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MODEL COMPLETENESS OF THE THEORY OF HRUSHOVSKI'S PSEUDOPLANE ASSOCIATED TO 5/8

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ABSTRACT. Hrushovski's pseudoplane associated to rational number 5/8 has a model complete theory.

Hrushovski's amalgamation construction, model completeness
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1. INTRODUCTION

Generic structures constructed by the Hrushovski's amalgamation construction are known to have theories which are nearly model complete. If an amalgamation class has the full amalgamation property then its generic structure has a theory which is not model complete [2]. On the other hand, Hrushovski's strongly minimal structure constructed by the amalgamation construction, refuting a conjecture of Zilber has a model complete theory [5].

We have shown that the generic structure of $\mathbf{K}_f$ with a coefficient between 0 and 1 for the predimension function has a model complete theory under some assumption on f [8].

Hrushovski's original boundary function does not satisfy our assumption above. Nevertheless, we show the model completeness of the theory of the generic graph associated to 5/8.

We essentially use notation and terminology from Baldwin-Shi [3] and Wagner [12]. We also use some terminology from graph theory [4].

For a set X, $[X]^n$ denotes the set of all subsets of X of size n, and $|X|$ the cardinality of X.

We recall some of the basic notions in graph theory we use in this paper. These appear in [4]. Let G be a graph. $V(G)$ denotes the set of vertices of G and $E(G)$ the set of edges of G. $E(G)$ is a subset of $[V(G)]^2$. $|G|$ denotes $|V(G)|$. The *degree* of a vertex v is the number of edges at v. A vertex of degree 0 is *isolated*. A vertex of degree 1 is a *leaf*. G is a *path* $x_0x_1\dots x_k$ if $V(G)=\{x_0,x_1,\dots,x_k\}$ and $E(G)=\{x_0x_1,x_1x_2,\dots,x_{k-1}x_k\}$ where the x_i are all distinct. x_0 and x_k are *ends* of G. The number of edges of a path is its *length*. A path of length 0 is a single vertex. G is a *cycle* $x_0x_1\dots x_{k-1}x_0$ if $k\geq 3$, $V(G)=\{x_0,x_1,\dots,x_{k-1}\}$ and $E(G)=\{x_0x_1,x_1x_2,\dots,x_{k-2}x_{k-1},x_{k-1}x_0\}$ where the x_i are all distinct. The number of edges of a cycle is its *length*. A *girth* of a graph G is the length of the shortest cycle in G. A non-empty graph G is *connected* if any two of its vertices are linked by a path in G. A *connected component* of a graph G is a maximal connected subgraph of G. A *forest* is a graph not containing any cycles. A *tree* is a connected forest.

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To see a graph G as a structure in the model theoretic sense, it is a structure in language $\{E\}$ where E is a binary relation symbol. $V(G)$ will be the universe, and $E(G)$ will be the interpretation of E. The language $\{E\}$ will be called *the graph language*.

Suppose A is a graph. If $X \subseteq V(A)$, $A|X$ denotes the substructure B of A such that $V(B) = X$. If there is no ambiguity, X denotes $A|X$. We usually follow this convention. $B \subseteq A$ means that B is a substructure of A. A substructure of a graph is an induced subgraph in graph theory. $A|X$ is the same as $A[X]$ in Diestel's book [4].

We say that X is *connected* in A if X is a connected graph in the graph theoretical sense [4]. A maximal connected substructure of A is a *connected component* of A.

Let A, B, C be graphs such that $A \subseteq C$ and $B \subseteq C$. AB denotes $C|(V(A) \cup V(B))$, $A \cap B$ denotes $C|(V(A) \cap V(B))$, and $A - B$ denotes $C|(V(A) - V(B))$. If $A \cap B = \emptyset$, $E(A,B)$ denotes the set of edges xy such that $x \in A$ and $y \in B$. We put $e(A,B) = |E(A,B)|$. $E(A,B)$ and $e(A,B)$ depend on the graph in which we are working. When we are working in a graph G, we sometimes write $E_G(A,B)$ and $e_G(A,B)$ respectively.

Let D be a graph and A, B, and C substructures of D. We write $D = B \otimes_A C$ if $D = BC$, $B \cap C = A$, and $E(D) = E(B) \cup E(C)$. $E(D) = E(B) \cup E(C)$ means that there are no edges between $B - A$ and $C - A$. D is called a *free amalgam of B and C over A*. If A is empty, we write $D = B \otimes C$, and D is also called a *free amalgam of B and C*.

Definition 1.1. Let α be a real number such that $0 < \alpha < 1$.

(1) For a finite graph A, we define a predimension function δ by $\delta(A) = |A| - \alpha|E(A)|$.
(2) Let A and B be substructures of a common graph. Put $\delta(A/B) = \delta(AB) - \delta(B)$.

Definition 1.2. Let A and B be graphs with $A \subseteq B$, and suppose A is finite.
$A \leq B$ if whenever $A \subseteq X \subseteq B$ with X finite then $\delta(A) \leq \delta(X)$.
$A < B$ if whenever $A \subsetneq X \subseteq B$ with X finite then $\delta(A) < \delta(X)$.
We say that A is *closed* in B if $A < B$.

If α is irrational then $\leq$ and $<$ are the same relations, but they are different if α is a rational number. Our relation $<$ is often denoted by $\leqslant$ in the literature and some people use $\leq^*$ for our $<$. Since we want to use the relation $\leq$ as well, we use the symbol $<$ for the closed substructure relation.

Let $\mathbf{K}_\alpha$ be the class of all finite graphs A such that $\emptyset < A$.

The following facts appear in [3, 12, 13]. Some proofs are given in [11].

Fact 1.3. *Let A, B, C be finite substructures in a common graph.*

(1) *If $A \cap C$ is empty then $\delta(A/C) = \delta(A) - \alpha e(A,C)$.*
(2) *If $A \cap C$ is empty and $B \subseteq C$ then $\delta(A/B) \geq \delta(A/C)$.*
(3) *$A \leq B$ if and only if $\delta(X/A) \geq 0$ for any $X \subseteq B$.*
(4) *$A < B$ if and only if $\delta(X/A) > 0$ for any $X \subseteq B$ with $X - A$ non-empty.*
(5) *$A \leq A$.*
(6) *If $A \leq B$ then $A \cap C \leq B \cap C$.*
(7) *If $A \leq B$ and $B \leq C$ then $A \leq C$.*
(8) *If $A \leq C$ and $B \leq C$ then $A \cap B \leq C$.*
(9) *$A < A$.*
(10) *If $A < B$ then $A \cap C < B \cap C$.*
(11) *If $A < B$ and $B < C$ then $A < C$.*
(12) *If $A < C$ and $B < C$ then $A \cap B < C$.*

Fact 1.4. *Let $D = B \otimes_A C$.*

(1) $\delta(D/A) = \delta(B/A) + \delta(C/A)$.

(2) *If $A \leq C$ then $B \leq D$.*
(3) *If $A \leq B$ and $A \leq C$ then $A \leq D$.*
(4) *If $A < C$ then $B < D$.*
(5) *If $A < B$ and $A < C$ then $A < D$.*

Fact 1.5. (1) *Let A, B, C and D be graphs with $D = B \otimes C$ and $A \subseteq D$. Then $\delta(D/A) = \delta(B/A \cap B) + \delta(C/A \cap C)$.*
(2) *Let D be a graph and A a substructure of D. Let $\{D_1, D_2, \ldots, D_k\}$ be the set of all connected components of D where the D_i are all distinct. Then*

$$\delta(D/A) = \sum_{i=1}^{k} \delta(D_i/A \cap D_i).$$

Let B, C be graphs and $g : B \to C$ a graph embedding. g is a *closed embedding* of B into C if $g(B) < C$. Let A be a graph with $A \subseteq B$ and $A \subseteq C$. g is a *closed embedding over A* if g is a closed embedding and $g(x) = x$ for any $x \in A$.

In the rest of the paper, $\mathbf{K}$ denotes a class of finite graphs closed under isomorphisms.

Definition 1.6. Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$. $(\mathbf{K}, <)$ has the *amalgamation property* if for any finite graphs $A, B, C \in \mathbf{K}$, whenever $g_1 : A \to B$ and $g_2 : A \to C$ are closed embeddings then there is a graph $D \in \mathbf{K}$ and closed embeddings $h_1 : B \to D$ and $g_2 : C \to D$ such that $h_1 \circ g_1 = h_2 \circ g_2$.

$\mathbf{K}$ has the *hereditary property* if for any finite graphs A, B, whenever $A \subseteq B \in \mathbf{K}$ then $A \in \mathbf{K}$.

$\mathbf{K}$ is an *amalgamation class* if $\emptyset \in \mathbf{K}$ and $\mathbf{K}$ has the hereditary property and the amalgamation property.

A countable graph M is a *generic structure* of $(\mathbf{K}, <)$ if the following conditions are satisfied:

(1) If $A \subseteq M$ and A is finite then there exists a finite graph $B \subseteq M$ such that $A \subseteq B < M$.
(2) If $A \subseteq M$ then $A \in \mathbf{K}$.
(3) For any A, $B \in \mathbf{K}$, if $A < M$ and $A < B$ then there is a closed embedding of B into M over A.

Let A be a finite structure of M. By Fact 1.3 (12), there is a smallest B satisfying $A \subseteq B < M$, written $\mathrm{cl}(A)$. The set $\mathrm{cl}(A)$ is called a *closure* of A in M.

Fact 1.7 ([3, 12, 13]). *Let $(\mathbf{K}, <)$ be an amalgamation class. Then there is a generic structure of $(\mathbf{K}, <)$. Let M be a generic structure of $(\mathbf{K}, <)$. Then any isomorphism between finite closed substructures of M can be extended to an automorphism of M.*

Definition 1.8. Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$. A graph $A \in \mathbf{K}$ is *absolutely closed* in $\mathbf{K}$ if whenever $A \subseteq B \in \mathbf{K}$ then $A < B$.

Note that the notion of being absolutely closed in $\mathbf{K}$ is invariant under isomorphisms.

Fact 1.9 ([11]). *Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$ and M a generic structure of $(\mathbf{K}, <)$. Assume that M is countably saturated. Suppose for any $A \in \mathbf{K}$ there is $C \in \mathbf{K}$ such that $A < C$ and C is absolutely closed in $\mathbf{K}$. Then the theory of M is model complete.*

Definition 1.10. Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$. $(\mathbf{K}, <)$ has the *free amalgamation property* if whenever $D = B \otimes_A C$ with $B, C \in \mathbf{K}$, $A < B$ and $A < C$ then $D \in \mathbf{K}$.

By Fact 1.4 (4), we have the following.

Fact 1.11. *Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$. If $(\mathbf{K}, <)$ has the free amalgamation property then it has the amalgamation property.*

Definition 1.12. Let $\mathbb{R}^+$ be the set of non-negative real numbers. Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing concave (convex upward) unbounded function. Assume that $f(0) = 0$, and $f(1) \leq 1$. We assume that f is piecewise smooth. $f'_+(x)$ denotes the right-hand derivative at x. We have $f(x+h) \leq f(x) + f'_+(x)h$ for $h > 0$. Define $\mathbf{K}_f$ as follows:

$$\mathbf{K}_f = \{A \in \mathbf{K}_\alpha \mid B \subseteq A \Rightarrow \delta(B) \geq f(|B|)\}.$$

Note that if $\mathbf{K}_f$ is an amalgamation class then the generic structure of $(\mathbf{K}_f, <)$ has a countably categorical theory [13].

Definition 1.13. Let R, S be sets and $\mu : R \to S$ a map. For $Z \subseteq [R]^m$, put

$$\mu(Z) = \{\{\mu(x_1), \dots, \mu(x_m)\} \mid \{x_1, \dots, x_m\} \in Z\}.$$

Let B, C, and D be graphs and X a set of vertices. We write $D = B \rtimes_X C$ if $C|X$ has no edges and the following hold:

(1) $V(D) = V(B) \cup V(C)$.
(2) $X = V(B) \cap V(C)$.
(3) $E(D) = E(B) \cup E(C)$.

Since we are assuming that C has no edges on X, B is a usual substructure of D but C may not be a substructure of D in general. If B has no edges on X, then D is the free amalgam of B and C over X.

Fact 1.14. *Let D be a graph with $D = B \rtimes_X C$.*

(1) $\delta(D/B) = \delta(C/X)$.
(2) $\delta(D) = \delta(B) + \delta(C/X)$.

Fact 1.15. *Let D be a graph with $D = B \rtimes_X C$.*

(1) *If $C|X < C$ then $B < D$.*
(2) *If $C|X \leq C$ then $B \leq D$.*

2. Zero-Extensions

Definition 2.1. Let A and B be graphs. B is a *zero-extension of* A if $A \leq B$ and $\delta(B/A) = 0$. B is a *minimal zero-extension of* A if B is a proper zero-extension of A and minimal with this property. In this case, $A \subsetneq U \subsetneq B$ implies $A < U$.

B is a *biminimal zero-extension of* A if B is a minimal zero-extension of A and whenever $A' \subseteq A$ and $\delta(B - A/A') = 0$ then $A' = A$.

We will use the following facts many times.

Fact 2.2. *Let A be a substructure of a graph B. The following are equivalent:*

(1) *B is a biminimal zero-extension of A.*
(2) *$\delta(B/A) = 0$ and whenever $D \subsetneq B$ then $A \cap D < D$.*

Fact 2.3. *Let $D = B \otimes_A C$ where B and C are zero-extensions of A. Then D is a zero-extension of A.*

Proof. We have $A \leq D$ by Fact 1.4 (3). We have $\delta(D/A) = 0$ by Fact 1.4 (1). $\square$

3. A Hrushovski's Boundary Function

Definition 3.1 ([6]). Let α be a positive real number. We define x_n, e_n, k_n, d_n for integers $n \geq 1$ by induction as follows: Put $x_1 = 2$ and $e_1 = 1$. Assume that x_n and e_n are defined. Let r_n be a smallest rational number r such that $r = k/d > \alpha$ with $d \leq e_n$ where k and d are positive integers. Let k_n and d_n be coprime positive integers with $k_n/d_n = r_n$. Finally, let $x_{n+1} = x_n + k_n$, and $e_{n+1} = e_n + d_n$.

Let $a_0 = (0,0)$, and $a_n = (x_n, x_n - e_n\alpha)$ for $n \geq 1$. Let f be a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ whose graph on interval $[x_n, x_{n+1}]$ with $n \geq 0$ is a line segment connecting a_n and a_{n+1}. We call f a *Hrushovski's boundary function associated to α.*

Example 3.2. Let $\alpha = 5/8$. Then we have a following chart:

n	1	2	3	4	5	6	7
x_n	2	3	4	6	8	10	17
e_n	1	2	3	6	9	12	23
k_n	1	1	2	2	2	7	12
d_n	1	1	3	3	3	11	19

Fact 3.3 ([6]). *Let f be a Hrushovski's boundary function associated to α. Then f is strictly increasing and concave, and $(\mathbf{K}_f, <)$ has the free amalgamation property. Therefore, there is a generic structure of $(\mathbf{K}_f, <)$. Any one point structure is absolutely closed in $\mathbf{K}_f$.*

Proposition 3.4. *Let f be a Hrushovski's boundary function associated to α. If α is a rational number then f is unbounded.*

Proof. Let x_n, e_n, k_n, d_n and a_n be as in Definition 3.1. Let y_n be the y-coordinate of a_n. Then $y_{n+1} - y_n = k_n - d_n\alpha > 0$ since $k_n/d_n > \alpha$. Suppose $\alpha = m/d$. Then $k_n - d_n\alpha \geq 1/d$. Therefore, $f(x_n) = y_n \geq n/d$. Hence $\lim_{n\to\infty} f(x_n) = \infty$. □

4. Model Completeness

Let f be a Hrushovski's boundary function associated to $5/8$. Let M be a generic structure of $(\mathbf{K}_f, <)$. We show that the theory of M is model complete. In the rest of the paper, we assume that $\alpha = 5/8$.

In order to discuss if a given graph is in $\mathbf{K}_f$ or not, the following definition will be convenient.

Definition 4.1. Let B be a graph and $c \geq 0$ an integer. B is *normal* to f if $\delta(B) \geq f(|B|)$. B is *c-normal* to f if $\delta(B) \geq f(|B| + c)$. B is *c-critical to f* if B is c-normal to f and c is maximal with this property.

The following three lemmas are immediate from the definitions.

Lemma 4.2 ([11]). *Let A be a finite graph.*

(1) *Suppose A is normal to f and non-empty. Then $\delta(A) > 0$.*
(2) *$A \in \mathbf{K}_f$ if and only if every substructure of A is normal to f.*
(3) *Let c and c' be integers such that $0 \leq c \leq c'$. If A is c'-normal to f then A is c-normal to f, and in particular, A is normal to f.*
(4) *Let A be normal to f. Let n be an integer such that $\delta(A) \geq f(n)$ but $\delta(A) < f(n+1)$. Such an n uniquely exists. Let $c = n - |A|$. Then A is c-critical to f. c is a unique integer u such that A is u-critical to f.*
(5) *Let B be another graph such that $\delta(A) = \delta(B)$, $|A| \leq |B|$ and A and B are normal to f. Then B is c-critical to f if and only if A is $(|B| - |A| + c)$-critical to f.*

Lemma 4.3. *Assume $\alpha = 5/8$. Let $B \in \mathbf{K}_f$. Suppose $|B| \geq 10$ and B is c-critical to f with $0 \leq c < 5$. Then B is absolutely closed in $\mathbf{K}_f$.*

Proof. We have $f(|B|+5) > \delta(B)$. Since $\alpha = 5/8$, there are no positive integers x, y such that $x - y\alpha = 0$ with $x < 5$. Hence, there are no extension C of B with $\delta(C/B) = 0$ with $|C - B| < 5$.

Suppose there is an extension C of B with $\delta(C/B) < 0$, $C \in \mathbf{K}_f$ and $|C - B| < 5$. Then we have $\delta(C/B) \leq 1/8$. Then $f(|B|) < f(|C|) \leq \delta(B) - 1/8$. But since $10 \leq |B|$, we have $f'_+(|B|) \leq f'_+(10) = 1/56$ by Example 3.2. Therefore,

$$f(|B|+5) \leq f(|B|) + 5f'_+(|B|) < \delta(B) - 1/8 + 5/56 < \delta(B).$$

A contradiction. □

Lemma 4.4. *Let A, U be graphs such that $A \subseteq U$, $\delta(A) \leq \delta(U)$, and A is $|U - A|$-normal to f. Then U is normal to f.*

Proof. $\delta(U) \geq \delta(A) \geq f(|A| + |U - A|) = f(|U|)$. □

Lemma 4.5. (1) *Let $C = A \otimes_p B$ where p is a single vertex and $A, B \in \mathbf{K}_f$. Then $C \in \mathbf{K}_f$.*
(2) *Any finite forest belongs to $\mathbf{K}_f$.*
(3) *Any cycle of length 6 or more belongs to $\mathbf{K}_f$.*

Proof. (1) Since one point structure is absolutely closed in $\mathbf{K}_f$, we have $p < A$ and $p < B$. Therefore, $C \in \mathbf{K}_f$ by the free amalgamation property.

(2) follows by induction on the number of vertices using (1).

(3) Any paths belongs to $\mathbf{K}_f$ by (2). In a path of length 3 or more, the end vertices is closed in the path with $\alpha = 5/8$. Amalgamating 2 paths of length 3 or more over its end vertices produces a cycle of length 6 or more. Hence, it belongs to $\mathbf{K}_f$ by the free amalgamation property. Any cycle of length 6 or more can be produced in this way. □

Lemma 4.6. *Let $B = A \rtimes_{\{x,y\}} P$ where $P = x \cdots y$ is a path. Suppose $A \in \mathbf{K}_f$.*
(1) *If the distance of x and y is 3 or more in A and the length of P is 3 then $B \in \mathbf{K}_f$.*
(2) *If the distance of x and y is 3 or more in A and the length of P is 3 or more then $B \in \mathbf{K}_f$.*
(3) *Suppose the distance of x and y is 2 or more in A and the length of P is 4 or more then $B \in \mathbf{K}_f$.*
(4) *Suppose the distance of x and y is 1 or more in A and the length of P is 5 or more then $B \in \mathbf{K}_f$.*

Proof. (1) Suppose P has length 3. We can write $P = xuvy$. Let U be a substructure of B. $U \cap A$ is normal to f because $A \in \mathbf{K}_f$. If $U = (U \cap A) \otimes_x xu$ or $U = (U \cap A) \otimes_y vy$ then $U \in \mathbf{K}_f$ by Lemma 4.5.

Suppose $U = (U \cap A) \otimes_{\{x,y\}} xuvy$. We have

$$\delta(U) = \delta(U \cap A) + 2 - 3\alpha.$$

Let $t = |U \cap A|$. If $t \geq 4$ then $f'_+(t) \leq 1 - \frac{3}{2}\alpha$ and

$$f(|U|) = f(t+2) \leq f(t) + 2f'_+(t) \leq \delta(U \cap A) + 2 - 3\alpha = \delta(U).$$

Suppose $t \leq 3$. This means that $U \cap A = xy$ or $U \cap A = xyw$ with $w \in A$. Since x and y has distance 3 or more, w is not connected to x or y. Therefore, U is a path or $U = xuvy \otimes w$. Hence, $U \in \mathbf{K}_f$ by Lemma 4.5.

FIGURE 1. A twig for $5/8$

(2)–(4) We can write $P = x\cdots x'uvy$. Let $A' = A \otimes_x x\cdots x'$. Then $A' \in \mathbf{K}_f$ by Lemma 4.5. Also, the distance between x' and y is 3 or more by the assumption. Now, $B = A' \otimes_{\{x',y\}} x'uvy$. B belongs to $\mathbf{K}_f$ by (1). □

Lemma 4.7. *Let $B = A \rtimes_{\{x,y,z\}} S$ where $S = xx'c \otimes_c yy'c \otimes zz'c$. Suppose $A \in \mathbf{K}_f$ and each pair of vertices in $\{x,y,z\}$ has distance 2 or more. Then $B \in \mathbf{K}_f$ also.*

Proof. Let U be a substructure of B. $U \cap A$ is normal to f because $A \in \mathbf{K}_f$.

Suppose $V(S)$ is not a subset of $V(U)$. Then U can be obtained from $U \cap A$ by amalgamations over 1 vertex, and connecting two points from x, y, z by a path of length 4. In this case, U is normal to $\mathbf{K}_f$ by Lemma 4.6.

Suppose $V(S)$ is a subset of $V(U)$. Then U is an extension of $U \cap A$ by 4 vertices and 6 edges. We have $|U| - |U \cap A| = 4$ and $\delta(U) - \delta(U \cap A) = 4 - 6\alpha$.

Case $|U \cap A| = 3$. This means that $V(U \cap A) = \{x,y,z\}$. Since each pair from $\{x,y,z\}$ has distance 2 or more in A, there are no edges among them. So, $U = S$ is a tree and thus $U \in \mathbf{K}_f$, and therefore U is normal to f.

Case $|U \cap A| \geq 4$. Then

$$f'_+(|U \cap A|) \leq f'_+(4) = 1 - (3/2)\alpha = \frac{4-6\alpha}{4} = \frac{\delta(U) - \delta(U \cap A)}{|U| - |U \cap A|}.$$

Therefore, $\delta(U) \geq f(|U|)$. This means that U is normal to f.

Now, we see that $B \in \mathbf{K}_f$. □

Definition 4.8. (Twig and Wreath)

Let $s = \langle 3/8, -2/8, 3/8, -2/8, -2/8 \rangle$. Note that $1 - \alpha = 3/8$ and $1 - 2\alpha = -2/8$ for $\alpha = 5/8$. We assume that s is indexed by 0, 1, 2, 3, 4. s is a special sequence for $5/8$ defined in [11]. For any $l < 4$, we have $0 < \sum_{i=0}^{l} s(i) < \alpha = 5/8$ and $\sum_{i=0}^{4} s(i) = 0$. Let s^k be a concatenation of k s's. That is, s^k denotes a function g on $\{i \in \mathbb{Z} \mid 0 \leq i < 5k\}$ such that $g(x) = s(x \bmod 5)$. For any $i \leq j < 5k$, we have $|\sum_{u=i}^{j} s(u)| < \alpha = 5/8$.

A graph W is called a *twig associated to s* if W can be written as $W = BF$ with substructures B and F having the following properties:

(1) B is a path $b_0b_1b_2b_3b_4$ of length 5.
(3) $V(F) = \{f_0, f_1, f_3, f_4\}$ and F has no edges.
(3) Each $f_i \in F$ is adjacent to b_i and a leaf of W.

See Figure 1.

Let D be a substructure of W. $F(D)$ denotes $F \cap D$.

Let $k \geq 2$. A graph W is called a *wreath associated to s^k* if W can be written as $W = BF$ with the following properties:

(1) B is a cycle $b_0b_1\cdots b_{5k-1}b_0$ of length $5k$.
(2) $V(F) = \cup_{i=0}^{k-1}\{f_{5i+1}, f_{5i+3}, f_{5i+4}\}$ and F has no edges.
(3) Each $f_l \in F$ is adjacent to b_l.
(4) Each $f \in F$ is a leaf of B.

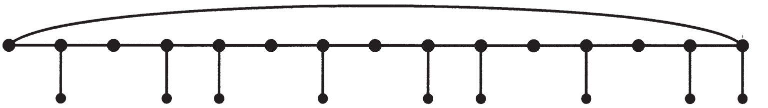

FIGURE 2. A wreath for $5/8$

See Figure 2.

We also say that W is a wreath for $5/8$ without referring to s^k.

$H(W)$ denotes the set $\{f_{5i+3} \mid 0 \le i < k\}$. Let D be a substructure of W. $F(D)$ denotes $F \cap D$.

By Lemma 4.5, we have the following.

Lemma 4.9. *Any twig for* $5/8$ *belongs to* $\mathbf{K}_f$. *Let* W *be a wreath for* $5/8$. *If the girth of* W *is 10 or more then* W *belongs to* $\mathbf{K}_f$.

Fact 4.10 ([11]). *Let* W *be a twig or a wreath for* $5/8$. *Then* W *is a biminimal zero-extension of* $F(W)$. *In particular, if* D *is a proper substructure of* W *then* $F(D) < D$ *by Fact 2.2.*

If $B = A \rtimes_{F(W)} W$ *then* B *is a minimal zero-extension of* A. *Moreover, if* $F(W) = V(A)$ *then* B *is a biminimal zero-extension of* A.

Definition 4.11. We call B a *special extension of* A *over* P if $B = (AP) \rtimes_{F(W)} W$ where W is a twig or a wreath for $5/8$, P has no edges, $AP = A \otimes P$, and $V(A) \cap F(W)$ is a proper subset of $F(W)$.

We call C a *semi-special extension of* A *over* P if we can write $C = B_1 \otimes_{AP} B_2 \otimes_{AP} \cdots \otimes_{AP} B_n$ where each B_i is a special extension of A over P.

Lemma 4.12. *Let* C *be a semi-special extension of* A. *Then* $A < C$.

Proof. Let C be as in the definition of a semi-special extension of A. Suppose $A \subsetneq U \subseteq C$. We can write $B_i = AP \rtimes_{F(W_i)} W_i$ for some twig or wreath W_i. So, we can write

$$U = (U \cap B_1) \otimes_{U \cap (AP)} \cdots \otimes_{U \cap (AP)} (U \cap B_n).$$

If $U \cap (A \otimes P)$ is a proper extension of A then $\delta(A) < \delta(U \cap (AP))$. Since $AP \le B_i$ for each i, we have $U \cap (AP) \le U \cap B_i$. Therefore, $U \cap (AP) \le U$. Hence, $\delta(A) < \delta(U)$.

Suppose $U \cap (AP) = A$. Then $V(U) \cap F(W_i)$ is a proper subset of $F(W_i)$. Hence, $U \cap (AP) = U \cap A < U \cap B_i$. Since U is a proper extension of A, there is j such that $U \cap B_j$ is a proper extension of $U \cap A$. Hence, $\delta(U/U \cap A) \ge \delta(U \cap B_j/U \cap A) > 0$.

We have shown that $A < C$. □

Lemma 4.13. *Let* A *be a graph in* $\mathbf{K}_f$ *with* $|A| \ge 2$. *Suppose* $0 \le k \le |A|$. *Then there is a semi-special extension* $D = C \otimes_{AP} B$ *of* A *over* P *such that* $D \in \mathbf{K}_f$, $|B - (AP)| = 5|A|$ *and* $|C - (AP)| = 5k$.

Proof. We prove the lemma in the case that $|A| = 3$ and $k = 2$. It will be easy to write down a proof for general cases.

We show that a wreath W_1 with girth $5|A|$ and a wreath W_2 with girth $5k$ can be properly attached to A will be a semi-special extension of A over some P. Recall $H(W)$ from the definition of a wreath. $H(W_1)$ will be $V(A)$, and $H(W_2)$ will be a subset of $V(A)$. $V(P)$ will be $F(W_1) - H(W_1)$. Hence, $|P|$ will be $2|A|$. In case $k = 1$, W_2 will be a twig, and $F(W_2)$ can be disjoint from A.

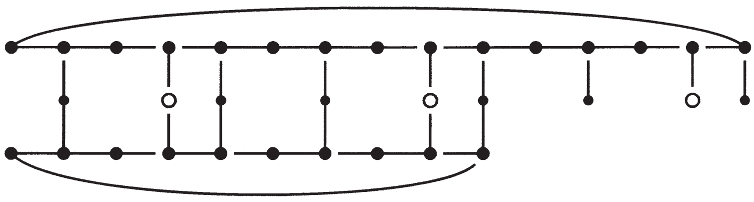

FIGURE 3. A semi-special extension.

Let $V(A) = \{a_3, a_8, a_{13}\}$. Note that $V(A)$ is indexed by $\{5i+3 \mid i = 0,1,2\}$. Attach new paths a_3b_3, a_8b_8, $a_{13}b_{13}$ to A. Here, b_3, b_8, b_{13} are new vertices. Let A_1 be the resulting graph. Then A_1 belongs to $\mathbf{K}_f$ by Lemma 4.5.

Connect b_3 and b_8 by a new path $b_3b_4b_5b_6b_7b_8$, b_8 and b_{13} by a new path $b_8b_9b_{10}b_{11}b_{12}b_{13}$, and b_{13} and b_3 by a new path $b_{13}b_{14}b_0b_1b_2b_3$. Let D_0 be the resulting graph. D_0 belongs to $\mathbf{K}_f$ by Lemma 4.6.

Now, attach new paths a_3c_3 to D_0. The resulting graph belongs to $\mathbf{K}_f$ by Lemma 4.5.

Connect c_3 and b_4 by new path $c_3c_4p_4b_4$.

Connect c_4 and b_6 by new path $c_4c_5c_6p_6b_6$.

Connect c_6 and a_8 by new path $c_6c_7c_8a_8$.

The resulting graph belongs to $\mathbf{K}_f$ by Lemma 4.6.

We can repeat this part if k is large.

Now, connect c_8 and b_9 by new path $c_8c_9p_9b_9$. The resulting graph belongs to $\mathbf{K}_f$ by Lemma 4.6.

Connect 3 vertices c_9, b_1, c_3 by structure $c_9c_0c_1p_1b_1 \otimes_{c_1} c_1c_2c_3$. The resulting graph belongs to $\mathbf{K}_f$ by Lemma 4.7.

Finally, attach new path b_ip_i at b_i for $i = 11$, 13, 14. The resulting graph D belongs to $\mathbf{K}_f$ by Lemma 4.5, and it is a desired graph.

See Figure 3. White circles are the vertices of A. The upper part is W_1 and the lower part is W_2. □

Lemma 4.14. *Let $D = C \otimes_{AP} B$ be a semi-special extension of A over P. Assume that $D \in \mathbf{K}_p$ and B is a extension by a wreath W with $F(W) = V(AP)$. Let*

$$G = C \otimes_{AP} B_1 \otimes_{AP} B_2 \otimes_{AP} \cdots \otimes_{AP} B_n$$

where $B_i \cong_{AP} B$ for each $i = 1, \ldots, n$. If G is normal to f then $G \in \mathbf{K}_f$.

Proof. Note that $C \otimes_{AP} B$ and $C \otimes_{AP} B_j$ for $j \geq 1$ are isomorphic over C. So, $C \otimes_{AP} B_j$ belongs to $\mathbf{K}_f$ for any $j \geq 1$.

We have $B = (AP) \rtimes_{F(W)} W$ with $F(W) = V(AP)$. Let W_i for $i \geq 1$ be a wreath isomorphic to W such that $B_i = (AP) \rtimes_{F(W_i)} W_i$.

Suppose $U \subseteq G$.

Case $AP \subseteq U$. Since G is normal to f, U is normal to f by Lemma 4.4.

Case $A \not\subseteq U$. Then $U \cap A$ is a proper subset of A. For each i with $0 \leq i \leq n$, put $U_i = U \cap B_i$. Then for $i \geq 1$, we have $U_i = (U \cap AP) \rtimes_{F(D_i)} D_i$ where $F(D_i)$ is a proper subset of $F(W_i) = V(AP)$. Hence, $F(D_i) < D_i$ by Lemma 4.10 for each $i \geq 1$. We have $U \cap C < (U \cap C) \rtimes_{F(D_i)} D_i$ by Lemma 1.15. Put $U_i' = (U \cap C) \rtimes_{F(D_i)} D_i$. Then $U \cap C < U_i'$. Note that it is possible that $U \cap C = U_i'$. Since $(U \cap C) \rtimes_{F(D_i)} D_i = (U \cap C) \otimes_{U \cap (AP)} U_i$, we have

$$U = U_1' \otimes_{U \cap C} \cdots \otimes_{U \cap C} U_n'.$$

Since $U_i' = (U \cap C) \otimes_{U \cap A} U_i$ is a substructure of $C \otimes_A B_i \in \mathbf{K}_f$, U_i' belongs to $\mathbf{K}_f$ for $i = 1$, $\ldots$, n. Therefore, U belongs to $\mathbf{K}_f$ by the free amalgamation property. □

Theorem 4.15. *Let f be a Hrushovski's boundary function associated to $5/8$. Let M be the generic structure of $(\mathbf{K}_f, <)$. Then the theory of M is model complete.*

Proof. Suppose $A \in \mathbf{K}_f$. We show that there is a graph G in $\mathbf{K}_f$ such that $A < G$ and G is absolutely closed in $\mathbf{K}_f$. Then we get the theorem by Fact 1.9.

Since a one point structure is absolutely closed, we can assume that $|A| \geq 2$.

Let B be a special extension of A by a wreath W for $5/8$ with $H(W) = A$. Let P be a substructure of B with $V(P) = F(W) - V(A)$. We have $B = AP \rtimes_{V(AP)} W$. We have $\delta(B) = \delta(AP)$ and $|B - AP| = 5|A|$.

By Lemma 4.13, B belongs to $\mathbf{K}_f$. Hence, B is normal to f.

Let n be such that $\delta(B) \geq f(n)$ but $\delta(B) < f(n)$.

Let $n - |AP| = 5|A|l + m$ with $0 \leq m < 5|A|$, and $m = 5k + r$ with $0 \leq r < 5$.

By Lemma 4.13, there is $D \in \mathbf{K}_f$ such that $D = C \otimes_{AP} B$ where C is also a special extension of A with $|C - (AP)| = 5k$.

Let

$$G = C \otimes_{AP} B_1 \otimes_{AP} B_2 \otimes_{AP} \cdots \otimes_{AP} B_l$$

where $B_i \cong_{AP} B$ for each $i = 1, \ldots, l$. Then $|G| = |AP| + 5k + 5|A|l = n - r$. Hence, G is normal to f. By Lemma 4.14, G belongs to $\mathbf{K}_f$. G is r-critical and $0 \leq r < 5$. Also, we have $|G| \geq |B| > 5|A| \geq 10$. Hence, G is absolutely closed in $\mathbf{K}_f$ by Lemma 4.3.

G is a semi-special extension of A over P. Therefore, $A < C$ by Proposition 4.12. □

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