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On the speed of hereditary properties of graphs

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Abstract

We give a short proof of C. Terry's result[3] on the jump to the fastest speed of hereditary properties.

1 Introduction and Preliminaries

Let L be a finite relational language. A hereditary L-property is a hereditary class H of finite L-structures which is closed under isomorphism. The universe of an L-structure A is denoted by $||A||$. Let $[n]$ be the n-point set $\{0, 1, ..., n-1\}$. For each $n \in \omega$, let $H_n = \{A \in H : ||A|| = [n]\}$. The speed of H is the function $n \mapsto |H_n|$. C. Terry proved that there is a gap on the speed and it is characterized by a kind of VC-dimension of H, as follows.

Theorem 1 (Terry[3]). Suppose L is a finite relational language of maximum arity $r \geq 2$, and H is a hereditary L-property. Then either

1. $VC^*_{r-1}(H) < \infty$ and there is an $\epsilon > 0$ such that for sufficiently large n, $|H_n| \leq 2^{n^{r-\epsilon}}$, or
2. $VC^*_{r-1}(H) = \infty$ and there is a constant $C > 0$ such that $|H_n| = 2^{Cn^r + o(n^r)}$.

This kind of result was well known for graph properties, i.e. for classes of graphs(for example, see introduction of [3], or Theorem 2 in [1]). Terry's work gives a generalization of them to the cases of hyper (not simple, not undirected) graphs. One of the main technique in her proof is VC^*-dimension introduced in her paper and a kind of Sauer-Shelah's lemma for product sets proved by Chernikov, Palacin and the author[2]. Note that Terry's result is about the cases of $r \geq 1$, however, $r = 1$ case is immediate (by the original Sauer-Shelah's lemma), hence we discuss only on the cases of $r \geq 2$ here.

In this article, we will give a simple proof of the theorem. In fact, it is almost directly proven from the generalized Sauer-Shelah's lemma with some model theoretic trick.

In the rest of this section, we recall some basic definitions and facts on VC$_r$-dimension and hereditary properties. One can also check [2] for the details of VC$_r$-dimension and the generalized Sauer-Shelah's lemma.

Definition 2. Let $r \in \omega$ and $\mathcal{C} \subset \mathcal{P}(\omega^r)$.

1. For $A \subset \omega^r$, $\mathcal{C}|A = \{A \cap B : B \in \mathcal{C}\}$.
2. $A \subset \omega^r$ is said to be shattered if $\mathcal{C}|A = \mathcal{P}(A)$.
3. A subset $A \subset \omega^r$ is called a box of size d if $A = A_0 \times ... \times A_{r-1}$ and $|A_i| = d$ for all $i < r$.
4. The VC$_r$-dimension $VC_r(\mathcal{C})$ of $\mathcal{C}$ is the maximum natural number d such that there is a box $A \subset \omega^r$ of size d such that A is shattered by $\mathcal{C}$.
5. The shatter function $\pi_{\mathcal{C}}$ is the function $\omega \to \omega$ such that $\pi_{\mathcal{C}}(n) = \max\{|\mathcal{C}|A| : A \subset \omega^r \text{ is a box of size } n\}$.

The following fact is a generalization of Sauer-Shelah's lemma.

Fact 3 (Chernikov, Palacin, and T.)**.** Let $\mathcal{C} \subset \mathcal{P}(\omega^r)$. Then either

1. $VC_r(\mathcal{C}) = \infty$ and $\pi_{\mathcal{C}}(n) = 2^{n^r}$ for every n, or
2. $VC_r(\mathcal{C}) = d < \infty$ and there is $\epsilon > 0$ such that $\pi_{\mathcal{C}}(n) < 2^{n^{r-\epsilon}}$ for sufficiently large n. Here, ϵ depends only on d and r.

Note that ϵ in the above fact was explicitly given in [2].

Next we define VC$_r$-dimension for hereditary L-properties. Suppose that L is a finite relational language of maximum arity $r \geq 2$.

Definition 4. Let H be a class of finite L-structures.

1. H is said to be hereditary if $A \in H$ and $B \subset A$ then $B \in H$. (Here $B \subset A$ means that B is a substructure, in other words induced subgraph, of A.)
2. H is said to be hereditary L-property if H is a hereditary class which is closed under isomorphism.

3. H^c is the class of finite L-structures which is not in H.
4. $Th(H) = \{\forall x_1, ..., x_n(x_1...x_n \ncong B) : B \in H^c, n \in \omega\}$.

Notice that $Th(H)$ is a set of first order L-sentences and for any finite L-structure A, A satisfies $Th(H)$ if and only if $A \in H$. To see this, suppose $A \in H$. If A contains a substructure $B \in H^c$, then B must be in H since H is hereditary, which implies a contradiction. Hence every substructure of A cannot be in H^c. The converse is immediate.

Remark 5. Let A be a finite L-structure. If there is infinite $M \models Th(H)$ with $A \subset M$ then $A \in H$. However, the converse is not always true. For example, let H be the class of every finite graph G such that if $|G| > 2$ then G has no edge. In this case $K_2 \in H$ but there is no infinite $M \models Th(H)$ containing K_2.

Remark 6. Let T be an L-theory and $\varphi(x_0, ..., x_r)$ be an L-formula. It is known that the following are equivalent:

1. There is an infinite model $M \models T$ such that $\mathcal{C}_\varphi = \{\varphi(a, M^r) : a \in M\}$ has VC$_r$-dimension ∞.
2. φ has Independent Property(IP).
3. There is an infinite model $M \models T$ and $A_i \subset M$ $(i < r)$ such that $(A_0, ..., A_{r-1}; \varphi)$ is isomorphic to the r-partite random r-hypergraph.

Definition 7. We define the VC$_r$-dimension d of an $(r+1)$-ary formula $\varphi(x_0, ..., x_r)$ by the VC$_r$-dimension of $\mathcal{C}_\varphi$ in in Remark 6. Hence $\varphi(x_0, ..., x_r)$ has infinite VC$_r$-dimension if and only if one of the conditions in Remark 6 holds.

It is easy to see that if a formula $\varphi(x_0, ..., x_r)$ is given by adding dummy variables to a formula $\psi(x_0, .., x_k)$ with $k < r$, then φ has finite VC$_r$-dimension. It is also known that if φ is given as a boolean combination of formulas with finite VC$_r$-dimension, then φ also has finite VC$_r$-dimension.

2 A proof of the main theorem

In this section, we give a proof of the following:

Theorem 8. Let L be a finite relational language with maximum arity r. Let H be a hereditary L-property. Then either

1. Every relation $R \in L$ has finite VC$_{r-1}$-dimension in $Th(H)$ and there is $\epsilon > 0$ such that $|H_n| < 2^{n^{r-\epsilon}}$ for sufficiently large n.
2. Some relation $R \in L$ has infinite VC$_{r-1}$-dimension in $Th(H)$ and there is a constant $C > 0$ such that $|H_n| \geq 2^{n^{Cr}}$ for every n.

Moreover, ϵ depends only on r and the maximum number d of VC$_{r-1}$-dimensions of $R \in L$

Before starting the proof, we need some definitions.

Definition 9. Let H be a hereditary L-property.

1. Let $R \in L$ and $A \in H$. An restricted structure $A|R$ is the $\{R\}$-structure which is obtained by forgetting other relations on A.
2. $H|R = \{A|R : A \in H\}$.
3. For each $R \in L$ with arity r, we put $\mathcal{C}(R) = \{f(R) \subset \omega^r : f : M \to \omega$ is an injection, $M \models Th(H)\}$, where $f(R) = \{(f(a_0), ..., f(a_{r-1})) : R(a_0, ..., a_{r-1})$ holds$\}$.

Remark 10. Let H be a hereditary L-property and $L = \{R_0, ..., R_{k-1}\}$. The following are easy to check.

1. If $L = \{R\}$, then $|H_n| = |\mathcal{C}(R)|[n]^r| \leq \pi_{\mathcal{C}(R)}(n)$.
2. $|(H|R_0)_n| \leq |H_n| \leq |(H|R_0)_n| \times ... \times |(H|R_{k-1})_n|$.

Now we prove the main theorem.

Proof. Suppose that $R \in L$ has infinite VC$_{r-1}$-dimension. By Remark 6 and the subsequent discussion, we can assume

- R is r-ary relation,
- there is $M \models Th(H)$ and $A_i \subset M$ $(i < r)$ such that $(A_0, ..., A_{r-1}; R)$ is isomorphic to the r-partite random r-hypergraph.

We show that for any n, $|H_n| \geq 2^{m^r}$ where $\frac{n}{r} \geq m \in \omega$. Suppose $rm \leq n$. By the second item of Remark 10, we can assume $L = \{R\}$. Let $X = X_0 \sqcup ... \sqcup X_{r-1}$ be a set of (r-partite) verticies such that $|X_i| = m$ for all $i < r$. The number of r-partite r-uniform hypergraph on X is 2^{m^r}, since edges R is determined as a subset of $\Pi_i X_i$. Since every r-partite r-uniform hypergraph (X, R) can be embeddable into the r-partite random r-hypergraph, $(X, R) \models Th(H)$. Hence $n \geq rm = |X|$ implies that $|H_n| \geq 2^{m^r}$.

Conversely, suppose that every $R \in L$ has finite VC_{r-1}-dimension $\leq d$ (if it is needed, by adding dummy variable). We'll show that there is $\epsilon = \epsilon(r,d) > 0$ such that $|H_n| \leq 2^{n^{r-\epsilon}}$ for sufficiently large n. Again by Remark 10, we can assume $L = \{R(x_0, ..., x_{r-1})\}$ and $|H_n| = \pi_{\mathcal{C}(R)}(n)$. Suppose that there is no $\epsilon > 0$ such that $\pi_{\mathcal{C}(R)}(n) \geq 2^{n^{r-\epsilon}}$ for sufficiently large n. Then by Fact 3, $\mathcal{C}(R)$ must have infinite VC_r-dimension and hence for every n there is a box $A_0 \times ... \times A_{r-1} \subset \omega^r$ of size n which is shattered by $\mathcal{C}(R)$. This means, by the definition of $\mathcal{C}(R)$, (by adding edges on each part if necessary) every r-partite r-uniform hypergraph satisfies $Th(H)$. By the compactness theorem, there is $M \models Th(H)$ and $A_i \subset M$ $(i < r)$ such that $(A_0, ..., A_{r-1}; R)$ is isomorphic to the r-partite random r-hypergraph. This contradicts to the finiteness of VC_{r-1}-dimension of R. $\square$

References

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[3] Terry, C. "VC_l-dimension and the jump to the fastest speed of a hereditary L-property." Proceedings of the American Mathematical Society (2018).