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The Church–Rosser Theorem and Analysis of Reduction Length*

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Abstract

The Church–Rosser theorem in the type-free λ-calculus is well investigated both for β-equality and β-reduction. We provide a new proof of the theorem for β-equality simply with Takahashi's translation. Based on this, we analyze quantitative properties of witnesses of the Church–Rosser theorem by using the notion of parallel reduction. In particular, upper bounds for reduction sequences on the theorem are obtained as the fourth level of the Grzegorczyk hierarchy, i.e., non-elementary recursive functions. Moreover, the proof method developed here can be applied to other reduction systems such as λ-calculus with $\beta\eta$-reduction, Girard's system **F**, Gödel's system **T**, combinatory logic, and Pure Type Systems as well.

1 Introduction

1.1 Background

The Church–Rosser theorem [4] is one of the most fundamental properties on rewriting systems, which guarantees uniqueness of the computational result and consistency of a formal system. For instance, for proof trees and formulae of logic the unique normal forms of the corresponding terms and types in a Pure Type System (PTS) can be chosen as their denotations [26] via the Curry–Howard isomorphism.

The confluence property states that if $M \twoheadrightarrow N_1$ and $M \twoheadrightarrow N_2$ then there exists P such that $N_1 \twoheadrightarrow P$ and $N_2 \twoheadrightarrow P$. Here, we write $\twoheadrightarrow$ for the reflexive and transitive closure of one-step reduction $\rightarrow$. There have been well-known proof techniques of the theorem: tracing the residuals of redexes along a sequence of reductions [4, 1, 13] and working with parallel reduction by the method of Tait and Martin-Löf [5, 1, 13, 23]. Recently, a simple proof of the theorem is also known with Takahashi's translation [24] (the Gross–Knuth reduction strategy [1]), but with no use of parallel reduction [17, 6, 16, 18].

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On the other hand. the Church–Rosser theorem states that if $M \longleftrightarrow N$ then there exists P such that $M \twoheadrightarrow P$ and $N \twoheadrightarrow P$. Here, we write[1] $M \longleftrightarrow N$ iff M is obtained from N by a finite series of reductions ($\twoheadrightarrow$) and reversed reductions ($\twoheadleftarrow$). It is well known that the Church–Rosser theorem follows repeated application of the confluence property, so that each peak can be made one by one into its own valley and finally one gorge.

One of our motivations is to analyze quantitative properties in general of reduction systems. For instance, measures for developments are investigated by Hindley [12] and de Vrijer [22]. Statman [21] proved that deciding the $\beta\eta$-equality of typable λ-terms is not elementary recursive. Schwichtenberg [19] analysed the complexity of normalization in the simply typed λ-calculus, and showed that the number of reduction steps necessary to reach the normal form is bounded by a function at the fourth level of the Grzegorczyk hierarchy ε^4 [11], i.e., a non-elementary recursive function. Later Beckmann [3] determined the exact bounds for the reduction length of a term in the simply typed λ-calculus. Xi [27] showed bounds for the number of reduction steps on the standardization theorem, and its application to normalization. Ketema and Simonsen [14] extensively studied valley sizes of confluence and the Church–Rosser property in term rewriting and λ-calculus as a function of given term sizes and reduction lengths. However, a bound in at least the fifth level of the Grzegorczyk hierarchy has been conjectured [14] for the complexity of finding common reducts for a β-equality in λ-calculus. Our main goal in this paper is to show that an upper bound function for the Church–Rosser theorem of λ-calculus with β-equality is to be in the fourth level of the Grzegorczyk hierarchy which is the same level as finding common reducts for the confluence property [14].

In this study, we are interested in quantitative analysis of *witnesses*[2] of the Church–Rosser theorem: how to find common reducts with the least size relating to space and with the least number of reduction steps relating to time. For the theorem for β-equality ($M \longleftrightarrow N$ implies $M \to^{l_1} P$ and $N \to^{l_2} P$ for some P), we study functions that set bounds on the least size of a common reduct P, and the least number of reduction steps l_1 and l_2 required to arrive at a common reduct, involving the term sizes of M and N. and the length of $\longleftrightarrow$.

1.2 New results of this paper

In this paper, first we investigate the Church–Rosser theorem in the type-free λ-calculus by means of Takahashi's translation. Although confluence and the Church–Rosser theorem are equivalent to each other, confluence is a special case of the Church–Rosser property. Our investigation shows that a common reduct of M and N with $M \longleftrightarrow N$ is determined by (i) M and the number of occurrences of reductions ($\to$) appearing in $\longleftrightarrow$, and also by (ii) N and that of reversed reductions ($\leftarrow$). The key lemma plays an important role and reveals a new invariant involved in the equality $\longleftrightarrow$, independently of an exponential

[1] In the literature [1, 13], the relation of β-equality is written by $=_\beta$ instead of $\longleftrightarrow$.

[2] Here, common reducts bear witnesses to the existential statement.

combination of reduction and reversed reduction. Next, the characterization of the Church–Rosser theorem makes it possible to analyse how large common reducts are in terms of iteration of Takahashi's translations, and how many reduction-steps are required to obtain them by means of the notion of parallel reduction. In this way, we obtain an upper bound function for the theorem of λ-calculus with β-equality in the fourth level of the Grzegorczyk hierarchy. Moreover, the same method can be applied for analyzing quantitative properties of other reduction systems such as Girard's system **F** and Gödel's system **T**.

1.3 Outline of paper

This paper is organized as follows. Section 1 is devoted to background, related work, motivation, and new results of the paper. Section 2 gives preliminaries including basic definitions and the key lemma and proposition. Section 3 analyzes term size and reduction length in the case of β-reduction, and provides measure functions for upper bounds. Based on the results section 4 demonstrates a quantitative analysis of some (but not all)[3] of the witnesses of the Church–Rosser theorem. Section 5 applies the method developed in sections 3 and 4 to λ-calculus with $\beta\eta$-reduction, Girard's system **F**, and Gödel's system **T** as well. Section 6 concludes with remarks and further work. This paper is an extended and revised version of the paper [8].

2 Preliminaries

First, λ-terms and β-reduction are defined referring to the standard texts [1, 13] for the definitions and related notions.

Definition 1 (λ-terms)

$$M, N, P, Q \in \Lambda ::= x \mid (\lambda x.M) \mid (M\, N)$$

We write $M \equiv N$ for the syntactical identity under renaming of bound variables. The set of free variables in M is denoted by $\mathrm{FV}(M)$.

Definition 2 (β-reduction) *One step β-reduction $\rightarrow$ is defined as usual.*

1. $(\lambda x.M)N \rightarrow M[x := N]$.
2. *If $M \rightarrow N$ then $PM \rightarrow PN$, $MP \rightarrow MP$, and $\lambda x.M \rightarrow \lambda x.N$.*

We write $\twoheadrightarrow$ for the reflexive and transitive closure of $\rightarrow$ (called β-reduction). Note that $M \twoheadrightarrow N$ iff there exists a finite sequence of terms $M_0, \ldots, M_n$ ($n \geq 0$) such that $M \equiv M_0 \rightarrow \cdots \rightarrow M_n \equiv N$. For this case we also write $M \rightarrow^n N$.

We denote the reflexive, transitive and symmetric closure of $\rightarrow$ (called β-convertibility) by $\longleftrightarrow$. Note that $M \longleftrightarrow N$ iff there exists a finite sequence of terms $M_0, \ldots, M_n$ ($n \geq 0$) such that $M \equiv M_0 \leftrightarrow \cdots \leftrightarrow M_n \equiv N$ where

[3] See Theorem 1 in [8] for other witnesses.

$\leftrightarrow$ is the symmetric closure of $\rightarrow$, namely either $M_i \rightarrow M_{i+1}$ or $M_{i+1} \rightarrow M_i$ $(i = 0, \ldots, n)$. Here, $\rightarrow$ in the former case $M_i \rightarrow M_{i+1}$ is called a *right arrow*, and that in the latter case is called a *left arrow*, denoted also by $M_i \leftarrow M_{i+1}$. If the number of occurrences of left arrows is l and that of right arrows is r along the conversion sequence, then we denote this by $M \overset{l}{\leftarrow}\!\!\overset{r}{\rightarrow} N$. By $\sharp L[j,k]$ we mean the number of occurrences of left arrows between M_j and M_k $(0 \leq j \leq k \leq n)$ in the sequence.

Next, Takahashi's translation [24] and its iteration are defined.

Definition 3 (Takahashi's * [24] and iteration) *1.* $x^* = x$.

2. $((\lambda x.M)N)^* = M^*[x := N^*]$.

3. $(M\,N)^* = (M^*\,N^*)$.

4. $(\lambda x.M)^* = \lambda x.M^*$.

The third case above is available provided that (MN) *is not a redex. We write* M^{n*} *for the n-fold iteration of the translation * as follows* [25].

1. $M^{0*} = M$,

2. $M^{n*} = (M^{(n-1)*})^*$.

Then we have the following properties of Lemma 1. According to the literature, Loader [17] treated the second and third properties for proving confluence of λ-calculus with β-reduction. Dehornoy and van Oostrom [7] called the properties Z-property, and demonstrated a number of examples with the Z-property. See also [16, 18] for examples and an extension on the properties.

Lemma 1 *1.* $M^*[x := N^*] \twoheadrightarrow (M[x := N])^*$.

2. If $M \rightarrow N$ *then* $M^* \twoheadrightarrow N^*$.

3. If $M \rightarrow N$ *then* $N \twoheadrightarrow M^*$.

Proof. The first property is proved by induction on M. The second and third properties are proved by induction on the derivation of $M \rightarrow N$. □

Now the Church–Rosser theorem [4] can be proved as the following proposition [8] by using the key lemma [8].

Lemma 2 ([8]) *If* $M \overset{l}{\leftarrow}\!\!\overset{r}{\rightarrow} N$ *then we have both* $M \twoheadrightarrow N^{l*}$ *and* $M^{r*} \twoheadleftarrow N$.

Proof. By induction on the length of $(l + r)$, together with Lemma 1. □

Proposition 1 ([8]) *If* $M \overset{l}{\leftarrow}\!\!\overset{r}{\rightarrow} N$*, then there exists a term* P *such that* $M \twoheadrightarrow P^{k*}$ *and* $P^{k*} \twoheadleftarrow N$ *where* $k = \sharp L[0,r]$.

Proof. Let $k = \sharp L[0,r]$ and $n = l + r$. At the length r from the left, we can divide the conversion sequence $M \overset{l}{\leftarrow}\!\!\overset{r}{\rightarrow} N$ into two sub-sequences such that $M \equiv M_0 \overset{k}{\leftarrow}\!\!\overset{k_r}{\rightarrow} M_r$ with $k_r = r - k$, and $M_r \overset{k_l}{\leftarrow}\!\!\overset{k}{\rightarrow} M_n \equiv N$ with $k_l = l - k$. Then, from Lemma 2, we have $M \twoheadrightarrow M_r^{k*}$ for the left sequence, and $M_r^{k*} \twoheadleftarrow N$ for the right sequence. Hence, we obtain a common reduct M_r^{k*} of M, N. □

We note that $0 \leq \sharp L[0,r] \leq \min\{l, r\}$.

3 Quantitative analysis of term size and reduction length

Following Lemma 2 and Proposition 1, we analyze quantitative properties of the Church–Rosser theorem. For this analysis the basic properties of term size and reduction length are summarized here through parallel reduction.

Definition 4 (Term Size) *1.* $|x| = 1$.

2. $|\lambda x.M| = 1 + |M|$.

3. $|MN| = 1 + |M| + |N|$.

We write $\sharp(x \in M)$ for the number of free occurrences of the variable x in M.

Lemma 3 *1.* $\sharp(x \in M) \leq 2^{-1}(|M| + 1)$.

2. $|M[x := N]| = |M| + \sharp(x \in M) \times (|N| - 1)$.

Proof. Both are proved by straightforward induction on M. □

Proposition 2 *If $M \to^n N$ $(n \geq 1)$ then $|N| < \mathsf{Size}(|M|, n)$ where*

$$\mathsf{Size}(m, n) = 8\left(\frac{m}{8}\right)^{2^n}.$$

Proof. By induction on the length n. □

It should be remarked that the denominator 8 of Size is almost strict, in the sense that we have $|(\lambda x.xx)(\lambda x.xx)| = 9$ and $\lim_{n\to\infty} \mathsf{Size}(|M|, n) \leq 8$ for $|M| \leq 8$.

The notion of parallel reduction is defined inductively following [23, 24].

Definition 5 (Parallel β-reduction [23, 24]) *1.* $x \Rightarrow x$.

2. $\lambda x.M \Rightarrow \lambda x.N$ *if* $M \Rightarrow N$.

3. $M_1M_2 \Rightarrow N_1N_2$ *if* $M_1 \Rightarrow N_1$ *and* $M_2 \Rightarrow N_2$.

4. $(\lambda x.M_1)M_2 \Rightarrow N_1[x := N_2]$ *if* $M_1 \Rightarrow N_1$ *and* $M_2 \Rightarrow N_2$.

From the definitions, if $M \to N$ then $M \Rightarrow N$, and if $M \Rightarrow N$ then $M \twoheadrightarrow N$. We write $M \Rightarrow^n N$ if $M \equiv M_0 \Rightarrow M_1 \Rightarrow \cdots \Rightarrow M_n \equiv N$ for some $n \geq 0$ and M_i $(i = 0, 1, \ldots, n)$. We also write $M \overset{l\quad r}{\Longleftrightarrow} N$. if $M \equiv M_0 \Leftrightarrow M_1 \Leftrightarrow \cdots \Leftrightarrow M_n \equiv N$ for some $n \geq 0$ and M_i $(i = 0, 1, \ldots, n)$ where $\Leftrightarrow$ denotes either $\Rightarrow$ or $\Leftarrow$ together with r the number of occurrences of $\Rightarrow$ and l that of $\Leftarrow$. By $\sharp L[j, k]$ we mean the number of occurrences of $\Leftarrow$ between M_j and M_k $(0 \leq j \leq k \leq n)$ in the sequence where $n = l + r$.

The first property [24] of Lemma 4, called triangle property [7], is proved by induction on M, and accordingly the second property can be proved as well.

Lemma 4 *1. If $M \Rightarrow N$ then $N \Rightarrow M^*$.*

2. *If* $M \Rightarrow N$ *then* $M^* \Rightarrow N^*$.

Based on Proposition 2, we adopt a bound function $F_2(x) = \sqrt{2}^x$. We define the n-fold iteration of function $f(x)$, denoted by $f^*(x,n)$, as follows.

Definition 6 (Iteration of $f(x)$) 1. $f^*(x,0) = x$,

2. $f^*(x,n) = f(f^*(x,n-1))$.

According to convention [19, 3], in the case of $f(x) = 2^x$ we write $\mathbf{2}_n(x)$ instead of $f^*(x,n)$. In the case of $f(x) = F_2(x)$, we may write $\sqrt{\mathbf{2}}_n(x)$ rather than $F_2^*(x,n)$. From the definition, $\sqrt{\mathbf{2}}_n(x)$ belongs to the fourth level of the Grzegorczyk hierarchy. For $x \geq 8$, we have the basic properties such that $2\sqrt{\mathbf{2}}_n(x) \leq \sqrt{\mathbf{2}}_{n+1}(x)$ by induction on n. and then for $x \geq 8$,

$$\sum_{i=0}^{n} \sqrt{\mathbf{2}}_i(x) \leq 2\sqrt{\mathbf{2}}_n(x).$$

Proposition 3 1. *If* $M \Rightarrow N$, *then* $M \rightarrow^l N$ *where* $l \leq 3^{-1}(|M| - 1)$.

2. *If* $M \Rightarrow N$, *then* $|N| \leq \sqrt{2}^{|M|}$ *for* $|M| \geq 4$.

3. *If* $M \Rightarrow^n N$ $(n \geq 1)$, *then* $M \rightarrow^l N$ *where* $l < \sqrt{\mathbf{2}}_{n-1}(|M|)$ *for* $|M| \geq 4$.

Proof.

1. By induction on the derivation of $M \Rightarrow N$. We show one of the cases here.
   (a) Case of $(\lambda x.M_1)M_2 \Rightarrow N_1[x := N_2]$ from $M_1 \Rightarrow N_1$ and $M_2 \Rightarrow N_2$: From the induction hypotheses, we have $M_1 \rightarrow^m N_1$ with $m \leq 3^{-1}(|M_1| - 1)$, and $M_2 \rightarrow^n N_2$ with $n \leq 3^{-1}(|M_2| - 1)$. Then we get
   $$(\lambda x.M_1)M_2 \rightarrow^m (\lambda x.N_1)M_2 \rightarrow^n (\lambda x.N_1)N_2 \rightarrow N_1[x := N_2],$$
   where $m + n + 1 \leq 3^{-1}(|M_1| + |M_2| + 1) = 3^{-1}(|(\lambda x.M_1)M_2| - 1)$.
2. By induction on $|M|$. We note that $|N| = |M|$ for $|M| = 1, 2, 3$.
3. Suppose $M \equiv M_0 \Rightarrow M_1 \Rightarrow \cdots \Rightarrow M_n \equiv N$. Then we have $M_0 \rightarrow^{l_1} M_1 \rightarrow^{l_2} \cdots \rightarrow^{l_n} M_n$ for some $l_1, l_2, \ldots, l_n$. If $|M_i| \leq 3$ for some i $(1 \leq i \leq n-1)$, then $|M_j| = |M_i|$ for each $j \geq i$, and hence we get $l_j = 0$ for each $j \geq i$. Now we consider the case where $|M_i| \geq 4$ for every $i = 0, 1, \ldots, n-1$. Then we have $l_i \leq 3^{-1}(|M_{i-1}| - 1)$ and $|M_{i-1}| \leq \sqrt{\mathbf{2}}_{i-1}(|M|)$ $(i = 1, 2, \ldots, n)$. Therefore, l is bounded as follows provided $|M| \geq 8$.
   $$\begin{aligned} l &= \sum_{i=1}^{n} l_i \\ &\leq \frac{1}{3}\sum_{i=0}^{n-1}\left(\sqrt{\mathbf{2}}_i(|M|) - 1\right) \\ &\leq \frac{1}{3}\left(2\sqrt{\mathbf{2}}_{n-1}(|M|) - n\right). \end{aligned}$$

Finally, $\sqrt{\mathbf{2}}_{n-1}(|M|)$ can be applied even to $|M| \geq 4$, since $\sqrt{\mathbf{2}}_i(x) \leq \sqrt{\mathbf{2}}_i(y)$ for $x \leq y$. □

Lemma 5 *If $M \overset{l}{\Longleftarrow}\!\!\overset{r}{\Longrightarrow} N$ then $M \Rightarrow^{l+r} N^{l*}$ and $N \Rightarrow^{l+r} M^{r*}$.*

Proof. By induction on the length $(l+r)$ with Lemma 4. □

Proposition 4 *If $M \overset{l}{\Longleftarrow}\!\!\overset{r}{\Longrightarrow} N$ then there exists a term P such that $M \Rightarrow^r P^{k*}$ and $N \Rightarrow^l P^{k*}$ with $k = \sharp L[0, r]$.*

Proof. Let $n = l + r$. Then at the length r from the left, we divide the sequence $M \overset{l}{\Longleftarrow}\!\!\overset{r}{\Longrightarrow} N$ into two sub-sequences such that $M \equiv M_0 \overset{k}{\Longleftarrow}\!\!\overset{k_r}{\Longrightarrow} M_r$ with $k_r = r - k$, and $M_r \overset{k_l}{\Longleftarrow}\!\!\overset{k}{\Longrightarrow} M_n \equiv N$ with $k_l = l - k$. Hence, we obtain $M \Rightarrow^r M_r^{k*}$ and $N \Rightarrow^l M_r^{k*}$ by Lemma 5. □

4 Quantitative analysis of the Church–Rosser theorem for β-equality

Based on the results in the previous section, we analyze quantitative properties of Lemma 2 and Proposition 1, respectively.

Proposition 5 *If $M \overset{l}{\longleftarrow}\!\!\overset{r}{\longrightarrow} N$ then $M \rightarrow^m M^{r*}$ and $N \rightarrow^n M^{r*}$ where $m \leq \sqrt{\mathbf{2}}_{r-1}(|M|)$, $n \leq \sqrt{\mathbf{2}}_{l+r-1}(|N|)$, and $|M^{r*}| \leq \sqrt{\mathbf{2}}_r(|M|)$ provided $|M|, |N| \geq 4$.*

Proof. Suppose that $M \overset{l}{\longleftarrow}\!\!\overset{r}{\longrightarrow} N$. Then we have $M \overset{l}{\Longleftarrow}\!\!\overset{r}{\Longrightarrow} N$, and hence $N \Rightarrow^{l+r} M^{r*}$ from Lemma 5 and $N \rightarrow^n M^{r*}$ with $n \leq \sqrt{\mathbf{2}}_{l+r-1}(|N|)$ from Proposition 3. On the other hand, we have $M \Rightarrow M$ from the definition. Then $M \Rightarrow M^*$ and $M \Rightarrow^r M^{r*}$ from Lemma 4, and hence $M \rightarrow^m M^{r*}$ with $m \leq \sqrt{\mathbf{2}}_{r-1}(|M|)$ and $|M^{r*}| \leq \sqrt{\mathbf{2}}_r(|M|)$ from Proposition 3. □

Theorem 1 *If $M \overset{l}{\longleftarrow}\!\!\overset{r}{\longrightarrow} N$ then there exists a term P such that $M \rightarrow^m P^{k*}$ and $N \rightarrow^n P^{k*}$ where $k = \sharp L[0, r]$, $m \leq \sqrt{\mathbf{2}}_{r-1}(|M|)$, $n \leq \sqrt{\mathbf{2}}_{l-1}(|N|)$, and $|P^{k*}| \leq \min\{\sqrt{\mathbf{2}}_r(|M|), \sqrt{\mathbf{2}}_l(|N|)\}$ provided $|M|, |N| \geq 4$.*

Proof. From Proposition 4 we can take $P \equiv M_r$, and then apply Proposition 3. □

A simple example is given as in [3]. The Church numerals $\mathbf{c}_n = \lambda f x.f^n(x)$ are defined [1], where $F^0(x) = x$, and $F^{n+1}(x) = F(F^n(x))$. Define Q_i and P_i as follows: $Q_1 = \mathbf{c}_2$, $Q_n = Q_{n-1}\mathbf{c}_2$, and $P_n = (\lambda v_1 \ldots v_n v.v)v_1 \ldots v_n y$ where $v_1, \ldots, v_n, v, y$ are fresh variables. Then $Q_n \rightarrow^a \mathbf{c}_{\mathbf{2}_n(1)}$ with $a = 3\sum_{i=1}^{n-1} \mathbf{2}_i(1) - n + 1 \leq 6 \times \mathbf{2}_{n-1}(1)$. We have the following peak with $M \equiv Q_n P_n$ $(n \geq 2)$:

$$\begin{array}{llll} N_2 \equiv Q_n y \ \leftarrow^{n+1} & M & \rightarrow^a & M_1 \equiv \mathbf{c}_{\mathbf{2}_n(1)}((\lambda v_1 \ldots v_n v.v)v_1 \ldots v_n y) \\ & & \rightarrow^{n+1} & N_1 \equiv \lambda x.((\lambda v.v)y)^{\mathbf{2}_n(1)}(x). \end{array}$$

For this, we obtain the common reduct $M_1^{(n+1)*}$ by Theorem 1 and the valley:

$$N_2 \to^c M_1^{(n+1)*} \leftarrow^b N_1,$$

where $b = \mathbf{2}_n(1)$ and $c = 1 + a$. Observe that b is non-elementary with respect to n, i.e., the number of occurrences of $\leftarrow$, and c is elementary with respect to that of $\to$. While our bound functions provide the inequalities as follows: $b = \mathbf{2}_n(1) \leq \sqrt{\mathbf{2}}_n(|N_1|) = \sqrt{\mathbf{2}}_n(2 + 5 \times \mathbf{2}_n(1))$ and $c = 1 + a \leq \sqrt{\mathbf{2}}_{a+n}(|N_2|) = \sqrt{\mathbf{2}}_{a+n}(8n + 1)$, both of which still belong to the fourth level since the level is closed under the composition of functions in the same level.

5 The Church–Rosser theorem for other reduction systems

We show that the methods developed in sections 3 and 4 still work for quantitative analysis of other reduction systems such as the λ-calculus with $\beta\eta$-reduction, Girard's system **F**, Gödel's system **T**, combinatory weak reduction, and Pure Types Systems as well. As a summary, we extract the common pattern of the theorems. For a reduction system with one-step reduction relation $\to$ and term size $|\ |$, suppose the following two conditions (A) for reduction and translation and (B) for measure functions.

(A): We have a binary relation $\Rightarrow$ on terms and a translation $*$ between terms as follows.

- (a) If $M \to N$ then $M \Rightarrow N$.
- (b) If $M \Rightarrow N$ then $M \twoheadrightarrow N$.
- (c) If $M \Rightarrow N$ then $N \Rightarrow M^*$.

(B): We have two monotonic functions $f, g : \mathbb{N} \to \mathbb{N}$ as follows.

If $M \Rightarrow N$ then $|N| \leq f(|M|)$ and $M \to^l N$ with $l \leq g(|M|)$, where f and g are respectively in the p-th and q-th levels of the Grzegorczyk hierarchy.

Then the following enriched form of the Church–Rosser theorem already holds.

Theorem 2 (Quantitative Church–Rosser Theorem) *If $M \overset{l}{\leftarrow}\!\!\overset{r}{\rightarrow} N$ then there exists a term P such that $M \to^m P^{k*}$ and $N \to^n P^{k*}$ where*

1. $k = \sharp L[0, r] \leq \min\{l, r\}$,
2. $m \leq \sum_{i=0}^{r-1} g(f^*(|M|, i))$, $\quad n \leq \sum_{i=0}^{l-1} g(f^*(|N|, i))$, *and*
3. *m, n are bounded by functions in the level of $\max\{p+1, q\}$ of the Grzegorczyk hierarchy.*

As an instance of the theorem, we can take λ-calculus with $\beta\eta$-reduction.

Corollary 1 (λ-calculus with $\beta\eta$) *If* $M \overset{l\quad r}{\underset{\beta\eta}{\longleftrightarrow}} N$ *then there exists a term* P *such that* $M \to_{\beta\eta}^{m} P^{k*}$ *and* $N \to_{\beta\eta}^{n} P^{k*}$ *where* $k = \sharp L[0,r]$, $m \leq \sqrt{\mathbf{2}}_{r-1}(|M|)$, $n \leq \sqrt{\mathbf{2}}_{l-1}(|N|)$, *and* $|P^{k*}| \leq \min\{\sqrt{\mathbf{2}}_r(|M|), \sqrt{\mathbf{2}}_l(|N|)\}$ *provided* $|M|, |N| \geq 4$.

Proof. From Theorem 2. For (A), take the parallel reduction $\Rightarrow_{\mathbf{F}}$ and Takahashi translation $*$ in [23], and take $f(x) = \sqrt{2}^{x}$ with $x \geq 4$, $g(x) = 3^{-1}(x-1)$ for (B). □

We write $\to_{\mathbf{F}}$ (respectively $\underset{\mathbf{F}}{\longleftrightarrow}$) for one-step reduction (respectively convertibility) of system $\mathbf{F}$ for both extensional and non-extensional ones [10].

Corollary 2 (Girard's system F) *If* $M \overset{l\quad r}{\underset{\mathbf{F}}{\longleftrightarrow}} N$ *then there exists a term* P *such that* $M \to_{\mathbf{F}}^{m} P^{k*}$ *and* $N \to_{\mathbf{F}}^{n} P^{k*}$ *where* $k = \sharp L[0,r]$, $m \leq \sqrt{\mathbf{2}}_{r-1}(|M|)$, $n \leq \sqrt{\mathbf{2}}_{l-1}(|N|)$, *and* $|P^{k*}| \leq \min\{\sqrt{\mathbf{2}}_r(|M|), \sqrt{\mathbf{2}}_l(|N|)\}$ *provided* $|M|, |N| \geq 4$.

Proof. From Theorem 2. For (A), take the parallel reduction $\Rightarrow_{\mathbf{F}}$ and Takahashi translation $*$ in [24], and take $f(x) = \sqrt{2}^{x}$ with $x \geq 4$, $g(x) = 3^{-1}(x-1)$ for (B). □

We write $\to_{\mathbf{T}}$ (respectively $\underset{\mathbf{T}}{\longleftrightarrow}$) for one-step reduction (respectively convertibility) of system $\mathbf{T}$ [10].

Corollary 3 (Gödel's system T) *If* $M \overset{l\quad r}{\underset{\mathbf{T}}{\longleftrightarrow}} N$ *then there exists a term* P *such that* $M \to_{\mathbf{T}}^{m} P^{k*}$ *and* $N \to_{\mathbf{T}}^{n} P^{k*}$ *where* $k = \sharp L[0,r]$, $m \leq \sqrt{\mathbf{2}}_{r-1}(|M|)$, $n \leq \sqrt{\mathbf{2}}_{l-1}(|N|)$, *and* $|P^{k*}| \leq \min\{\sqrt{\mathbf{2}}_r(|M|), \sqrt{\mathbf{2}}_l(|N|)\}$ *provided* $|M|, |N| \geq 4$.

Proof. From Theorem 2. For (A), take the parallel reduction $\Rightarrow_{\mathbf{T}}$ and Takahashi translation $*$ in [24], and $f(x) = \sqrt{2}^{x}$ with $x \geq 4$, $g(x) = 3^{-1}(x-1)$ for (B). □

We show another example of combinatory weak reduction, see also [13] for the basic definitions. We use the notations $\overset{l\quad r}{\underset{\mathbf{w}}{\longleftrightarrow}}$ and $\rhd^{m}$ for terms denoted by X, Y, Z as follows.

$$X, Y, Z ::= x \mid \mathsf{K} \mid \mathsf{S} \mid (X\,Y)$$

Corollary 4 (Combinatory logic) *If* $X \overset{l\quad r}{\underset{\mathbf{w}}{\longleftrightarrow}} Y$ *then there exists a term* Z *such that* $X \rhd^{m} Z^{k*}$ *and* $Y \rhd^{n} Z^{k*}$ *where* $k = \sharp L[0,r]$, $m \leq 2^{-1} \times \mathbf{2}_{r-1}(|X|)$, $n \leq 2^{-1} \times \mathbf{2}_{l-1}(|Y|)$, *and* $|Z^{k*}| \leq 2^{-1} \times \min\{\mathbf{2}_r(|X|), \mathbf{2}_l(|Y|)\}$.

Proof. For (B), take $f(x) = 2^{x-1}$, and $g(x) = 4^{-1}(x-1)$. □

We show yet another example of Pure Type Systems (PTSs), see also [2] for the basic definitions. For PTSs with β-reduction, the Church–Rosser property on well-typed terms follows that for pseudo-terms denoted by T, U and the subject reduction property.

$$T, U ::= x \mid c \mid (\lambda x{:}T.U) \mid (T\,U) \mid (\Pi x{:}T.U)$$

Corollary 5 (Pure Type Systems) *If* $T \overset{l\quad r}{\underset{\beta}{\longleftrightarrow}} U$ *then there exists a term* P *such that* $T \to^m P^{k*}$ *and* $U \to^n P^{k*}$ *where* $k = \sharp L[0,r]$, $m \leq \mathbf{2}_r(|T|)$, *and* $n \leq \mathbf{2}_l(|U|)$.

Proof. For (B), take $f(x) = 2^x$, and $g(x) = x$. □

6 Concluding remarks and further work

Although a bound in at least the fifth level of the Grzegorczyk hierarchy had been conjectured [14], it is in the fourth level of the hierarchy that our bound function is obtained for the valley size of the theorem for β-equality and $\beta\eta$-equality.

Based on Lemma 2, we revealed that common reducts of $M \overset{l\quad r}{\longleftrightarrow} N$ can be determined by M^{r*} with r the number of occurrences of $\to$, N^{l*} with l that of $\leftarrow$, and also M_r^{k*} with $k = \sharp L[0,r]$, although we have $\frac{(l+r)!}{l!r!}$ patterns of combinations of $\to$ and $\leftarrow$ for $\overset{l\quad r}{\longleftrightarrow}$.

For a quantitative analysis of the Church–Rosser theorem for β-reduction we provided a measure function $F_2(x) = \sqrt{2}^x$ based on Proposition 2. Our bound is given in terms of Takahashi's translation and analyzed via the notion of parallel reduction[4] [23, 24] which makes technical proofs simpler, compared with a previous version [8]. In [24] Takahashi showed that the notion is useful for proving not only confluence but also other fundamental theorems. In addition, here this is indeed useful even for a quantitative analysis of reduction.

The first property of Proposition 3 essentially corresponds to the complete sequential reduction relative to a minimal sequence [5], so-called minimal complete development [12, 13] that yields shortest complete developments [15, 20]. Under the reduction, iteration of the exponential function leads to a non-elementary recursive function as described by $F_2^*(x, n)$ for bounds on term size and reduction length. Proposition 3 and Lemma 4 should be investigated further from a viewpoint of reduction paths [9].

Moreover, all of the quantitative properties in sections 3 and 4 can be applied straightforwardly to the Church-Rosser theorem for $\beta\eta$-equality. It is known that the triangle property [23] is equivalent to the Z-property in general [6], and hence for the λ-calculus with $\beta\eta$-reduction, the corresponding properties to Lemma 2 and Proposition 1 hold still where $\longleftrightarrow$ (respectively $\twoheadrightarrow$) is replaced with $\underset{\beta\eta}{\longleftrightarrow}$ (respectively $\twoheadrightarrow_{\beta\eta}$). In fact, section 5 demonstrates that our approach in sections 3 and 4 has a lot of potential for analysing quantitative properties not only of system **F** and system **T**, but also of other reduction systems with x-reduction following Theorem 2, where the corresponding Lemma 2 and Proposition 1 with $\underset{\mathsf{x}}{\longleftrightarrow}$ and $\twoheadrightarrow_{\mathsf{x}}$ respectively hold from the condition (A) and play an important role. It turns out that the properties of Propositions 2

[4]We note that in [14], the notion of parallel rewriting which is similar to but different from this is applied successfully to orthogonal TRSs for investigating upper bounds on valley sizes.

and 3, and Theorem 2 are invariant for the typical examples of λ-calculi under the appropriate definitions of term size.

A connection to typed calculi should be remarked. Following noteworthy investigations [19, 17, 3], the exact bounds for the reduction length in the simply typed λ-calculus is known as $\mathbf{2}_{\mathsf{g}(M)}(\mathsf{l}(M))$ [3][5] where the degree $\mathsf{g}(M) = \max\{\mathsf{lv}(A) \mid A \text{ is a type of a subterm of } M\}$ and the level (rank) $\mathsf{lv}(A)$ of a type A are defined as usual such that $\mathsf{lv}(X) = 0$ for atomic types and $\mathsf{lv}(A \to B) = \max\{1 + \mathsf{lv}(A), \mathsf{lv}(B)\}$. For typed λ-terms, normal terms provide a common reduct to which the reduction length is still bounded by the function. From the point of the $*$-translation, there exists a natural number n such that M^{n*} serves a normal term of well-typed $M =_\beta N$. Here, the number n of iteration is given by the depth $\mathsf{d}(M) = \max\{\mathsf{dp}(A) \mid A \text{ is a type of a subterm of } M\}$, where the depth $\mathsf{dp}(A)$ of a type A is defined as usual such that $\mathsf{dp}(X) = 0$ for atomic types and $\mathsf{dp}(A \to B) = 1 + \max\{\mathsf{dp}(A), \mathsf{dp}(B)\}$. If M contains a redex $(\lambda x^A.M_1)M_2$ with type B, then for any redex $(\lambda y^{A'}.M_1')M_2'$ with type B' in M^* we have either $(A' \to B') = A$ or $(A' \to B') = B$, see also [17, 27]. Hence, for well-typed $M =_\beta N$ we have $M \to^l M^{\mathsf{d}(M)*}$ such that $M^{\mathsf{d}(M)*}$ is a normal term, i.e., common reduct with $l \le F_2^*(|M|, \mathsf{d}(M) - 1)$ by Proposition 3. According to the literature [19, 17, 3] the number of reduction steps to common reducts of $M =_\beta N$ is bounded by non-elementary functions depending on the level of types and the length of terms, while our bound functions depend on the size of terms and the iteration number of $*$ which relates directly to the depth of types or the length of equality. This subject should be investigated further for a wide variety of reduction systems.

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References

[1] H. P. Barendregt, The Lambda Calculus. Its Syntax and Semantics, North-Holland, revised edition, 1984.

[2] H. P. Barendregt, Lambda Calculi with Types, Edited by S. Abramsky, D.M. Gabbay and T.S.E. Maibaum. Handbook of Logic in Computer Science Vol. II, Oxford University Press, 1992.

[3] A. Beckmann, Exact bound for lengths of reductions in typed λ-calculus, J. Symb. Log. 66 (2001) 1277–1285.

[4] A. Church and J. B. Rosser, Some properties of conversion, Transactions of the American Mathematical Society 39 (3) (1936) 472–482.

[5] The length l is defined as usual as follows. $\mathsf{l}(x) = 1$, $\mathsf{l}(\lambda x.M) = 1 + \mathsf{l}(M)$, $\mathsf{l}(MN) = \mathsf{l}(M) + \mathsf{l}(N)$.

[5] H. B. Curry, R. Feys, W. Craig, Combinatory Logic, Volume1, North-Holland, Third Printing, 1974.

[6] P. Dehornoy, V. van Oostrom, Z, Draft: For Your Mind Only, July 27, 2008.

[7] P. Dehornoy, V. van Oostrom, Z: proving confluence by monotonic single-step upper bound functions, in: Logical Models of Reasoning and Computation, Steklov Mathematical Institute, Moscow. May 5–8, 2008.

[8] K. Fujita, On Upper Bounds on the Church–Rosser Theorem, Electronic Proceedings in Theoretical Computer Science 235 (2017) 16–31 (Third International Workshop on Rewriting Techniques for Program Transformation and Evaluation, Porto, Portugal, 23rd June 2016).

[9] K. Fujita. A Formal System of Reduction Paths. The Mathematical Society of Japan, Proceedings, Foundations of Mathematics and History of Mathematics, pp. 37–38, September 11–14, 2017.

[10] J.-Y. Girard, Y. Lafont, P. Taylor, Proofs and Types, Cambridge University Press, 1989.

[11] A. Grzegorczyk, Some classes of recursive functions, ROZPRAWY MATEMATYCZNE IV, pp. 1–48, 1953.

[12] J. R. Hindley, Reductions of residuals are finite, Transactions of the American Mathematical Society 240 (1978) 345–361.

[13] J. R. Hindley, J. P. Seldin, Lambda-calculus and Combinators, An Introduction, Cambridge University Press, 2008.

[14] J. Ketema, J. G. Simonsen, Least Upper Bounds on the Size of Confluence and Church-Rosser Diagrams in Term Rewriting and λ-Calculus, ACM Transactions on Computational Logic 14 (4) (2013) 1–28.

[15] Z. Khasidashvili, β-reductions and β-developments with the least number of steps, in: P. Martin-Löf, G. Mints (Eds.), International Conference on Computer Logic, COLOG-88, Tallinn. USSR, December, 1988, Proceedings, no. 417 in LNCS (1990) 105–111.

[16] Y. Komori, N. Matsuda, F. Yamakawa, A Simplified Proof of the Church-Rosser Theorem, Studia Logica 102 (2014) 175–183.

[17] R. Loader, Notes on Simply Typed Lambda Calculus, Tech. Rep. ECS-LFCS-98-381, Edinburgh, 1998.

[18] K. Nakazawa, K. Fujita, Compositional Z: Confluence proofs for permutative conversion, Studia Logica 104 (2016) 1205–1224.

[19] H. Schwichtenberg, Complexity of normalization in the pure lambda-calculus. In A. S. Troelstra and D. van Dalen editors, THE L.E.J.BROUWER CENTENARY SYMPOSIUM (1982) 453–457.

[20] M. H. Sørensen, A note on shortest developments, Log. Meth. in Comput. Science 3 (4:2) (2007) 1–8.

[21] R. Statman, The typed λ-calculus is not elementary recursive, Theoret. Comput. Sci. 9 (1979) 73–81.

[22] R. de Vrijer, A direct proof of the finite developments theorem. J. Symb. Log. 50-2 (1985) 339–343.

[23] M. Takahashi, Parallel reductions in λ-calculus, J. Symb. Comput. 7 (1989) 113–123.

[24] M. Takahashi: Parallel reductions in λ-calculus, Inf. Comput. 118 (1995) 120–127.

[25] M. Takahashi, Theory of Computation: Computability and Lambda Calculus, Kindai Kagaku Shya, 1991.

[26] H. Tonino, K. Fujita, On the adequacy of representing higher order intuitionistic logic as a pure type system, Ann. Pure Appl. Logic 57 (1992) 251–276.

[27] H. Xi, Upper bounds for standardizations and an application, J. Symb. Log. 64 (1999) 291–303.