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An Alternative Proof of 1-Generic Splittings

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Abstract

Wu (2006) showed that every nonzero computably enumerable degree splits into two 1-generic degrees, and therefore, no two computably enumerable degrees bound the same class of 1-generic degrees. By relativizing this result with respect to the Lachlan set, it can be shown that (*) every nonzero d.c.e degree splits into four 1-generic degrees. Here, a set A is d.c.e. (or, 2-c.e.) if there are two computably enumerable sets B and C such that $A = B - C$ (set difference). Turing degree of a d.c.e. set is called a d.c.e. degree. By (*), no two d.c.e. degrees bound the same class of 1-generic degrees. Chong and Yu (2016) improved the result (*). In fact, it is split into two 1-generic degrees. In this note, we propose a construction with rollbacks of stages. By means of this construction, we give an alternative proof of (*).

Keywords: 1-generic set; Ershov hierarchy; d.c.e. set; 2-c.e. set

1 Introduction

A set G of natural numbers is 1-generic if for any computably enumerable set W, G has a finite initial segment a such that either (i) a belongs to W or (ii) no extension of a belongs to W. Turing degree of a 1-generic set is called a 1-generic degree.

Among many interesting properties of 1-generic degrees, it is well-known that $\mathbf{0}'$, the degree of halting problem, splits into two 1-generic degrees. More precisely, there is a pair of 1-generic sets whose join is Turing equivalent to the halting problem. The join of two sets A and B is $\{2n : n \in A\} \cup \{2n+1 : n \in B\}$, and denoted by $A \oplus B$.

Wu [11] extended the above fact to all nonzero c.e. degrees. Every nonzero c.e. degree splits into two 1-generic degrees.

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We investigate this result under a relaxed assumption. Suppose that a set A of natural numbers is a set difference of two c.e. sets, in other words, there are two computably enumerable sets B and C such that $A = B - C$. Such A is called a *d.c.e. set.* Turing degree of A is called a *d.c.e. degree.*There exists a degree **a** such that a d.c.e. set belongs to **a** but no c.e. set belongs to **a** [2, 5].

It is known that A is d.c.e. if and only if A has an approximation sequence $\{A_s\}_{s\in\omega}$ with number of mind changes at most two. To be more precise, $\{A_s\}_{s\in\omega}$ is a computable sequence of finite sets, each natural number x belongs to A if and only if x belongs to all but finitely many A_s, A_0 is empty, and for each x there are at most two s such that $x \in (A_{s+1} - A_s) \cup (A_s - A_{s+1})$.

We show that every nonzero d.c.e. degree splits into at most four 1-generic degrees.

Main Theorem *Suppose that A is a non-computable d.c.e. set. Then, there exist 1-generic sets G_0, G_1, G_2 and G_3 such that A is Turing equivalent to $G_0 \oplus G_1 \oplus G_2 \oplus G_3$.* . The major differences between our construction and the original construction by Wu are the following. (1) We introduce real stages and ostensible stages to use in construction. (2) We define a set H of real stages, and we make use of the set for construction and verification.

We are going to sketch the above-mentioned (1) and (2). Firstly, we describe (1). Our construction has real stages and ostensible stages. The former are the stages in a usual finite injury priority argument. Each real stage r has its ostensible stage $s(r)$. For many r, we have $s(r+1) = s(r)+1$. However, for some r, we have $s(r+1) < s(r)$. We may roll ostensible stages back.

For example, suppose that $A_0 = \emptyset$, $A_1 = \{a_1\}$, $A_2 = \{a_1, a_2\}$, $A_3 = \{a_1\}$, $A_4 = \emptyset$, $A_5 = \{a_5\}$, and $A_6 = \{a_5, a_6\}$. At each real stage $r \leq 3$, the corresponding ostensible stage is $s = r$. We perform our construction using A_s. At real stage $r = 3$, a_2 drops out of A_3. We roll important parameters back to their values at the end of real stage $r = 1$, the moment just before a_2 is enumerated into A. This time we skip the operation we performed at real stage $r = 2$. At real stage $r = 4$, the ostensible stage is 3. At real stage $r = 5$, the ostensible stage is 4. Thus, we look at A_4. Then a_1 drops out of A_4. We roll important parameters back to their values at the end of real stage $r = 0$, the moment just before a_1 is enumerated into A. This time we skip the operation we performed at real stage $r = 1$. At real stage $r = 6$, the ostensible is 2. Figure 1 illustrates the ostensible stage s in the above example as a function of the real stage r.

Secondly, we describe (2).When an element of A is drop out,we put "This will drop out of A"mark on the element.The set H of real stage is defined with reference to "This will drop out of A"marks.The set H is infinite and Π^0_1.By means of H,we show that G_0 and G_1 are 1-generic and A is Turing equivalent to $G_0 \oplus G_1 \oplus H$.Since H is equivalent to a c.e.set, by Wu's result, H is Turing equivalent to join of two 1-generic sets.Thus, **a** splits into at most four 1-generic degrees.

In section 2, we present our notation. We give technical details of our proof of the main theorem in section 3. As a corollary, we get that no two d.c.e.

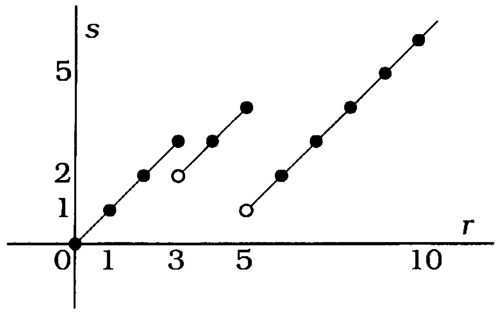

Figure 1: **ostensible stage s as a function of real stage r**

degrees bound the same class of 1-generic degrees.

2 Notation

Definition 2.1. Suppose that A is a set of natural numbers. For a positive integer k, A is *k-c.e.* if there exists a comptable function $f : \omega \times \omega \to \{0,1\}$ such that for each natural number x, the following hold.

- $f(x, 0) = 0$.
- $\exists n \ \forall s \geq n \ f(x,s) = A(x)$. (We say "$\lim_n f(x,s) = A(x)$".)
- $|\{s : f(x,s) \neq f(x,s+1)\}| \leq k$. (We say "the number of mind changes at x is at most k".)

A set is *co-k-c.e.* if the complement is k-c.e.

The sequence $\{A_s\}_{s\in\omega}$ (with $A_s(x) = f(x,s)$) is called an *approximation sequence* of A.

1-c.e. sets are computably enumerable sets. It is known that a set A is 2-c.e. if and only if it is d.c.e.

k-c.e. sets for positive integers k forms the finite levels of the *Ershov hierarchy* [6, 7, 8, 9]. The Ershov hierarchy does not collapse. In particular, it is known that there exists a set A such that both of A and its complement are 2-c.e. and neither A nor its complement is c.e.

Given natural numbers a, b, we let $[a, b]$ denote the closed interval of natural numbers.

3 Construction

3.1 A brief review of Wu's original construction

Suppose that A is a c.e. set and $\{A_s\}_{s\in\omega}$ is an approximation sequence of A with the number of mind changes at most one. We let $\mathbf{a}$ denote the Turing degree of A.

By means of $\{A_s\}_{s\in\omega}$, Wu [11] performs a finite injury priority argument, and splits $\mathbf{a}$ into two 1-generic degrees. We review the requirements in Wu's original construction.

In the following, our notation is similar to that of the original one, but we have some exceptions. In particular, 1-generic sets are written as A_0 and A_1 in [11], but we write them as G_0 and G_1.

At stage s, strings $\alpha_{0,s}$, $\alpha_{1,s}$ and Γ_s are defined. G_0 and G_1 are defined as limits of $\alpha_{0,s}$ and $\alpha_{1,s}$ $(s \to \infty)$, respectively. For example, for each natural number x, $G_0(x) = \lim_s \alpha_{0,s}(x)$. Γ_s plays a role of an approximation of an initial segment of A. We define a finite function γ_s from the domain of Γ_s to ω. γ_s plays a role of use. We let Γ denote the limit of Γ_s $(s \to \infty)$.

Three types of requirements are considered. The suffix e runs over all natural numbers and i runs over $\{0, 1\}$.

$\mathcal{C}$: Γ gives a reduction of A to $G_0 \oplus G_1$. (Coding)
$\mathcal{P}$: $G_0 \oplus G_1 \leq_T A$. (Permitting)
$\mathcal{G}_{e,i}$: $\{e\}^{G_i}(e)\downarrow \ \vee\ (\exists \sigma \subseteq G_i)\ (\forall \tau \supseteq \sigma)\ \{e\}^{\tau}(e)\uparrow$ (Genericity)

In order to satisfy $\mathcal{G}_{e,i}$, we run $\text{cycle}_{e,i}(n)$, for some natural numbers n. For each n, $\text{cycle}_{e,i}(n)$ has three parameters $k^n_{e,i}$, $\sigma^n_{e,i}$ and $\tau^n_{e,i}$. These are called threshold, pretarget and target, respectively. If there is no worry of misreading, we omit subscripts e, i.

The following (1)–(4) describe the move of cycle(n).

(1) We choose a big number k^n at stage s_n. If the domain of Γ_{s_n-1}, which is a natural number, is less than $k^n + 1 (= \{0, \dots, k^n\})$ then we extend the domain of Γ_{s_n} to $k^n + 1$ by filling in each missing part, say x, by $A_{s_n}(x)$. For each such x, we define $\gamma_{s_n}(x)$ as a big number. We extend α_{i,s_n-1} to σ^n of length $\gamma_{s_n}(k^n)$ by concatenating 0s after α_{i,s_n-1}. We call σ^n the *pretarget* of cycle(n). We define $\alpha_{i,s_n} = \sigma^n$, and $\alpha_{1-i,s_n} = \alpha_{1-i,s_n-1}$.

(2) At each stage $s > s_n$, we search for an extension τ^n of the pretarget such that $|\tau^n| < s$ and $\{e\}^{\tau}_s(e)\downarrow$, where the superscript τ denotes τ^n. We call such a τ^n a *target* of cycle(n) (if such τ^n exists).

(3) Suppose that at stage $u > s$, we find a target τ^n. For each $y < k^n$ such that $f(y)$ is not defined, we define $f_{e,i}(y)$ to be $A_u(y)$. We start cycle($n+1$) (we take out insurance against failure of cycle(n)).

(4) Suppose that at a stage $t_n > u$, a natural number $y < k^n$ is enumerated into A. Then we put $\gamma_{t_n-1}(y)$ into G_{1-i}, in other words, we let $\alpha_{1-i}(\gamma_{t_n-1}(y)) = 1$. We undefine $\Gamma_{t_n}(x)$ for each $x \geq y$, and define $\alpha_{i,t_n} = \tau^n$. We declare that $\mathcal{G}_{e,i}$ is satisfied via the target $\tau^n_{e,i}$.

If (1) happens at stage s_n, and at stage $s' > s_n$, a natural number $y < k^n$ is enumerated into A and we do not have found the target yet, then we say that cycle(n) is *refreshed* at stage s'.

If (4) happens, we say that $\tau^n_{e,i}$ is *realized* at stage t_n.

We say that $\mathcal{G}_{e,i}$ requires attention at stage s if one of the following (i)–(iv) happens.

(i) We do not have started any cycle for $\mathcal{G}_{e,i}$ yet by stage s.
(ii) For some n, $\text{cycle}_{e,i}(n)$ is refreshed at stage s.
(iii) For some n, we find the target $\tau^n_{e,i}$ at stage s.
(iv) For some n, the target $\tau^n_{e,i}$ is realized at stage s.

We fix a bijection $\langle\ ,\ \rangle : \omega^2 \to \omega$ such that both $\langle\ ,\ \rangle$ and its inverse is computable. If $\langle e,i\rangle < \langle e',i'\rangle$, we say that $\mathcal{G}_{e,i}$ has higher priority than $\mathcal{G}_{e',i'}$.

Whenever $\mathcal{G}_{e,i}$ requires attention, all $\mathcal{G}_{e',i'}$ with lower priority are initialized. All $\text{cycle}_{e',i'}(n)$ are also initialized.

3.2 Components of our construction

We are going to explain components of our construction. We make an agreement on an approximation sequence. We sketch concepts of ostensible stages, real stages, "Do not enter range of gamma" mark, and "This will drop out of A" mark. We modify requirements in Wu's original construction.

Suppose that A is a d.c.e. set of natural numbers. There exists an approximation sequence $\{A_s\}_{s\in\omega}$ of A such that the following hold.

1. $A_0 = \emptyset$.
2. $\lim_s A_s = A$.
3. For each s, $A_{2s} \triangle A_{2s+1}$ is a singleton, and $A_{2s+1} = A_{2(s+1)}$. ($X \triangle Y$ denotes the *symmetric difference* $(X - Y) \cup (Y - X)$.)

We fix such a sequence. We denote the unique element of $A_{2s} \triangle A_{2s+1}$ by a_{s+1}.

Our construction has real stages and ostensible stages. A real stage is a stage in the usual sense of priority arguments. An ostensible stage s is a function of a real stage r. The difference $r - s$ is always a nonnegative even number. In the usual case, every time a real stage increases one, the corresponding ostensible stage increases one. When a certain condition holds, we decrease the ostensible stage. In the following description of construction, stage means an ostensible stage, unless specified. We abbreviate real stage unless there is a risk of confusion.

During the construction, we mark some natural numbers. We have two kinds of marks. One is "Do not enter range of gamma" mark. If a natural number x has this mark at stage s (with real stage r), at stage $s+1$ (with real stage $r+1$), x can not be in the range of γ_{s+1}.

The other is "This will drop out of A" mark. When $a_{s+1} \in A_s - A_{s+1}$, We roll important parameters back to their values at the end of real stage, say r', the moment just before a_{s+1} is enumerated into A. We put "This will drop out of A" mark on a_{s+1}.

We have some exceptions of rollback. Once a natural number n receives "Do not enter range of gamma" mark, n holds the mark even if we roll other parameters back. The same convention applies to "This will drop out of A" mark.

In the initial setting, both of the real stage and the ostensible stage are 0.

3.3 Main body of our construction

Ostensible stage 0 (with real stage 0):

Start cycle(0) of requirement $\mathcal{G}_{0,0}$ as follows.

(a) Choose $k^0_{0,0}$ as a big number. In particular, $k^0_{0,0} > a_1$.

(b) We define functions $\Gamma_0 : [0, k^0_{0,0}] \to \{0,1\}$ and $\gamma_0 : [0, k^0_{0,0}] \to \omega$. For each $x \leq k^0_{0,0}$, $\Gamma_0(x) := 0$. $\gamma_0(x)$ are big enough and in increasing order.

(c) We define $\sigma^0_{0,0}[0]$, the pretarget of cycle(0), as the following function.

$$\sigma^0_{0,0}[0] : [0, \gamma_0(k^0_{0,0})) \to \{0\}$$

We define $\alpha_{0,0} := \sigma^0_{0,0}[0]$ and $\alpha_{1,0} := \emptyset$.

Go to next ostensible stage (with next real stage).

Ostensible stage $s+1$ (with real stage $r+1$):

I. s is even (enumeration of A):

Case IA: a_{s+1} has "This will drop out of A" mark. Go to next ostensible stage (with next real stage).

Case IB: a_{s+1} does not have "This will drop out of A" mark and $a_{s+1} \in A_{s+1} - A_s$.

Take the least $\langle e, i \rangle < s+1$ such that $\mathcal{G}_{e,i}$ requires attention. Then we have $a_{s+1} \leq k^n_{e,i}$.

For some n, one of the following holds.

(i) $\tau^n_{e,i}$ is realized by a_{s+1} at stage $s+1$ (with real stage $r+1$).

(ii) cycle(n) is refreshed at stage $s+1$ (with real stage $r+1$).

We choose the least such n. We define $\alpha_{1-i,s+1}, \Gamma_{s+1}$ and γ_{s+1} as follows.

$\alpha_{1-i,s+1} := (\alpha_{1-i,s} \upharpoonright \gamma_s(a_{s+1}))^\frown \langle 1 \rangle$

$\Gamma_{s+1} := \Gamma_s \upharpoonright a_{s+1}$

$\gamma_{s+1} := \gamma_s \upharpoonright a_{s+1}$

To be more precise, we define $\alpha_{1-i,s+1}$ as follows. For each $x < \gamma_s(a_{s+1})$, if $\alpha_{1-i,s}(x)$ is defined we define $\alpha_{1-i,s+1}(x) = \alpha_{1-i,s}(x)$; otherwise, we let $\alpha_{1-i,s+1}(x) = 0$. We define $\alpha_{1-i,s+1}(\gamma_s(a_{s+1})) = 1$.

We initialize all the strategies with lower priority, and cancel all cycles(m) of $\mathcal{G}_{e,i}$ with $m > n$.

Subcase IB (i): (i) happens.

$\alpha_{i,s+1} := \tau^n_{e,i}[s]$

We put "Do not enter range of gamma" marks on all natural numbers in the domain of the target minus pretarget, in other words, we put the marks on all natural numbers in $\mathrm{dom}(\tau^n_{e,i}[s] - \sigma^n_{e,i}[s])$.

Subcase IB (ii): Otherwise. Thus, (i) does not happen and (ii) happens.

We redfine Γ_{s+1} as a function $[0, k^n_{e,i}] \to \{0,1\}$. For each $x < a_{s+1}$, we have defined $\Gamma_{s+1}(x)$. For each x such that $a_{s+1} \leq x \leq k^n_{e,i}$ and x has "This will drop out of A" mark, we define $\Gamma_{s+1}(x) := 0$. Otherwise, we define $\Gamma_{s+1}(x) := A_{s+1}(x)$.

We redfine γ_{s+1} as a function $[0, k^n_{e,i}] \to \omega$. For each $x < a_{s+1}$, we have defined $\gamma_{s+1}(x)$. We define $\gamma_{s+1}(x)$, for $x = a_{s+1}, \cdots, k^n_{e,i}$ in this order, obeying the following rules.

- If there is a stage t with real stage $r' < r$ and a natural number x' such that $\gamma_t(x')$ is defined, then $\gamma_{s+1}(x) \geq \gamma_t(x')$. In particular, if $x' < x$ then $\gamma_{s+1}(x)$ is bigger enough than $\gamma_t(x')$.
- $\gamma_{s+1}(x)$ is bigger enough than $\gamma_{s+1}(x-1)$.
- We choose a natural number without "Do not enter range of gamma" mark as the value of $\gamma_{s+1}(x)$.

We define $\sigma^n_{e,i}[s+1] := \alpha_{i,s}{}^\frown\langle 0, \cdots, 0\rangle$ with domain $[0, \gamma_{s+1}(k^n_{e,i}))$.
$\alpha_{i,s+1} = \sigma^n_{e,i}[s+1]$.

In both Subcases IB (i) and IB (ii), go to next stage (with next real stage).

Case IC: Otherwise. That is, a_{s+1} does not have "This will drop out of A" mark and we have $a_{s+1} \in A_s - A_{s+1}$.

Recall that the real stage is $r+1$.

We observe the construction from the beginning, and find the largest $r_0 < r$ such that, letting $t_0 + 1$ be the ostensible stage at real stage $r_0 + 1$, we have $t_0 < s$ and $a_{t_0+1} = a_{s+1} \in A_{t_0+1} - A_{t_0}$.

We call $(r+1, s+1)$ and (r_0+1, t_0+1) *local time at origin* and *local time at destination* of this Case IC, respectively.

We define $\alpha_{0,s+1}$, $\alpha_{1,s+1}$, Γ_{s+1} and γ_{s+1} (of real stage $r+1$) as α_{0,t_0}, α_{1,t_0}, Γ_{t_0} and γ_{t_0} (of real stage r_0), respectively.

We redefine status of requirements, cycles, targets and pretargets as those in stage t_0 with real stage r_0. We do not erase (the two kinds of) marks written up to real stage r.

We put "This will drop out of A" mark on a_{s+1}.

Go to ostensible stage $t_0 + 2$ with real stage $r+2$. In real stage $r+2$, for example, α_{0,t_0} denotes $\alpha_{0,s+1}$ defined above.

II. s is odd (defining permission markers, pretargets and targets):

Choose the least $\langle e, i\rangle < s+1$ such that $\mathcal{G}_{e,i}$ requires attention. Then for some n, one of the following holds.

(i) $\mathcal{G}_{e,i}$ is not in a cycle.

(ii) $\mathcal{G}_{e,i}$ founds a target.

Subcase II (i): (i) happens.

Start cycle(0) of requirement $\mathcal{G}_{e,i}$ as follows.

(a) Choose $k^0_{e,i}$ as a big number. In particular, $k^0_{e,i} > a_{s+2}$.

(b) We extend Γ_s to a function $\Gamma_{s+1} : [0, k^0_{e,\imath}] \to \{0,1\}$, and extend γ_s to a function $\gamma_{s+1} : [0, k^0_{e,i}] \to \omega$.

Suppose $x \leq k^0_{e,\imath}$ and we do not have defined $\Gamma_s(x)$. If x has "This will drop out of A" mark, we define $\Gamma_{s+1}(x) := 0$. Otherwise, we define $\Gamma_{s+1}(x) := A_{s+1}(x)$.

We define $\gamma_{s+1}(x)$ in the same way as Subcase IB (ii).

(c) We define $\sigma^0_{e,\imath}[0]$, the pretarget of cycle(0), and $\alpha_{\imath,s+1}$ in the same way as Subcase IB (ii).

We define $\alpha_{1-\imath,s+1} := \alpha_{1-\imath,s}$.

Go to next stage (with next real stage).

Subase II (ii): Otherwise. Then (i) does not happen and (ii) happens.

Choose the largest n such that cycle(n) of $\mathcal{G}_{e,0}$ founds a target, and let $\tau^n_{e,\imath}$ be the target.

For $z < k^n_{e,\imath}$, if $f_{e,\imath}$ is undefined then define $f_{e,\imath}(z) := A_{s+1}(z)$ and declare that $\tau^n_{e,\imath}$ is waiting for an A-permission below $k^n_{e,\imath}$.

Start cycle($n+1$) of $\mathcal{G}_{e,\imath}$ as follows.

(a) Choose $k^{n+1}_{e,\imath}$ as a big number. In particular, $k^{n+1}_{e,\imath} > a_{s+2}$.

(b) We extend Γ_s to a function $\Gamma_{s+1} : [0, k^{n+1}_{e,\imath}] \to \{0,1\}$, and extend γ_s to a function $\gamma_{s+1} : [0, k^{n+1}_{e,\imath}] \to \omega$, in the same way as in Subcase II (ii).

(c) We define $\sigma^{n+1}_{e,\imath}[0]$, the pretarget of cycle(n+1), and $\alpha_{\imath,s+1}$ in the same way as Subcase IB (ii).

We define $\alpha_{1-\imath,s+1} := \alpha_{1-\imath,s}$.

For each $i \in \{0,1\}$, we define $G_\imath$ as $\lim_s \alpha_\imath, s$.

This completes the construction of G_0 and G_1.

Remark. We have defined $k^n_{e,\imath}$ as a large enough number. In particular, $k^0_{e,\imath} > a_1$ (ostensible stage 0), $k^0_{e,\imath} > a_{s+2}$ (ostensible stage $s+1$, II (i)) and $k^{n+1}_{e,\imath} > a_{s+2}$ (ostensible stage $s+1$, II (ii)). Hence, in ostensible stage $s+1$ Case IB, there is at least one $\langle e,i\rangle < s+1$ such that $\mathcal{G}_{e,\imath}$ requires attention. In addition, when ostensible stage $s+1$ II begins for odd s, there are at least $(s-1)/2$ many $\langle e,i\rangle < s+1$ such that $\mathcal{G}_{e,\imath}$ is not in a cycle, hence there is at least one $\langle e,i\rangle < s+1$ such that $\mathcal{G}_{e,\imath}$ requires attention.

4 Basic property of stage regression

We overview basic property of our construction with focus on regression of stages.

By induction on t, we get the following: For each natural number t, letting k be the number of real stages whose ostensible stage $= t$, k is positive and finite.

Definition 4.1. 1. For each natural number s, let $\varrho_0(s)$ denote the least real stage whose ostensible stage is s.

2. For each natural number s, let $\varrho_1(s)$ denote the largest real stage r such that the ostensible stage of real stage $r+1$ is $\leq s+1$.

3. A real stage r is *nice* if $r = \varrho_1(s)$ for some s.

Example 4.1. The following are some basic properties of ϱ_0 and ϱ_1.

- $t < s$ if and only if $\varrho_0(t) < \varrho_0(s)$.
- For each natural number s, $\varrho_1(s)$ equals the largest real stage r such that the ostensible stage of real stage $r+1$ is $s+1$.
- If $r > \varrho_1(s)$ then the ostensible stage at real stage $r+1$ is larger than $s+1$. Therefore, $t < s$ if and only if $\varrho_1(t) < \varrho_1(s)$.
- If there is only one real stage where ostensible stage is $t+1$ then $\varrho_0(t) = \varrho_1(t)$.
- Suppose that Case IC happens with local time at origin $(r+1, s+1)$ and local time at destination (r_0+1, t_0+1). In addition, suppose that $r+1 = \varrho_1(t+1)$ for some t. In this case, $r+1 = \varrho_1(t_0+1)$. Note that $r+1 \neq \varrho_1(s+1)$.

Proposition 4.1. *Suppose that r_0^*, r_1 and r_1^* are natural numbers such that $r_0^* < r_1 < r_1^*$. Let t_0^*+1, t_1+1 and t_1^*+1 denote the ostensible stages of of real stages r_0^*+1, r_1+1 and r_1^*+1, respectively. Suppose that Case IC happens with local time at origin (r_1^*+1, t_1^*+1) and local time at destination (r_0^*+1, t_0^*+1). Then the following hold.*

1. $a_{t_0^*+1} \in A_{t_1+1}$
2. $a_{t_0^*+1} \notin A$
3. $t_0^* < t_1 < t_1^*$

Proof. (1) Let $a := a_{t_0^*+1}$. By the above assumption on Case IC, we have $a \in A_{t_0^*+1} - A_{t_0^*}$ and $a \in A_{t_1^*} - A_{t_1^*+1}$.

Since $r_1 < r_1^*$, a does not have "This will drop out of A" mark at real stage r_1+1.

Therefore, if $a \notin A_{t_1+1}$ then a should be enumerated into A after real stage r_1+1 before real stage r_1^*+1. However, then the destination r_0^*+1 is larger than r_1+1, which contradicts to our assumption of $r_0^* < r_1$.

Hence we have $a \in A_{t_1+1}$.

(2) Since the number of mind changes is at most 2, for all t such that $t < t_0^*$ or $t \geq t_1^*$, we have $a \notin A_{t+1}$. Therefore, we have $a \notin A$.

(3) However, by (1), we have $a \in A_{t_1+1}$. Hence $t_0^* \leq t_1 < t_1^*$.

If $t_0^* = t_1$ then the destination r_0^*+1 is at least r_1+1, a contradiction.

Therefore, we have $t_0^* < t_1 < t_1^*$. □

Definition 4.2. We define a set H of natural numbers as follows, where we let $t_i + 1$ denote ostensible stage of real stage $r_i + 1$ (for $i = 1, 2$).

$$H := \{r_1 + 1 : \text{For every } x \in A_{t_1+1} \text{ such that } x \text{ does not have "This will drop out of } A\text{" mark at the end of real stage } r_1 + 1, \forall r_2 > r_1\ x \in A_{t_2+1}.\}$$

Proposition 4.2. *1. H is a Π^0_1 set.*

2. If r_1+1 is a real stage, t_1+1 is its ostensible stage and $r_1+1 = \varrho_1(t_1+1)$, then $r_1 + 1 \in H$.

3. H is an infinite set (provided that A is an infinite set).

4. $H \leq_T A$

Proof. (1) is obvious.

(2) Assume for a contradiction that $r_1 + 1 = \varrho_1(t_1 + 1)$ and ostensible stage of real stage $r_1 + 1 = t_1 + 1$, and $r_1 + 1 \notin H$.

Then, there exists $x \in A_{t_1+1}$ such that x does not have "This will drop out of A" mark at real stage $r_1 + 1$, and for some $s + 1 > t_1 + 1$, $x \notin A_{s+1}$.

Fix such x. Choose the least real stage $r_2+1 > r_1+1$ such that, letting t_2+1 be ostensible stage of real stage $r_2 + 1$, we have $t_2 + 1 > t_1 + 1$ and $x \notin A_{t_2+1}$.

It is easy to see that x does not have the mark at real stage $r_2 + 1$. Thus, Case IC happens with local time at origin $(r_2 + 1, t_2 + 1)$ and local time at destination $(r_{0,2} + 1, t_{0,2} + 1)$ for some $r_{0,2}$ and $t_{0,2}$.

Since $x \in A_{t_1+1}$, for some real stage $r_3 + 1 \leq r_1 + 1$, letting its ostensible be $t_3 + 1$, we have $t_3 + 1 \leq t_2 + 1$ and we have $x = a_{t_3+1}$. Choose the least such r_3.

Recall that $r_{0,2} + 1$ is the largest $r + 1 < r_2 + 1$ such that x is enumerated into A. Therefore, we have $r_{0,2} = r_3$ and $t_{0,2} = t_3$. Thus, we have $t_{0,2} \leq t_1$, in other words, $t_{0,2} + 2 \leq t_1 + 2$.

By the definition of ϱ_1, it holds that $\varrho_1(t_1 + 1) \geq r_2 + 1$.

On the other hand, $r_2 + 1 > r_1 + 1$, and by our assumption, $r_1 + 1 = \varrho_1(t_{0,1} + 1)$. Hence, $\varrho_1(t_1 + 1) > \varrho_1(t_1 + 1)$, a contradiction.

(3) If $x \in A$ then x is in $A_{s+1} - A_s$ for a unique s, and for all $t > s$, $x \in A_t$. Thus, if $r + 1$ is the largest real stage whose ostensible stage is $s + 1$, then $r + 1 = \varrho_1(s + 1)$. Under our assumption that A is an infinite set, there are infinitely many such x and thus, such r. By (2), all the such r belong to H.

(4) It is easy to see that H equals the following set, where we let $t_1 + 1$ denote ostensible stage of real stage $r_1 + 1$.

$$\{r_1 + 1 : \text{For every } x \in A_{t_1+1} \text{ such that } x \text{ does not have "This will drop out of } A\text{" mark at real stage } r_1 + 1, x \in A.\}$$

□

Lemma 4.1. *If the assumptions of Proposition 4.1 except for "$r_0^* < r_1$" hold, and in addition, if it holds that $r_1 + 1 \in H$, then we have $r_0^* > r_1$ and $t_0^* > t_1$.*

Proof. Suppose that the assumptions of Propostion 4.1 hold, and let a be $a_{t_0^*+1}$ at the end of real stage $r_0^* + 1$. Then, by Propostion 4.1 and its proof, a does not have "This will drop out of A" mark at real stage $r_1 + 1$, $a \in A_{t_1+1}$ and $a \notin A$. Then by the definition of H, we have $r_1 + 1 \notin H$.

In summary, under the assumptions of Proposition 4.1 except for "$r_0^* < r_1$", we have the following.

$$r_0^* < r_1 \;\rightarrow\; r_1 + 1 \notin H$$

By taking the contrapositive, $r_1 + 1 \in H$ implies $r_0^* \geq r_1$.

Now, assume the assumptions of Proposition 4.1 except for "$r_0^* < r_1$" and assume that $r_1 + 1 \in H$.

By the definition of H, $r_1 + 1$ is not a real stage of a destination of Case IC, while $r_0^* + 1$ is a real stage of a destination. Therefore, $r_0^* \neq r_1$. Hence we have $r_0^* > r_1$.

According to the above discussion, after real stage $r_1 + 1$, there is no chance that an ostensible stage has a value less than $t_1 + 2$. Thus $t_0^* > t_1$. □

Proposition 4.3. *In the following, r_i, r_i', t_i and t_i' are natural numbers ($i \in \{0, 1\}$). Suppose that the following hold.*

- *r_1 and r_1' are even numbers.*
- *$r_1 < r_1'$*
- *Case IC happens with local time at origin $(r_1 + 1, t_1 + 1)$ and local time at destination $(r_0 + 1, t_0 + 1)$.*
- *Case IC happens with local time at origin $(r_1' + 1, t_1' + 1)$ and local time at destination $(r_0' + 1, t_0' + 1)$.*

Then $r_0' + 1 < r_0 + 1$ or $r_0' + 1 > r_1 + 1$.

Proof. Both of r_1 and r_1' are even numbers and we have $r_1 + 2 \leq r_1' + 1$, thus it holds that $r_1 + 2 < r_1' + 1$.

Now, assume for a contradiction that $r_0 + 1 \leq r_0' + 1 \leq r_1 + 1$. Then the following holds.

$$r_0 + 1 \leq r_0' + 1 < r_1 + 2 < r_1' + 1 \tag{1}$$

Let a denote $a_{t_0'+1}$. We are going to show $a \notin A_{t_0+2}$. We apply Propositon 4.1 to this setting, where we substitute $(r_0 + 1, t_0 + 1)$, $(r_0' + 1, t_0' + 1)$ and $(r_1 + 1, t_1 + 1)$ for $(r_0 + 1, t_0 + 1)$, $(r_1 + 1, t_1 + 1)$ and $(r_2 + 1, t_2 + 1)$ of Propositon 4.1, respectively. By Propositon 4.1, we have $t_0 < t_0'$. Since both of t_0 and t_0' are even, we have $t_0 + 2 \leq t_0'$, thus we have $t_0 + 2 < t_0' + 1$. On the other hand, since the number of mind changes is at most 2, $t_0' + 1$ is the least t such that $a \in A_t$. Hence, $a \notin A_{t_0+2}$.

At real stage r_1+2, the ostensible stage is t_0+2, thus by the above pragraph, a is not enumerated into A at real stage r_1+2. In addition, at real stage r_1+2, a does not have "This will drop out of A" mark yet [proof: among real stages larger than r'_0+1, r'_1+1 is the least one at which a has the mark. However, by (1), $r_1+2<r'_1+1$.]. However, a causes Case IC at real stage r'_1+1.

Therefore, there exists r such that $r_1+2<r+1<r'_1+1$ and a is enumerated into A at real stage $r+1$. By (1), we have $r'_0+1<r+1<r'_1+1$. This contradicts to our assumption that r'_0+1 is the local time at destination of Case IC at r'_1+1. □

5 Verification

5.1 Genericity

Proposition 5.1. *Suppose that r_1, t_1, n, e and r_0 are natural numbers with the following propeties.*

- *r_1 is even, and at real stage r_1, ostensible stage is t_1.*
- *r_0+1 is the largest real stage such that $r_0+1\le r_1$, and pretarget $\sigma^n_{e,i}$ (for these particular n and e) is defined at real stage r_0+1.*
- *At real stage r_1+1,* cycle(n) *of requirement $\mathcal{G}_{e,i}$ receives attention.*

Then, $\gamma_{t_1}(k^n_{e,i})$ (at real stage r_1) equals $\gamma_{t_0+1}(k^n_{e,i})$ (at real stage r_0+1).

Proof. Suppose that r_2 is a natural number such that $r_0+1<r_2+1\le r_1$. Let t_2+1 be ostensible stage of real stage r_2+1. Suppose that for every r such that $r_0+1<r+1<r_2+1$, at real stage $r+1$,

By the third item of the assumptions, cycle(n) of requirement $\mathcal{G}_{e,i}$ is not canceled at real stage r_2+1. Therefore, if Case IB happens at real stage r_2+1, we have $a_{t_2+1}\ge k^n_{e,i}$. Therefore, the value of $\gamma(k^n_{e,i})$ does not chage at real stage r_2+1.

Next, we observe the case where Case IC happens with local time at origin (r_2+1, t_2+1) and local time at destination (r'_0+1, t'_0+1), for some (r'_0+1, t'_0+1). Again, by the third item of the assumptions, $\sigma^n_{e,i}$ is not erased at real stage r_2+1. Thus, we have $r'_0>r_0$. Therefore, $\gamma_{t_1}(k^n_{e,i})$ (at real stage r_2+1) equals $\gamma_{t_0+1}(k^n_{e,i})$ (at real stage r_0+1). □

Lemma 5.1. *For each e, the following hold.*

1. *There are only finitely many nice real stages at which $\mathcal{G}_{e,i}$ is initialized.*
2. *There are only finitely many nice real stages at which $\mathcal{G}_{e,i}$ runs new cycles.*
3. *There are only finitely many nice real stages at which $\mathcal{G}_{e,i}$ requires attention.*
4. *$\mathcal{G}_{e,i}$ is satisfied. Thus, G_i is 1-generic.*

Proof. Suppose that e is given and that (1)–(4) hold for all $e' < e$.

There is a natural number s such that for $t > s$, at real stage $\varrho_1(t)$, no requirement $e' < e$ requires attention.

By concentrating on nice real stages, we can prove (1)–(3) in similar ways to Wu's original proofs of Lemma 4.1 (1)–(3) in [11].

Now, assume the following.

1. $r_1 + 1 > \varrho_1(s+1)$
2. $t_1 + 1$ is ostensible stage of real stage $r_1 + 1$.
3. $r_1 + 1$ is the last real stage at which $\mathcal{G}_{e,\imath}$ requires attention.
4. $r_1 + 1$ is a nice stage. In other words, $r_1 + 1 = \varrho_1(u+1)$ for some u.
5. At real stage r_1+1, Case IB (i) happens for cycle(n), and $\alpha_{\imath,t_1+1}$ is defined as $\tau^n_{e,\imath}[t_1]$.

On the one hand, $r_1+1 = \varrho_1(u+1)$. On the other hand, at real stage r_1+2, ostensible stage is t_1+2. Hence, we have $u = t_1$. Thus, $r_1+1 = \varrho_1(t_1+1)$. By Proposition 4.2, we have $r_1 + 1 \in H$.

Hence, by Lemma 4.1, if Case IC happens, say with local time at origin (r', t') and local time at destination (r'_0, t'_0), we have $r_1 \leq r'_0$.

In addition, by Proposition 4.3, the above r'_0 avoids each interval that previous Case IC made.

Thus, Case IC after real stage $r_1 + 1$ does not injure the value of $\alpha_\imath(x)$.

It is enough to investigate Case IB.

Claim 1. Suppose that at a real stage $r_2+1 > r_1+1$, letting t_2 be ostensible stage of real stage $r_2 + 1$, Case IB happens. Then, it holds that $\alpha_{\imath,t_2+1}$ (of real stage $r_2 + 1$) extends $\alpha_{\imath,t_1+1}$ (of real stage $r_1 + 1$).

Proof of Claim 1: Suppose that x is in the domain of $\alpha_{\imath,t+1}$. By Proposition 5.1, we have two cases: (i) x is in the domain of the pretarget $\sigma^n_{e,\imath}[t]$, that is, $x < \gamma_t(k^n_{e,\imath})$; (ii) x is in the domain of the target minus pretarget (that is, $\tau^n_{e,\imath}[t] - \sigma^n_{e,\imath}[t]$) and $x \geq \gamma_t(k^n_{e,\imath})$.

Now, suppose $r_2 + 1 > r_1 + 1$. We are going to show that the value of $\alpha_0(x)$ is not injured at real stage $r_2 + 1$.

By our choice of t, we have $a_{t_2+1} \geq k^n_{e,\imath}$, therefore the following hold.

$$\gamma_{t_2}(a_{t_2+1}) \geq \gamma_{t_2}(k^n_{e,\imath}) \geq \gamma_{t_1}(k^n_{e,\imath}) \tag{2}$$

In the case of (i), $\gamma_{t_2}(a_{t_2+1}) > x$, thus the value of $\alpha_\imath(x)$ is not injured at real stage $r_2 + 1$.

In the case of (ii), by our rule on "Do not enter range of gamma" mark, $x \neq \gamma_u(a_{u+1})$. Thus the value of $\alpha_0(x)$ is not injured because of $\alpha_\imath(\gamma_u(a_{u+1}))$ is set as 1.

In addition, in the case of (ii), by (2), $\gamma_{t_2}(a_{t_2+1})$ is not in the domain of the pretarget $\sigma^n_{e,\imath}[t_1]$ (of real stage r_1). Again, by our rule on "Do not enter range of gamma" mark, $\gamma_{t_2}(a_{t_2+1})$ is not in the domain of the target $\tau^n_{e,\imath}[t_1]$ minus pretarget $\sigma^n_{e,\imath}[t_1]$. In summary, $\gamma_{t_2}(a_{t_2+1})$ is larger than the largest value in the domain of the target $\tau^n_{e,\imath}[t_1]$.

However, the least value of the domain of target $\tau^-_{-,\imath}$ (of real stage r_2) minus pretarget $\sigma^-_{-,\imath}$ (of real stage r_2) is $\gamma_{t_2}(a_{t_2+1})$. Therefore, the value of $\alpha_\imath(x)$ is not injured because of x is in the domain of target $\tau^-_{-,\imath}$ minus pretarget $\sigma^-_{-,\imath}$. Q.E.D. (Claim 1)

By means of Claim 1, we can prove (4) in a similar way to Wu's original proof of Lemma 4.1 (4) in [11]. □

5.2 Reduction of G to A

Lemma 5.2. $G_0 \oplus G_1 \leq_T A$.

Proof. By Proposition 4.2, we have $H \leq_T A$. Thus, it is enough to show that $G_\imath \leq_T A \oplus H$ for each $i = 0, 1$. Let $i \in \{0, 1\}$.

Throughout this proof, unless otherwise specified, γ, Γ and α with sufix $t_1 + 1$ denote those at real stage $r_1 + 1$.

Given x, by means of oracle $A \oplus H$, we find (r_1, t_1) of the following properties.

- r_1 is an even number and $r_1 + 1 \in H$
- Ostensible stage of real stage $r_1 + 1$ is $t_1 + 1$.
- $A \upharpoonright (x + 1) = A_{t_1+1} \upharpoonright (x + 1)$.
- For each $i \in \{0, 1\}$, x is in the domain of $\alpha_{\imath,t_1+1}$.

Our goal is to show that $\alpha_{\imath,t_1+1}(x) = G_\imath(x)$.

For each natural number $r \geq r_1$, we define assertion $\Psi(r + 1)$ as follows.

$\Psi(r+1)$: "Let $t+1$ denote the ostensible stage of real stage $r+1$. $\gamma_{t+1} \upharpoonright (x+1)$ ($\Gamma_{t+1} \upharpoonright (x + 1)$ and $\alpha_{\imath,t+1}(x)$, respectively) at the end of real stage $r + 1$ equals $\gamma_{t_1+1} \upharpoonright (x + 1)$ ($\Gamma_{t_1+1} \upharpoonright (x + 1)$ and $\alpha_{\imath,t_1+1}(x)$, respectively)."

Letting r_1 be the first term, we define a sequence $\{r_\imath\}_{\imath=1}^{\infty}$ as follows. We define $r_{\imath+1}$ accroding to the following two cases.

The first case is when for some r_1^*, r_0^*, t_1^* and t_0^*, Case IC happens with local time at origin $(r_1^* + 1, t_1^* + 1)$ and local time at destination $(r_0^* + 1, t_0^* + 1)$ and we have $r_\imath = r_0^*$. In this case we define $r_{\imath+1}$ as $r_1^* + 1$ (thus $r_{\imath+1} + 1 = r_1^* + 2$).

Otherwise, we define $r_{\imath+1}$ as $r_\imath + 1$ (thus $r_{i+1} + 1 = r_\imath + 2$).

It is enough to show the following claim.

Claim 1. For every $i \geq 1$, $\Psi(r_\imath + 1)$ holds.

We show the claim by induction. The base case $\Psi(r_1 + 1)$ is obvious.

Throughout the discussion on induction step, we always keep in our mind that Lemma 4.1 holds, and that Cases IB or IC can happen only at real stages $r+1$ such that r is even.

Suppose that $i > 1$ and that $\Psi(r_j + 1)$ holds for all j such that $1 \le j \le i$. Let $r = r_{i+1}$. Let $t+1$ denote the ostensible stage at real stage $r+1$.

We investigate the following three cases.

Case 1: $r > r_i + 1$.

Case 2: $r = r_i + 1$, Case IB holds at real stage $r+1$, and $a := a_{t+1} \le x$.

Case 3: None of the following two cases happens.

We look at the easiest case first.

Case 3: By our definition of $\{r_i\}_{i=1}^{\infty}$, Case IC does not hold at real stage $r+1$. Thus, either r_i is odd, Case IA happens at real stage $r+1$ or Case IB happens at real stage $r+1$ but $a := a_{t+1} > x$. Hence, $\gamma \restriction (x+1), \Gamma \restriction (x+1), \alpha_0(x)$ and $\alpha_1(x)$ at real stage $r+1$ are the same as those at real stage r_i. Therefore $\Psi(r+1)$ holds.

Then we look at the other two cases.

Case 1: Since both r_1 and r_i are even and $r_1 < r_i$, we have $r_1 + 2 \le r_i$. Thus we have $i \ge 3$ and $r_i - 1 = r_{i-1}$. By induction hypothesis, $\Psi(r_{i-1}+1)$, in other words, $\Psi(r_i)$ holds.

By our definition of $\{r_i\}_{i=1}^{\infty}$ and the definition of Case IC, $\gamma \restriction (x+1), \Gamma \restriction (x+1), \alpha_0(x)$ and $\alpha_1(x)$ at the end of real stage $r+1$ are the same as those at real stage r_i. Hence, by the previous paragraph, $\Psi(r+1)$ holds.

Case 2: In this case we have $a \notin A_t$, and a does not have the "This will drop out of A" mark at real stage $r+1$. Thus a does not have the mark at real stage $r_1 + 1$. By the definition of H, we have $a \notin A_{t_1+1}$. By our assumption, $a \notin A$.

However, we have $a \in A_{t+1}$. Therefore at some real stage, say r_1^*+1 $(> r+1)$, Case IC happens and the real stage of its destination, say r_0^*+1 is $r+1$ or smaller.

By our assumption of $r = r_i + 1$, we have $r_0^* + 1 \ne r+1$. Hence $r_0^*+1 < r+1$. This contradicts to our definition of $\{r_i\}_{i=1}^{\infty}$ (such r should be skipped).

Thus we have shown Claim 1. Hence, the lemma holds. □

5.3 Reduction of A to G

Proposition 5.2. *Suppose a natural number x is given. Then, for all but finitely many real stages $r+1$, letting $t+1$ be the ostensible stage, we have $\Gamma_{t+1}(x) \downarrow = A_{t+1}(x) = A(x)$.*

Proof. Throughout this proof, $t+1$ denotes the ostensible stage of real stage $r+1$.

By our construction, at each real stage $r+1$ and for each y in the domain of Γ_{t+1}, we have either $\Gamma_{t+1}(y) = A_{t+1}(y)$ or $\Gamma_{t+1}(y) = A(y)$.

On the other hand, for all but finitely many real stages $r+1$, we have $a_{t+1} > x$, hence the following hold.

- $A_{t+1}(x) = A(x)$

- $\Gamma_{t+1}(x) \simeq \Gamma_t(x)$ (The left-hand side is defined if and only if the right-hand side is defined, and in this case they have the same value.)

In particular, among such real stages, there exists $r'+1$, letting $t'+1$ be its ostensible stage, such that Case IB happens at real stage $r'+1$. Since $a_{t'+1} > x$, we have $x < k^n_{e,0}$, where e and n are indices of the requirement and the cycle payed attention, and therefore x is in the domain of $\Gamma_{t'+1}$.

By the previous three paragrahps, for each $r \geq r'$, we have $\Gamma_{t+1}(x) \downarrow = A_{t+1}(x) = A(x)$ at real stages $r+1$. □

Lemma 5.3. $A \leq_T G_0 \oplus G_1 \oplus H$.

Proof. Given x, by means of oracle $G_0 \oplus G_1 \oplus H$, we find r_1 and t_1 satisfying the following (1)–(5).

1. At real stage $r_1 + 1$, the ostensible stage is $t_1 + 1$.
2. For all $y \leq x$, $\gamma_{t_1}(y) \downarrow$ at real stage r_1.
3. For all $y \leq x$, $\gamma_{t_1+1}(y) \downarrow$ at real stage $r_1 + 1$, and it equals $\gamma_{t_1}(y)$ of real stage r_1.
4. For each $i \in \{0,1\}$, $\forall w \leq \gamma_{t_1+1}(x)$, $\alpha_{i,t_1+1}(w) \downarrow = G_i(w)$.
5. $a_{t_1+1} > x$.

We output $\Gamma_{t_1+1}(x)$ of real stage $r_1 + 1$.

We are going to show that our output equals $\lim_s \Gamma_s(x)$. By Proposition 5.2, $\lim_s \Gamma_s(x) = \lim_s A_s(x) = A(x)$.

Thus, A is Turing reducible to $G_0 \oplus G_1$.

To complete the proof, it is enough to show that for each $t_2 > t_1$, at real stage $\varrho_1(t_2+1)$, $\Gamma_{t_2+1}(x) \downarrow = \Gamma_{t_1+1}(x)$.

Let assertion $\Psi(r+1)$ and sequence $\{r_i\}_{i=1}^{\infty}$ be those defined in the proof of Lemma 5.2. It is enough to show the following claim.

Claim 1. For every $i \geq 1$, $\Psi(r_i + 1)$ holds.

This is the same claim as before, however we are going to prove it under different assumptions. The base case is the same as before. At induction step, we investigate the same three cases as in the proof of Lemma 5.2. Then the same proofs for Cases 3 and 1 work.

Case 2: $r = r_i + 1$, Case IB holds at real stage $r+1$, and $a := a_{t+1} \leq x$.

As in the proof of Lemma 5.2, we have $a \notin A_{t_1+1}$. Thus at the end of real stage $r_1 + 1$, we have $\alpha_{1,t_1+1}(\gamma_{t_1}(a)) = 0$.

On the other hand, we have $\alpha_{1,t+1}(\gamma_t(a)) = 1$ at the end of real stage $r+1$. By induction hypothesis, we have $\gamma_{t_1}(a) = \gamma_t(a)$. Hence it holds that $\alpha_{1,t+1}(\gamma_{t_1}(a)) \neq \alpha_{1,t_1+1}(\gamma_{t_1}(a)) = G_1(\gamma_{t_1}(a))$. The last equality is by our assumption of the lemma.

Hence the wrong value of $\alpha_{1,t+1}(\gamma_{t_1}(a))$ should be canceled at a later stage. The only means to do this is Case IC. Therefore at some real stage, say $r_1^* + 1$ $(> r+1)$, Case IC happens and the real stage of its destination, say $r_0^* + 1$ is $r+1$ or smaller. The remainder of the proof is the same as that in the proof of Lemma 5.2.

Thus we have shown Claim 1. Hence, the lemma holds. □

5.4 1-Generic Splitting

Theorem 5.4. *(Main theorem) Suppose that A is a non-computable d.c.e. set. Then, there exist 1-generic sets G_0, G_1, G_2 and G_3 such that A is Turing equivalent to $G_0 \oplus G_1 \oplus G_2 \oplus G_3$.*

Proof. By Lemma 5.1, G_0 and G_1 are 1-generic sets. By Proposition 4.2 and Lemmas 5.2 and 5.3, $A \equiv_T G_0 \oplus G_1 \oplus H$.

In the case where H is computable, we have $A \equiv_T G_0 \oplus G_1$.

Otherwise, by Proposition 4.2, H is a Π^0_1 set that is non-computable. Therefore, Turing degree of H is a nonzero c.e. degree. Hence, by Wu's result [11], there are 1-generic sets G_2 and G_3 such that $H \equiv_T G_2 \oplus G_3$. Hence, $A \equiv_T G_0 \oplus G_1 \oplus G_2 \oplus G_3$. □

Corollary 5.5. *No two d.c.e. degrees bound the same class of 1-generic degrees.*

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