数理解析研究所講究録
第2083巻 2018年 76-87

A Higher-arity Sequent Calculus for Modal Linear Logic

Yosuke Fukuda

Graduate School of Informatics, Kyoto University

Akira Yoshimizu

French Institute for Research in Computer Science and Automation (INRIA)
and
Department of Computer Science and Engineering, University of Bologna

Abstract

We propose a cut-free sequent calculus for multiplicative exponential linear logic with S4 necessity and possibility modalities. The calculus has the so-called "higher-arity" judgment to formalize the exponentials and S4 modalities neatly. All the properties follow from the model of proofs for a higher-arity sequent calculus of classical modal logic S4, which is also proposed and studied here.

1 Introduction

Each modal logic and linear logic is undoubtedly one of the most important non-classical logics, and from the late 90s, their computational counterparts have been used as a basis of the theory of programming languages via the Curry–Howard correspondence.

Several studies [9][2][16][7] have discovered that modal logic S4 corresponds to a typed λ-calculus that can use "computation" itself as a programmable object. It is also well-known that linear logic allows us to analyze the fine-grained structure of computation. For instance, Mackie [11] proposed a linear logical foundation for programming language implementation, called the Geometry of Interaction Machine (GoIM). His theory, as the name indicates, uses Girard's Geometry of Interaction semantics [6] to model the "dynamics" of program execution with only a few primitive operators, so that the theory gives a simple and compact execution environment.

In these contexts, we aim to integrate the two logics into one, named *modal linear logic*, to characterize a computational model for modal logic S4 in terms of the GoIM. We believe that the model sheds new light on the development of calculi for modal logic. However, the work in the present paper is still preliminary to obtain the computational model of modal linear logic; and thus we propose a proof theory of the logic as the first step toward the goal.

In the rest of the present paper, we first define a sequent calculus for classical modal logic S4, called $\mathbf{HLK_{S4}}$, and prove the cut-elimination theorem via the so-called G3-style sequent calculus which is known as a "structural-rule-free" system. Secondly, we propose a sequent calculus for modal linear logic, called $\mathbf{HMELL_{S4}}$, and discuss that all the properties can be shown by simply modifying the model of proofs for those of $\mathbf{HLK_{S4}}$. Finally, we mention logics which relate to modal logic and linear logic and their sequent calculus, and then we conclude our work and suggest some further directions.

2 Higher-arity sequent calculus for modal logic S4

We propose a sequent calculus for classical modal logic S4, named $\mathbf{HLK_{S4}}$. While many modal sequent calculi have been developed so far (cf., a survey by Poggiolesi [17]), our calculus has the following characteristics: (1) it has the so-called "higher-arity" judgment to formalize S4 modalities, independently of the propositional part; (2) it is defined as a Gentzen–Schütte style one-sided calculus (cf., Troelstra and Schwichtenberg's text [18]), to use the calculus as a basis for modal linear logic in a later section.

In what follows, we define the syntax and the inference rules of $\mathbf{HLK_{S4}}$ and prove the cut-elimination theorem by using a "structural-rule-free" sequent calculus.

Definition 1 (Formula). The set of *propositional variables*, written V_p, is assumed to be given. The set of *formulae* is defined by the following grammar:

$$A, B ::= p \mid \neg p \mid A \wedge B \mid A \vee B \mid \Box A \mid \Diamond A$$

where p is a meta-variable which ranges over V_p. We say that p and $\neg p$ are *literals*. The *negation* of a formula A, written $\neg A$, is defined by de Morgan duality as follows:

$$\begin{aligned} \neg(p) &\stackrel{def}{=} \neg p & \neg(\neg p) &\stackrel{def}{=} p \\ \neg(A \wedge B) &\stackrel{def}{=} (\neg A) \vee (\neg B) & \neg(A \vee B) &\stackrel{def}{=} (\neg A) \wedge (\neg B) \\ \neg(\Box A) &\stackrel{def}{=} \Diamond(\neg A) & \neg(\Diamond A) &\stackrel{def}{=} \Box(\neg A) \end{aligned}$$

The *size* of a formula A, written $|A|$, is defined to be the number of logical connectives occurring in A. For instance, $|\neg p \vee \Box p|$ is 3 and $|\neg(\neg p)|$ is 0.

Definition 2 (Judgment). A *context* Γ is defined to be a multi-set of formulae, also written as $A_0, \ldots, A_{n-1}$ as usual. Then, a *judgment* $\vdash \Delta; \Gamma$ is a pair that consists of two contexts Δ and Γ. We often say that Δ is the *modal part* and Γ is the *normal part* of a judgment $\vdash \Delta; \Gamma$.

Remark 1. The intuition of formulae is as usual. They represent propositional variable, its negation, conjunction, disjunction, necessity, and possibility, respectively. The intuition of a judgment $\vdash \Delta; \Gamma$ is described by the following formula:

$$(\bigvee \Diamond \Delta) \vee (\bigvee \Gamma)$$

where the notation $\Diamond \Sigma$ denotes the context $\Diamond A_0, \cdots, \Diamond A_{n-1}$ for a context $\Sigma \equiv A_0, \cdots, A_{n-1}$; and $\bigvee \Sigma$ denotes the formula $A_0 \vee \cdots \vee A_{n-1}$ for the same context Σ.

Definition 3 (Inference rule). The *inference rules* of $\mathbf{HLK_{S4}}$ are defined as follows:

$\boxed{\text{Axiom rule}}$

$$\frac{}{\vdash \emptyset; \neg A, A}\,\text{Ax} \qquad\qquad \frac{\vdash \Delta; \Gamma, A}{\vdash \Delta, A; \Gamma}\,\text{T}$$

$\boxed{\text{Logical rule}}$

$$\frac{\vdash \Delta; \Gamma, A \qquad \vdash \Delta; \Gamma, B}{\vdash \Delta; \Gamma, A \wedge B}\,\wedge \quad \frac{\vdash \Delta; \Gamma, A_i}{\vdash \Delta; \Gamma, A_1 \vee A_2}\,\vee_i \quad \frac{\vdash \Delta; A}{\vdash \Delta; \Box A}\,\Box \quad \frac{\vdash \Delta, A; \Gamma}{\vdash \Delta; \Gamma, \Diamond A}\,\Diamond$$

$\boxed{\text{Structural rule}}$

$$\frac{\vdash \Delta; \Gamma}{\vdash \Delta; \Gamma, A}\,\text{W} \quad \frac{\vdash \Delta; \Gamma}{\vdash \Delta, A; \Gamma}\,\Diamond\text{W} \quad \frac{\vdash \Delta; \Gamma, A, A}{\vdash \Delta; \Gamma, A}\,\text{C} \quad \frac{\vdash \Delta, A, A; \Gamma}{\vdash \Delta, A; \Gamma}\,\Diamond\text{C}$$

$$\frac{\vdash \Delta; \Gamma, A \qquad \vdash \Delta'; \Gamma', \neg A}{\vdash \Delta, \Delta'; \Gamma, \Gamma'}\,\text{Cut}$$

$$\frac{\vdash \Delta; A \qquad \vdash \Delta', \neg A; \Gamma'}{\vdash \Delta, \Delta'; \Gamma'}\,\Box\text{Cut} \qquad \frac{\vdash \Delta, A; \Gamma \qquad \vdash \Delta'; \neg A}{\vdash \Delta, \Delta'; \Gamma}\,\Diamond\text{Cut}$$

Note that we assume that $i \in \{1, 2\}$ in the rule $\vee_i$.

Definition 4. An *active formula* (of an application of an inference rule) is a formula denoted by a meta-variable in the corresponding inference schema. For instance, the active formulae of $\wedge$ are A, B, and $A \wedge B$; and the active formulae of $\Box$Cut are A and $\neg A$. A *cut formula* is a formula eliminated by an application of the cut rules (i.e., the active formulae A and $\neg A$ in the inference schema of Cut, $\Box$Cut, and $\Diamond$Cut).

Definition 5. The *height* of a derivation is defined to be the tree height when we see that derivation as a rooted tree, that is, the height is defined as the length of the longest path to a leaf node from the root node in the derivation.

Definition 6. If a rule R expresses that $\vdash \Delta; \Gamma$ implies $\vdash \Delta'; \Gamma'$, then R is said to be an *admissible* rule and we depict it as follows as a usual double-lined rule:

$$\frac{\vdash \Delta; \Gamma}{\vdash \Delta'; \Gamma'}\,R$$

Moreover, we say that R is *height-preserving* if it entails that $\vdash_n \Delta;\Gamma$ implies $\vdash_n \Delta';\Gamma'$, where the notation $\vdash_n \Delta;\Gamma$ means that $\vdash \Delta;\Gamma$ is derivable with a height of at most n.

Remark 2. The inference rules for propositional fragment, i.e., Ax, $\wedge$, $\vee$, W, C, and Cut are defined in the same way as the Gentzen–Schütte style one-sided calculus. The others are the rules for S4 and described as follows. The rule T formalizes the so-called axiom T (that is, $\vdash \Box A \supset A$) in the dual setting. The rule $\Box$ is the necessitation rule, and it intuitively formalizes the following: $\Box\Delta \vdash A$ implies $\Box\Delta \vdash \Box A$. Note that we need a restriction in $\Box$ that the normal part must be one formula to reject ill-formed formulae[1]. The rule $\Diamond$ translates the meta-level possibility into the object-level possibility $\Diamond$. The rules $\Diamond$W and $\Diamond$C are for weakening and contraction, respectively. The rules $\Box$Cut and $\Diamond$Cut are the cut rules for modal part, and we need these so that every local cut-elimination step preserves the provability of derivations.

Although $\mathbf{HLK_{S4}}$ is defined in a somewhat unusual way, the system is indeed the propositional fragment of classical S4. First, the following two claims readily hold:

Claim 1 (Soundness w.r.t. Kripke model). $\mathbf{HLK_{S4}}$ *is sound w.r.t. the standard Kripke model for classical S4 (e.g., the S4 model in Kripke's [10]).*

Claim 2 (Soudness w.r.t. Hibert-style). $\mathbf{HLK_{S4}}$ *proves all the axiomata of Hilbert-style version of classical S4 (e.g., the Hilbert-style in Troelstra and Schwichtenberg's [18])*

Then, we can obtain the following since the Hilbert-style enjoys the completeness theorem.

Corollary 1. *The system* $\mathbf{HLK_{S4}}$ *is the propositional fragment of classical S4.*

In the rest of this section, we use another sequent calculus, so-called *G3-style sequent calculus*, to prove the cut-elimination theorem. The G3-style sequent calculus, developed by Kleene [8] and later refined by Dragalin [5], is one style of formalization for sequent calculus. The fundamental feature is that the system has no primitive structural rules but all the rules are defined in a somewhat tricky way to satisfy the "height-preserving admissible" weakening and contraction rules. Thanks to the characteristic, the cut-elimination can be shown directly (i.e., not depending on the mix rule) since there is no difficulty to handle contraction. Thus it gives a simpler proof than the Gentzen-style sequent calculus.

We will define the G3-style version of $\mathbf{HLK_{S4}}$, called $\mathbf{G3\text{-}HLK_{S4}}$, and then explain how we define the rules and how it allows us to prove the cut-elimination theorem.

Definition 7 (G3-style rule). The *inference rules* of $\mathbf{G3\text{-}HLK_{S4}}$ are defined as follows:

$\boxed{\text{Axiom rule}}$

$$\frac{}{\vdash \Delta;\Gamma,\neg p,p}\,\mathrm{Ax} \qquad\qquad \frac{\vdash \Delta,A;\Gamma,A}{\vdash \Delta,A;\Gamma}\,\mathrm{T}$$

[1]For instance, we could prove the judgment $\vdash \Box p \vee \Box(\neg p)$ which is not provable in S4, if there were no restriction.

$\boxed{\text{Logical rule}}$

$$\frac{\vdash \Delta;\Gamma,A \quad \vdash \Delta;\Gamma,B}{\vdash \Delta;\Gamma,A\wedge B}\wedge \quad \frac{\vdash \Delta;\Gamma,A,B}{\vdash \Delta;\Gamma,A\vee B}\vee \quad \frac{\vdash \Delta;A}{\vdash \Delta;\Gamma,\Box A}\Box \quad \frac{\vdash \Delta,A;\Gamma}{\vdash \Delta;\Gamma,\Diamond A}\Diamond$$

Remark 3. As we mentioned, there are no primitive structural rules, and all the rules are defined to derive admissible structural rules. The important rules are Ax, T, $\vee$, and $\Box$. Ax restricts the active formulae to be literals to derive contraction, and the contexts Δ and Γ are for weakening. In T, there are two occurrences of A in the premise. Intuitively, we only need the right A, as the original T of $\mathbf{HLK_{S4}}$, but here we add the left A to make contraction admissible. Similarly, the active formulae A and B in $\vee$ are for contraction, and the context Γ in the conclusion of $\Box$ is for weakening.

Although the G3-style slightly differs from the original Gentzen-style, we can in what follows establish the general axiom rule, the height-preserving weakening and contraction, and the equivalence of provability w.r.t. the original system.

Lemma 1 (General axiom rule). $\vdash \Delta;\Gamma,\neg A,A$ *is derivable in* $\mathbf{G3\text{-}HLK_{S4}}$.

Proof. By induction on the size of the formula A. □

Lemma 2 (Height-preserving inversion). *The followings are height-preserving admissible rules:*

1. *If* $\vdash_n \Delta;\Gamma,A\wedge B$, *then* $\vdash_n \Delta;\Gamma,A$ *and* $\vdash_n \Delta;\Gamma,B$.
2. *If* $\vdash_n \Delta;\Gamma,A\vee B$, *then* $\vdash_n \Delta;\Gamma,A,B$.
3. *If* $\vdash_n \Delta;\Gamma,\Box A$, *then* $\vdash_n \Delta;\Gamma,A$.
4. *If* $\vdash_n \Delta;\Gamma,\Diamond A$, *then* $\vdash_n \Delta,A;\Gamma$.

Proof. By induction on the derivation. □

Lemma 3 (Height-preserving weakening and contraction). *The structural rules W, $\Diamond$W, C, and $\Diamond$C in* $\mathbf{HLK_{S4}}$ *are height-preserving admissible in* $\mathbf{G3\text{-}HLK_{S4}}$.

Proof. By induction on the derivation. □

Theorem 1 (Equivalence). *The provability of* $\mathbf{HLK_{S4}}$ *and* $\mathbf{G3\text{-}HLK_{S4}+Cut}$ *are equal, where* $\mathbf{G3\text{-}HLK_{S4}+Cut}$ *means an extension of* $\mathbf{G3\text{-}HLK_{S4}}$ *with Cut, $\Box$Cut, and $\Diamond$Cut.*

Proof. Both directions are shown by straightforward induction. □

Thanks to these properties, all we have to do next is to prove the admissibility of the cut rules in $\mathbf{G3\text{-}HLK_{S4}}$ to establish the cut-elimination theorem of $\mathbf{HLK_{S4}}$. We show the admissibility together with the following complexity measure.

Definition 8 (Cut-degree). The *cut-degree* of an application of the cut rules is defined as follows:

$$\begin{cases} 2 \times |A| & \text{if } A \text{ is the left cut formula of Cut} \\ 2 \times |A| + 1 & \text{if } A \text{ is the left cut formula of } \Box\text{Cut and } \Diamond\text{Cut} \end{cases}$$

Theorem 2 (Cut-elimination). *Cut, $\Box$Cut, and $\Diamond$Cut are admissible in* **G3-HLK$_{S4}$**.

Proof. Let Π and Π' be the derivations of the premises of a cut. For instance, if we take an application of Cut, Π and Π' are the derivation of $\vdash \Delta; \Gamma, A$ and that of $\vdash \Delta'; \Gamma', \neg A$, respectively. Then, the two admissibilities are shown simultaneously by main induction on the cut-degree, with sub induction the sum of the heights of Π and Π'. It is enough to show that for every cut, one of the following three holds (1) it can be reduced to a cut with a smaller cut-degree; (2) it can be reduced to a cut with a smaller height; (3) it can be eliminated immediately.

We only deal with several cases in what follows.

Case: Π ends with $\Box$ and Π' ends with $\Diamond$, and the cut formulae are both active. In this case, the translation is as follows:

$$\dfrac{\dfrac{\vdots \atop \vdash \Delta; A}{\vdash \Delta; \Gamma, \Box A}\,\Box \qquad \dfrac{\vdots \atop \vdash \Delta', \neg A; \Gamma'}{\vdash \Delta'; \Gamma', \Diamond(\neg A)}\,\Diamond}{\vdash \Delta, \Delta'; \Gamma, \Gamma'}\,\text{Cut} \quad \Longrightarrow \quad \dfrac{\vdots \atop \vdash \Delta; A \qquad \vdots \atop \vdash \Delta', A; \Gamma'}{\vdash \Delta, \Delta'; \Gamma, \Gamma'}\,\text{M.I.H}$$

where M.I.H means the application of the main induction hypothesis (i.e., the induction hypothesis w.r.t. the cut-degree). Here, the definition of cut-degree plays an essential role to use M.I.H.

Case: Π ends with $\vee$ but the cut formula in Π is not active. In this case, the translation is as follows:

$$\dfrac{\dfrac{\vdots \atop \vdash \Delta; \Gamma, B, C, A}{\vdash \Delta; \Gamma, B \vee C, A}\,\vee \qquad \vdots \atop \vdash \Delta'; \Gamma', \neg A}{\vdash \Delta, \Delta'; \Gamma, B \vee C, \Gamma'}\,\text{Cut} \quad \Longrightarrow \quad \dfrac{\dfrac{\vdots \atop \vdash \Delta; \Gamma, B, C, A \qquad \vdots \atop \vdash \Delta'; \Gamma', \neg A}{\vdash \Delta, \Delta'; \Gamma, B, C, \Gamma'}\,\text{S.I.H}}{\vdash \Delta, \Delta'; \Gamma, B \vee C, \Gamma'}$$

where S.I.H means the application of the sub induction hypothesis (i.e., the induction hypothesis w.r.t. the sum of the heights).

Case: Π ends with T and the cut formula in Π is active. The translation is as follows:

$$\dfrac{\dfrac{\vdots \atop \vdash \Delta, A; \Gamma, A}{\vdash \Delta, A; \Gamma}\,\text{T} \qquad \vdots \atop \vdash \Delta'; \neg A}{\vdash \Delta, \Delta'; \Gamma}\,\Diamond\text{Cut} \quad \Longrightarrow \quad \dfrac{\dfrac{\dfrac{\vdots \atop \vdash \Delta, A; \Gamma, A \qquad \vdots \atop \vdash \Delta'; \neg A}{\vdash \Delta, \Delta'; \Gamma, A}\,* \qquad \vdots \atop \vdash \Delta'; \neg A}{\vdash \Delta, \Delta', \Delta'; \Gamma}\,**}{\vdash \Delta, \Delta'; \Gamma}\,\Diamond\text{C}_{\Delta'}$$

where $*$ and $**$ mean the applications of S.I.H and M.I.H, respectively; and $\Diamond C_{\Delta'}$ means the multiple applications of $\Diamond$C to contract Δ'. $\Box$

Theorem 2 tells us that $\mathbf{G3\text{-}HLK_{S4}}$ and $\mathbf{G3\text{-}HLK_{S4} + Cut}$ are equivalent; and hence together with Theorem 1, the following main theorem holds as a corollary.

Corollary 2. *The cut rules Cut, $\Box$Cut, and $\Diamond$Cut are admissible in* $\mathbf{HLK_{S4}}$.

3 Higher-arity sequent calculus for modal linear logic

We consider an extension of MELL with the S4 modalities $\Box$ and $\Diamond$ in this section. We call the logic as *modal linear logic*, and propose its sequent calculus, named $\mathbf{HMELL_{S4}}$. Our $\mathbf{HMELL_{S4}}$ is based on two systems: one is $\mathbf{HLK_{S4}}$ presented above, and the other is Pfenning's LV [15] which can be seen as a higher-arity sequent calculus for linear logic.

Readers may think that the structure of linear logic causes some problems to create a sequent calculus. It is, however, not the case because that the exponentials (!, ?) can be seen as the S4 modalities ($\Box$, $\Diamond$) as Martini and Masini discussed [13]. For instance, let us consider in intuitionistic setting the following two-sided version of exponential rules:

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\, ! \qquad\qquad \frac{\Gamma \vdash A}{\Gamma \vdash ?A}\, ?$$

and if we change the symbols $(!, ?)$ to the symbols $(\Box, \Diamond)$, we have the followings:

$$\frac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\, \Box \qquad\qquad \frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A}\, \Diamond$$

These are, although formalized in two-sided version, what we defined for $(\Box, \Diamond)$ in $\mathbf{HLK_{S4}}$. The structure of exponentials and S4 modalities are almost the same[2] and hence we can apply the technique of $\mathbf{HLK_{S4}}$ to modal linear logic with a simple modification.

In what follows, we define $\mathbf{HMELL_{S4}}$ and its G3-style version $\mathbf{G3\text{-}HMELL_{S4}}$ and then prove the cut-elimination theorem as we did in the previous section.

Definition 9 (Formula). The set of *formulae* is defined by the following grammar:

$$A, B ::= p \mid p^{\perp} \mid A \otimes B \mid A \parr B \mid !A \mid ?A \mid \Box A \mid \Diamond A$$

The *dual* of a formula A, written $A^{\perp}$, is defined as usual for MELL-part; and $(\Box A)^{\perp}$ and $(\Diamond A)^{\perp}$ is defined to be $\Diamond(A^{\perp})$ and $\Box(A^{\perp})$, respectively.

Definition 10 (Judgment). A *judgment* $\vdash \Delta; \Gamma; \Sigma$ is a triple that consists of three contexts Δ, Γ, and Σ. We say that Δ, Γ, and Σ are the *modal part*, the *normal part*, and the *exponential part*, respectively.

[2]The only difference between exponentials and modalities is the applicability of structural rules. We can use weakening and contraction freely for $(\Box, \Diamond)$ in S4, but we can use those only for formulae of form $?A$ in MELL.

Remark 4. The formulae are defined as usual as MELL and S4. Note that, intuitively speaking, formulae without exponentials are assumed to be used exactly once in a derivation. The intuitive meaning of a judgment $\vdash \Delta; \Gamma; \Sigma$ is described by the following formula:

$$(\invamp \Diamond \Delta) \invamp (\invamp \Gamma) \invamp (\invamp ?\Sigma)$$

where the notation $?\Sigma$ means the context $?A_0, \cdots, ?A_{n-1}$ for a context $\Sigma \equiv A_0, \cdots, A_{n-1}$; and $\invamp\Sigma$ means the formula $A_0 \invamp \cdots \invamp A_{n-1}$ for the same context Σ.

Definition 11 (Inference rule). The *inference rules* of $\mathbf{HMELL_{S4}}$ are defined as follows:

$\boxed{\text{Axiom rule}}$

$$\frac{}{\vdash \emptyset; A^{\perp}, A; \emptyset}\,\mathrm{Ax} \qquad \frac{\vdash \Delta; \Gamma, A; \Sigma}{\vdash \Delta; \Gamma; \Sigma, A}\,\mathrm{D} \qquad \frac{\vdash \Delta; \Gamma, A; \Sigma}{\vdash \Delta, A; \Gamma; \Sigma}\,\mathrm{T}$$

$\boxed{\text{Logical rule}}$

$$\frac{\vdash \Delta; \Gamma, A; \Sigma \quad \vdash \Delta'; \Gamma', B; \Sigma'}{\vdash \Delta, \Delta'; \Gamma, \Gamma', A \otimes B; \Sigma, \Sigma'}\,\otimes \qquad \frac{\vdash \Delta; \Gamma, A, B; \Sigma}{\vdash \Delta; \Gamma, A \invamp B; \Sigma}\,\invamp$$

$$\frac{\vdash \emptyset; A; \Sigma}{\vdash \emptyset; !A; \Sigma}\,! \qquad \frac{\vdash \Delta; \Gamma; \Sigma, A}{\vdash \Delta; \Gamma, ?A; \Sigma}\,? \qquad \frac{\vdash \Delta; A; \emptyset}{\vdash \Delta; \Box A; \emptyset}\,\Box \qquad \frac{\vdash \Delta, A; \Gamma; \Sigma}{\vdash \Delta; \Gamma, \Diamond A; \Sigma}\,\Diamond$$

$\boxed{\text{Structural rule}}$

$$\frac{\vdash \Delta; \Gamma; \Sigma}{\vdash \Delta; \Gamma; \Sigma, A}\,?\mathrm{W} \qquad \frac{\vdash \Delta; \Gamma; \Sigma, A, A}{\vdash \Delta; \Gamma; \Sigma, A}\,?\mathrm{C} \qquad \frac{\vdash \Delta; \Gamma, A; \Sigma \quad \vdash \Delta'; \Gamma', A^{\perp}; \Sigma'}{\vdash \Delta, \Delta'; \Gamma, \Gamma'; \Sigma, \Sigma'}\,\mathrm{Cut}$$

$$\frac{\vdash \emptyset; A; \Sigma \quad \vdash \Delta'; \Gamma'; \Sigma', A^{\perp}}{\vdash \Delta'; \Gamma'; \Sigma, \Sigma'}\,!\mathrm{Cut} \qquad \frac{\vdash \Delta; \Gamma; \Sigma, A \quad \vdash \emptyset; A^{\perp}; \Sigma'}{\vdash \Delta; \Gamma; \Sigma, \Sigma'}\,?\mathrm{Cut}$$

$$\frac{\vdash \Delta; A; \emptyset \quad \vdash \Delta', A^{\perp}; \Gamma'; \Sigma'}{\vdash \Delta, \Delta'; \Gamma'; \Sigma'}\,\Box\mathrm{Cut} \qquad \frac{\vdash \Delta, A; \Gamma; \Sigma \quad \vdash \Delta'; A^{\perp}; \emptyset}{\vdash \Delta, \Delta'; \Gamma; \Sigma}\,\Diamond\mathrm{Cut}$$

Remark 5. Almost all of them can be understood in the same way as MELL or $\mathbf{HLK_{S4}}$. We only mention about the rules D, !, and ?. D is the rule for dereliction. ! introduces the exponential formula, and again we need the restriction that the modal part must be empty to reject ill-formed formulae. ? translates the meta-level ? modality to the object-level as we did for $\Diamond$ in $\mathbf{HLK_{S4}}$.

Definition 12 (G3-style rule). The *inference rules* of $\mathbf{G3\text{-}HMELL_{S4}}$ are obtained by discarding all the original structural rules from $\mathbf{HMELL_{S4}}$ and by replacing the original Ax, D, T, $\otimes$, and $\Box$ of $\mathbf{HMELL_{S4}}$ by the following rules:

$$\frac{}{\vdash \emptyset; A^{\perp}, A; \Sigma}\,\mathrm{Ax} \qquad \frac{\vdash \Delta; \Gamma, A; \Sigma, A}{\vdash \Delta; \Gamma; \Sigma, A}\,\mathrm{D}$$

$$\frac{\vdash \Delta; \Gamma, A; \Sigma \quad \vdash \Delta'; \Gamma', B; \Sigma}{\vdash \Delta, \Delta'; \Gamma, \Gamma', A \otimes B; \Sigma}\,\otimes \qquad \frac{\vdash \Delta; A; \emptyset}{\vdash \Delta; \Box A; \Sigma}\,\Box$$

Remark 6. All the G3-style rules are defined in the same way as we did for $\mathbf{G3\text{-}HLK_{S4}}$, but it is enough to add the "G3-style features" only to the exponential part since the weakening and contraction are admitted only for that part in $\mathbf{HMELL_{S4}}$. Therefore, we need the exponential part Σ in Ax for weakening but there is no need that the active formulae must be literals. The rule D is defined in a similar way as we did for T in $\mathbf{G3\text{-}HLK_{S4}}$ but the rule T itself remains the same as $\mathbf{HMELL_{S4}}$. The exponential part of $\otimes$ is defined as "context sharing" form to achieve height-preserving contraction, and we need the exponential part Σ in the rule $\Box$ for height-preserving weakening.

Now, we can prove the cut-elimination theorem. The proof proceeds in the same manner as described for $\mathbf{HLK_{S4}}$ since no difficulty is caused by the addition of the exponentials.

Lemma 4 (Height-preserving weakening and contraction). *The structural rules* $?W$ *and* $?C$ *of* $\mathbf{HMELL_{S4}}$ *are height-preserving admissible in* $\mathbf{G3\text{-}HMELL_{S4}}$.

Proof. By induction on the derivation. $\Box$

Definition 13 (Cut-degree). The *cut-degree* of an application of the cut rules is defined as follows:

$$\begin{cases} 2 \times |A| & \text{if } A \text{ is the left cut formula of Cut} \\ 2 \times |A| + 1 & \text{if } A \text{ is the left cut formula of !Cut, ?Cut, } \Box\text{Cut, and } \Diamond\text{Cut} \end{cases}$$

Theorem 3 (Cut-elimination). *The cut rules in* $\mathbf{G3\text{-}HMELL_{S4}}$ *are all admissible.*

Proof. Similar to the proof of the cut-elimination theorem for $\mathbf{G3\text{-}HLK_{S4}}$. $\Box$

Finally, we can obtain the following desired properties.

Corollary 3 (Equivalence). $\mathbf{HMELL_{S4}}$ *and* $\mathbf{G3\text{-}HMELL_{S4}}$ *are equivalent w.r.t. the provability of derivations.*

Corollary 4. *The cut rules in* $\mathbf{HMELL_{S4}}$ *are all admissible.*

Corollary 5 (Conservative extension). $\mathbf{HMELL_{S4}}$ *is a conservative extension of MELL, i.e., the followings are equivalent:*

- $\vdash (\parr\Gamma) \parr (\parr ?\Sigma)$ *is derivable in MELL*
- $\vdash \emptyset ; \Gamma ; \Sigma$ *is derivable in* $\mathbf{HMELL_{S4}}$

where Γ *and* Σ *have no occurrence of* $\Box A$ *or* $\Diamond A$.

4 Related work

Several featrues of $\mathbf{HLK_{S4}}$, $\mathbf{HMELL_{S4}}$ and their G3-style sequent calculi $\mathbf{G3\text{-}HLK_{S4}}$ and $\mathbf{G3\text{-}HMELL_{S4}}$ exist in some previous work. For $\mathbf{HLK_{S4}}$ and $\mathbf{G3\text{-}HLK_{S4}}$, our work can be seen as a refinement of the one-sided sequent calculus for S4 and its G3-style formalization by Troelstra and Schwichtenberg [18]. The essential difference is that our systems use higher-arity judgments to handle S4 modalities. Although the notion of higher-arity judgments are used in various work[3], our work is new in the sense that we can neatly formulate both exponentials and S4 modalities by using higher-arity judgments.

There are several studies on a logic named modal linear logic [1][3][12]. First two works aim to enrich expressiveness on resource usage. They add modalities to linear logic to formulate "presence of resources" (Archangelsky and Taitslin [1]) and "location and mobility of formula" (Biri and Galmiche [3]). The last one, Martin [12], has studied a mathematical investigation of linear logic with general modalities. Compared to them, our work is slightly different because we aim to analyze modal logic in a linear logical framework, whereas they intend to extend linear logic with modalities.

Linear logic with multi-modalities (or rather, multi-exponentials) has been considered by Danos et al. [4] and Nigam and Miller [14]. Both works deal with the so-called *subexponentials*, which is weaker than the ordinary exponentials, to characterize a proof structure of linear logic. While such exponentials do not relate modalities in modal logic directly, it should be interesting to investigate a similarity between their formalization and ours.

5 Conclusion

We have proposed the cut-free sequent calculus $\mathbf{HMELL_{S4}}$ for modal linear logic. The characteristic of the system is to have the higher-arity judgment to formalize both exponentials (!, ?) and S4 modalities ($\Box$, $\Diamond$). To prove the cut-elimination theorem, we created the G3-style version of $\mathbf{HMELL_{S4}}$ and discussed that the proof follows from what we did for the sequent calculus $\mathbf{HLK_{S4}}$ of classical modal logic S4.

As a further direction, we are working on the geometry of interaction semantics for $\mathbf{HMELL_{S4}}$ as we mentioned in the introduction. It should also be interesting to investigate variants of $\mathbf{HMELL_{S4}}$, e.g., the S5 extension of $\mathbf{HMELL_{S4}}$, the modal extensions of substructural logic, etc. We envisage that these system become a new basis to develop computational models for modal logic.

Acknowledgment

The authors thank Kensuke Kojima for fruitful discussion and Atsushi Igarashi for helpful comments on an earlier draft. The authors are also grateful to Kazushige Terui for pointing out some related work.

[3] A modal natural deduction by Pfenning and Davies [16] and a linear sequent calculus by Pfenning [15] are such examples. It is also known that several studies on modal logics use higher-arity judgments (see a survey by Poggiolesi [17]).

References

[1] D. A. Archangelsky and M. A. Taitslin. Modal Linear Logic. In A. Nerode and M. Taitslin, editors, *Logical Foundations of Computer Science - Tver '92*, pages 1–8, Berlin, Heidelberg, 1992. Springer Berlin Heidelberg.

[2] P. N. Benton, G. M. Bierman, and V. C. V. De Paiva. Computational Types from a Logical Perspective. *Journal of Fuctional Programming*, 8(2):177–193, Mar. 1998.

[3] N. Biri and D. Galmiche. A Modal Linear Logic for Distribution and Mobility - extended abstract -. In *Workshop on Linear Logic - FLOC'02*, Copenhagen, Danemark, 2002.

[4] V. Danos, J.-B. Joinet, and H. Schellinx. The Structure of Exponentials: Uncovering the Dynamics of Linear Logic Proofs. In G. Gottlob, A. Leitsch, and D. Mundici, editors, *Computational Logic and Proof Theory*, pages 159–171, Berlin, Heidelberg, 1993. Springer Berlin Heidelberg.

[5] A. Dragalin. *Mathematical Intuitionism: Introduction to Proof Theory*, volume 67 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, Rhode Island, 1988.

[6] J.-Y. Girard. Geometry of Interaction 1: Interpretation of System F. In R. Ferro, C. Bonotto, S. Valentini, and A. Zanardo, editors, *Logic Colloquium '88*, volume 127 of *Studies in Logic and the Foundations of Mathematics*, pages 221–260. Elsevier, 1989.

[7] D. Kimura and Y. Kakutani. Classical Natural Deduction for S4 Modal Logic. *New Generation Computing*, 29(1):61–86, Jan 2011.

[8] S. C. Kleene, N. de Bruijn, J. de Groot, and A. C. Zaanen. *Introduction to metamathematics*, volume 483. van Nostrand New York, 1952.

[9] S. Kobayashi. Monad as modality. *Theoretical Computer Science*, 175(1):29–74, 1997.

[10] S. A. Kripke. Semantical Analysis of Modal Logic I: Normal Propositional Calculi. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 9(56):67–96, 1963.

[11] I. Mackie. The Geometry of Interaction Machine. In *Proceedings of the 22nd ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, pages 198–208, San Francisco, California, USA, January 1995.

[12] A. J. Martin. Modal and Fixpoint Linear Logic. Master's thesis, University of Ottawa, Canada, 2002.

[13] S. Martini and A. Masini. A Modal View of Linear Logic. *Journal of Symbolic Logic*, 59(3):888–899, September 1994.

[14] V. Nigam and D. Miller. Algorithmic specifications in linear logic with subexponentials. In *Proceedings of the 11th ACM SIGPLAN Conference on Principles and Practice of Declarative Programming*, Coimbra, Portugal, September 2009.

[15] F. Pfenning. *Structural Cut Elimination in Linear Logic*. School of Computer Science, Carnegie Mellon University, 1994.

[16] F. Pfenning and R. Davies. A Judgmental Reconstruction of Modal Logic. *Mathematical Structures in Computer Science*, 11(4):511–540, 2001.

[17] F. Poggiolesi. *Gentzen Calculi for Modal Propositional Logic.* Trends in Logic. Springer Netherlands, 2010.

[18] A. S. Troelstra and H. Schwichtenberg. *Basic Proof Theory.* Cambridge University Press, New York, USA, 1996.

Graduate School of Informatics,
Kyoto University,
Kyoto 606-8502, JAPAN
E-mail address: `yfukuda@fos.kuis.kyoto-u.ac.jp`

京都大学大学院情報学研究科　福田 陽介

Department of Computer Science and Engineering,
University of Bologna,
Mura Anteo Zamboni 7, ITALY
E-mail address: `akira.yoshimizu@unibo.it`

フランス国立情報学自動制御研究所 (INRIA) / ボローニャ大学計算機理工学部　由水 輝