is still to insist on **both** global in time solutions **and** the domains being simply connected. This has a price:

- One need to allow $\mathbb{C}$ to be replaced by a multi-sheeted and branched Riemann covering surface $\mathcal{M}$, which contains $\Omega(0)$.

8 Main result

Theorem 8.1 (modulo **Conjecture** below). *Starting with any function f which is analytic in a neighborhood of $\overline{\mathbb{D}}$ and is normalized by $f(0) = 0$, $f'(0) > 0$, there is a global evolution in time satisfying $\dot{f} = \nabla(0)f$ in a weak sense.*

More precisely, there exists a Riemann surface $\mathcal{M}$ and a covering map $p : \mathcal{M} \to \mathbb{C}$ such that, for each t, $f(\cdot, t) \to \mathbb{C}$ lifts to

$$\tilde{f}(\cdot, t) : \mathbb{D} \to \mathcal{M}$$

and then becomes univalent. The image domains $\tilde{\Omega}(t) = \tilde{f}(\mathbb{D}, t)$ make up a global weak **LG** *evolution on $\mathcal{M}$.*

The evolution is not unique, but presumably there is a unique minimal choice, introducing no more branch points than necessary.

Proof. The steps of the proof are the following.

- Starting with $\Omega(0) = f(\mathbb{D})$, construct the weak solution $\Omega(t)$. Then there are no problems as long as $\Omega(t)$ remains simply connected, there is a corresponding conformal map $f(\cdot, t) : \mathbb{D} \to \Omega(t)$.

- Even if $\Omega(t)$ starts wrapping over itself, in the sense that the conformal mapping f from the unit disk is no longer univalent, there are no major problems as long as f stays locally univalent, i.e., there are no zeros of f' inside $\mathbb{D}$. The solution just proceeds on a non-branched covering surface.

- The real problem starts when zeros of f' penetrate $\partial\mathbb{D}$ and go into $\mathbb{D}$. Then it, first of all, takes some efforts to construct an appropriate branched Riemann surface on which the solution can proceed. Secondly it is, after the somewhat singular step of adding a branch point, difficult to control that the solution stays simply connected on the new surface. However once this is controlled the solution can, by repeating the procedure of adding branch points if necessary, be extended forever as a simply connected solution.

$\square$

A weak solution on a branched covering surface becomes weighted Laplacian growth in a uniformizing coordinate. The problem of keeping the domains simply connected for a short time after adding a branch point then boils down to the following statement.

Conjecture. Let g be analytic in a neighborhood of $\overline{\mathbb{D}}$ and let, for $t > 0$ sufficiently small, $\Omega(t) = \{u > 0\}$, where u is the smallest function satisfying

$$\begin{cases} u \geq 0, \\ \Delta u \leq |g|^2(1 - \chi_{\mathbb{D}}) - 2\pi t \delta_0. \end{cases}$$

Then (**claim**),

$$\Omega(t) = \{u > 0\}$$

is star-shaped with respect to the origin if $t > 0$ is sufficiently small. In particular $\Omega(t)$ is then simply connected.

□

- The crucial case is when g has zeros on $\partial\mathbb{D}$. Otherwise the conjecture is known to be true by stability estimates due to L. Caffarelli [1], [2]. These are based on having $\Delta u \geq c > 0$ near the free boundary.
- The conjecture may not seem very exciting since it is almost obvious that it must be true. Still we have not been able to prove it. Neither was Makoto Sakai, who ran into the same question when working on an inverse problem in potential theory. See his paper [26].

9 Riemann surface of square root

In this section, and next, we give examples of Laplacian growth on branched Riemann surfaces.

Example 9.1. $\mathcal{M}$ = Riemann surface of $\sqrt{z-1}$ = the two-sheeted surface

$$\mathcal{M} = (\mathbb{C} \setminus \{1\}) \cup \{1\} \cup (\mathbb{C} \setminus \{1\})$$

over $\mathbb{C}$. A local coordinate (actually global) on $\mathcal{M}$ is $\tilde{z} = \sqrt{z-1}$. The covering map is $p : \mathcal{M} \to \mathbb{C}$, $\tilde{z} \mapsto z = \tilde{z}^2 + 1$.

Laplacian growth $\tilde{f}(\cdot,t):\mathbb{D}\to\mathcal{M}$ started at $\tilde{z}=+\mathrm{i}$ becomes

$$\tilde{z}=\tilde{f}(\zeta,t)=\begin{cases}\sqrt{t\zeta-1}, & (0<t<1),\\ \sqrt{\frac{t(t\zeta-1)^2}{\zeta-t}} & (1<t<\infty),\end{cases}$$

and when pushed down to $\mathbb{C}$

$$z=f(\zeta,t)=\begin{cases}t\zeta, & (0<t<1),\\ \frac{\zeta(t^3\zeta-2t^2+1)}{\zeta-t} & (1<t<\infty).\end{cases}$$

Example 9.2. The derivative is

$$f'(\zeta,t)=\begin{cases}t & (0<t<1),\\ t\cdot\frac{(t\zeta-1)(t\zeta-2t^2+1)}{(\zeta-t)^2} & (1<t<\infty),\end{cases}$$

hence it adopts the factor $G(\zeta)=\frac{(t\zeta-1)(t\zeta-2t^2+1)}{(\zeta-t)^2}$ at critical time $t=1$. This has, for $t>1$,

- Zeros: $\omega_1=1/t$ (in $\mathbb{D}$), $\omega_2=2t-1/t$ (outside $\mathbb{D}$).
- Poles: $\zeta_1=\zeta_2=t$.

With suitable scaling G is a **contractive zero divisor** in the sense of H. Hedenmalm [13], [14] for Bergman space. This means for example that

$$h(0)=\int_{\mathbb{D}}h(z)|G(z)|^2dA(z)\quad\forall h\in\mathrm{Hol}(\overline{\mathbb{D}}).$$

Example 9.3. For a more general G, of the form

$$G(\zeta)=\frac{(\zeta-\omega_1)(\zeta-\omega_2)}{(\zeta-\zeta_1)^2},$$

one has an identity

$$\frac{1}{\pi}\int_{\mathbb{D}}h(z)|G(z)|^2dA(z)=a_0h(0)+a_1h(1/\bar{\zeta}_1)+c\int_0^{1/\bar{\zeta}_1}hG\,d\zeta.$$

If here $1/\bar{\zeta}_1=\omega_1$ (or $=\omega_2$) then $a_1=0$, and if $\zeta_1=\frac{1}{2}(\omega_1+\omega_2)$, then $c=0$. This is exactly what we had in Example 9.2, and it is what happens in general in the **LG** evolution when zeros of f' penetrate into $\mathbb{D}$: a pair of zeros and a double pole, subject to the above relations, are created.

10 Several evolutions of cardioid

We start **LG** with

$$f(\zeta,0)=\zeta-\frac{1}{2}\zeta^2, \tag{10.1}$$

for which $M_0 = 3/2$, $M_1 = -1/2$, $M_2 = M_3 = \cdots = 0$. For convenience we shall allow a more free (but monotone) relation between time t and M_0. Normalizing so that the leading coefficient in f is e^t we have a perfectly good global **LG** solution

$$f(\zeta,t)=e^t\zeta-\frac{1}{2}e^{-2t}\zeta^2, \quad 0<t<\infty,$$

for which $M_1, M_2, \dots$ remain fixed and

$$M_0=a_0^2+2|a_1|^2=e^{2t}+\frac{1}{2}e^{-4t}.$$

Above M_0 is a convex function of t for all $-\infty<t<\infty$ and it attains its minimum value at $t=0$. Therefore M_0 (essentially the area) increases also when t decreases from $t=0$, and we get a new **LG**-evolution by changing the sign of t:

$$f(\zeta,t)=e^{-t}\zeta-\frac{1}{2}e^{2t}\zeta^2, \quad 0<t<\infty.$$

This is however not univalent, f' has a zero $\omega_1(t)=e^{-3t}$ in $\mathbb{D}$, but f still satisfies the string equation. Also, $b_1(t)=f(\omega_1(t),t)$ does not stay fixed, so the $f(\cdot,t)$ are not conformal maps into a fixed Riemann surface.

Since $f'(\zeta,0)=1-\zeta$ has a zero $\omega_1=1$ on $\partial\mathbb{D}$ one might want to lift solutions to a Riemann surface with a branch point over $f(\omega_1,0)=\frac{1}{2}$, in order to make sure that one does not run into troubles. We then let

$$f'(\zeta,0)=(1-\zeta)\,\frac{(\zeta-1)(\zeta-1)}{(\zeta-1)^2}$$

continue as

$$f'(\zeta,t)=b(t)\,\frac{(\zeta-\omega_1(t))(\zeta-\omega_2(t))(\zeta-\omega_3(t))}{(\zeta-\zeta_1(t))^2},$$

with the zeros and poles related according to certain principles:

The requirements which determine this evolution are (see [7] for explanations):

- The reflected point of $\zeta_1(t)$ is to be a zero of f':

$$f'(1/\overline{\zeta_1(t)}, t) = 0.$$

- $f(\cdot, t)$ shall map the above point $1/\overline{\zeta_1(t)}$ to a point which does not move:

$$f(1/\overline{\zeta_1(t)}, t) = \text{constant} = f(1,0) = \frac{1}{2}.$$

- The moment M_1 is conserved in time:

$$M_1(t) = \mathrm{Res}_{\zeta=0}(ff^*f'd\zeta) = M_1(0) = -\frac{1}{2}.$$

- The dependence of $M_0(t)$ on t has to be specified.

The above may be worked out to a solution

$$f(\zeta, t) = \frac{b_1\zeta + b_2\zeta^2 + b_3\zeta^3}{\zeta_1 - \zeta}$$

where

$$\begin{cases} \zeta_1(t) = \sqrt{\frac{1}{2}(1 + 2e^t - e^{-2t})}, \\ b_1(t) = e^t, \\ b_2(t) = -\frac{1}{4\sqrt{2}}(1 + 2e^t + 3e^{-2t})\sqrt{1 + 2e^t - e^{-2t}}, \\ b_3(t) = \frac{1}{4}(2e^{-t} + e^{-2t} - e^{-4t}). \end{cases}$$

The relation between M_0 and t is here

$$M_0(t) = \frac{1}{8}(4e^{2t} + 2e^t + e^{-2t} + 6e^{-3t} + 2e^{-4t} - 3e^{-6t}).$$

An interesting aspect is that the above fully explicit solution $f(\zeta, t)$ is not only smooth at $t = 0$, it even has a real analytic continuation across $t = 0$. This extended solution, defined on $-\varepsilon < t < \infty$ (say), has the drawback that it has a pole inside $\mathbb{D}$ when $t < 0$. But in some sense it still represents suction out of the cardioid as t decreases to negative values.

In summary we have constructed, starting from (10.1), the following solutions to **PG** and **LK** (recall that **LK** $\Rightarrow$ **PG**):

- One univalent forward ($t \nearrow$) solution of **LK**.

- One non-univalent forward ($t \nearrow$) solution of **PG** (not satisfying **LK**).
- One non-univalent forward ($t \nearrow$) solution of **LK**.
- One backward ($t \searrow$) solution, with a pole inside $\mathbb{D}$, of **LK**. This solution is non-univalent, but $f(\partial\mathbb{D}, t)$ still consist of simple analytic curves.

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