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AROUND GENERIC LINEAR PERTURBATIONS

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1. Introduction

In this paper, ℓ, m and n stand for positive integers. Throughout this paper, unless otherwise stated, all manifolds and mappings belong to class C^∞ and all manifolds are without boundary. The purpose of this paper is to introduce some results shown in [2, 3].

Let $\pi : \mathbb{R}^m \to \mathbb{R}^\ell$, U and $F : U \to \mathbb{R}^\ell$ be a linear mapping, an open subset of $\mathbb{R}^m$ and a mapping, respectively.

Set

$$F_\pi = F + \pi.$$

Here, π in $F_\pi = F + \pi$ is restricted to the open set U.

Let $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ be the space consisting of all linear mappings of $\mathbb{R}^m$ into $\mathbb{R}^\ell$. Notice that we have the natural identification $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell$. An n-dimensional manifold is denoted by N.

In Section 2, two main theorems of [2] (Theorems 1 and 2) are introduced. Theorem 1 is as follows. Let $f : N \to U$ (resp., $F : U \to \mathbb{R}^\ell$) be an immersion (resp., a mapping). Generally, the composition $F \circ f$ does not necessarily yield a mapping which is transverse to a given subfiber-bundle of the jet bundle $J^1(N, \mathbb{R}^\ell)$. Nevertheless, Theorem 1 asserts that for any $\mathcal{A}^1$-invariant fiber, a generic mapping $F_\pi \circ f$ yields a mapping which is transverse to the subfiber-bundle of $J^1(N, \mathbb{R}^\ell)$ with the given fiber. Theorem 2 is a specialized transversality result on crossings of a generic mapping $F_\pi \circ f$, where $f : N \to U$ (resp., $F : U \to \mathbb{R}^\ell$) is a given injection (resp., a given mapping).

In Section 3, some applications of Theorems 1 and 2 are introduced.

In Section 4, the main result of [3] (Theorem 4) is introduced. Theorem 4 is as follows. In [4], John Mather proved that almost all linear projections from a submanifold of a vector space into a subspace are transverse with respect to a given modular submanifold. Theorem 4 is an improvement of the result. Namely, almost all linear perturbations of a smooth mapping from a submanifold of $\mathbb{R}^m$ into $\mathbb{R}^\ell$ yield a transverse mapping with respect to a given modular submanifold.

2. Composing generic linearly perturbed mappings and immersions/injections

In the following, we denote manifolds by N and P.

Definition 1. Let W be a submanifold of P, and let $g : N \to P$ be a mapping.

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(1) We say that $g : N \to P$ is *transverse* to W *at* q if $g(q) \notin W$ or in the case of $g(q) \in W$, the following holds:

$$dg_q(T_qN) + T_{g(q)}W = T_{g(q)}P.$$

(2) We say that $g : N \to P$ is *transverse* to W if for any $q \in N$, g is transverse to W at q.

We say that $g : N \to P$ is $\mathcal{A}$*-equivalent* to $h : N \to P$ if there exist two diffeomorphisms $\Phi : N \to N$ and $\Psi : P \to P$ such that $g = \Psi \circ h \circ \Phi^{-1}$.

Let $J^r(N, P)$ denote the space of r-jets of mappings of N into P. For a given mapping $g : N \to P$, the mapping $j^r g : N \to J^r(N, P)$ is given by $q \mapsto j^r g(q)$ (for details on $J^r(N, P)$ or $j^r g : N \to J^r(N, P)$, see for instance, [1]).

In order to state Theorem 1, it is sufficient to consider the case of $r = 1$ and $P = \mathbb{R}^\ell$. Let $\{(U_\lambda, \varphi_\lambda)\}_{\lambda\in\Lambda}$ denote a coordinate neighborhood system of N. Let $\Pi : J^1(N, \mathbb{R}^\ell) \to N \times \mathbb{R}^\ell$ denote the natural projection defined by $\Pi(j^1g(q)) = (q, g(q))$. Let $\Phi_\lambda : \Pi^{-1}(U_\lambda \times \mathbb{R}^\ell) \to \varphi_\lambda(U_\lambda) \times \mathbb{R}^\ell \times J^1(n, \ell)$ denote the homeomorphism given by

$$\Phi_\lambda \left(j^1g(q)\right) = \left(\varphi_\lambda(q), g(q), j^1(\psi_\lambda \circ g \circ \varphi_\lambda^{-1} \circ \widetilde{\varphi}_\lambda)(0)\right),$$

where $J^1(n, \ell) = \{j^1g(0) \mid g : (\mathbb{R}^n, 0) \to (\mathbb{R}^\ell, 0)\}$ and $\widetilde{\varphi}_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ (resp., $\psi_\lambda : \mathbb{R}^m \to \mathbb{R}^m$) is the translation defined by $\widetilde{\varphi}_\lambda(0) = \varphi_\lambda(q)$ (resp., $\psi_\lambda(g(q)) = 0$). Then, $\{(\Pi^{-1}(U_\lambda \times \mathbb{R}^\ell), \Phi_\lambda)\}_{\lambda\in\Lambda}$ is a coordinate neighborhood system of $J^1(N, \mathbb{R}^\ell)$. We say that a subset $X \subset J^1(n, \ell)$ is $\mathcal{A}^1$*-invariant* if for any $j^1g(0) \in X$, and for any two germs of diffeomorphisms $H : (\mathbb{R}^\ell, 0) \to (\mathbb{R}^\ell, 0)$ and $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, we get $j^1(H \circ g \circ h^{-1})(0) \in X$. Let X denote an $\mathcal{A}^1$-invariant submanifold of $J^1(n, \ell)$. Set

$$X(N, \mathbb{R}^\ell) = \bigcup_{\lambda\in\Lambda} \Phi_\lambda^{-1} \left(\varphi_\lambda(U_\lambda) \times \mathbb{R}^\ell \times X\right).$$

Then, $X(N, \mathbb{R}^\ell)$ is a subfiber-bundle of $J^1(N, \mathbb{R}^\ell)$ with the fiber X satisfying

$$\begin{aligned} \text{codim } X(N, \mathbb{R}^\ell) &= \dim J^1(N, \mathbb{R}^\ell) - \dim X(N, \mathbb{R}^\ell) \\ &= \dim J^1(n, \ell) - \dim X \\ &= \text{codim } X. \end{aligned}$$

Theorem 1 ([2]). *Let f be an immersion of N into an open subset U of $\mathbb{R}^m$, where N is a manifold of dimension n. Let $F : U \to \mathbb{R}^\ell$ be a mapping. If X is an $\mathcal{A}^1$-invariant submanifold of $J^1(n, \ell)$, then there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to $X(N, \mathbb{R}^\ell)$.*

Now, for the statement of Theorem 2, we will prepare some definitions. Set $N^{(s)} = \{(q_1, q_2, \ldots, q_s) \in N^s \mid q_i \neq q_j \ (i \neq j)\}$. Note that $N^{(s)}$ is an open submanifold of N^s. For a given mapping $g : N \to P$, let $g^{(s)} : N^{(s)} \to P^s$ be the mapping defined by

$$g^{(s)}(q_1, q_2, \ldots, q_s) = (g(q_1), g(q_2), \ldots, g(q_s)).$$

Set $\Delta_s = \{(y, \ldots, y) \in P^s \mid y \in P\}$. It is not hard to see that Δ_s is a submanifold of P^s satisfying

$$\text{codim } \Delta_s = \dim P^s - \dim \Delta_s = (s - 1)\dim P.$$

Definition 2. We say that $g : N \to P$ is a *mapping with normal crossings* if for any positive integer s ($s \geq 2$), $g^{(s)} : N^{(s)} \to P^s$ is transverse to Δ_s.

For any injection $f : N \to \mathbb{R}^m$, set

$$s_f = \max\left\{ s \;\middle|\; \forall (q_1, q_2, \ldots, q_s) \in N^{(s)}, \dim \sum_{i=2}^{s} \mathbb{R}\overrightarrow{f(q_1)f(q_i)} = s-1 \right\}.$$

Since the mapping f is injective, it follows that $2 \leq s_f$. Since $f(q_1), f(q_2), \ldots, f(q_{s_f})$ are points of $\mathbb{R}^m$, we have $s_f \leq m+1$. Hence, we get

$$2 \leq s_f \leq m+1.$$

Moreover, in the following, for a set X, we denote the number of its elements (or its cardinality) by $|X|$.

Theorem 2 ([2]). *Let f be an injection of N into an open subset U of $\mathbb{R}^m$, where N is a manifold of dimension n. Let $F : U \to \mathbb{R}^\ell$ be a mapping. Then, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, and for any s ($2 \leq s \leq s_f$), $(F_\pi \circ f)^{(s)} : N^{(s)} \to (\mathbb{R}^\ell)^s$ is transverse to Δ_s. Furthermore, if the mapping F_π satisfies that $|F_\pi^{-1}(y)| \leq s_f$ for any $y \in \mathbb{R}^\ell$, then $F_\pi \circ f : N \to \mathbb{R}^\ell$ is a mapping with normal crossings.*

3. Applications of Theorems 1 and 2

In Subsection 3.1 (resp., Subsection 3.2), applications of Theorem 1 (resp., Theorem 2) are stated.

3.1. Applications of Theorem 1.

Set

$$\Sigma^k = \left\{ j^1 g(0) \in J^1(n,\ell) \mid \text{corank } Jg(0) = k \right\},$$

where corank $Jg(0) = \min\{n,\ell\} - \text{rank } Jg(0)$ and $k = 1, 2, \ldots, \min\{n,\ell\}$. Then, Σ^k is an $\mathcal{A}^1$-invariant submanifold of $J^1(n,\ell)$. Set

$$\Sigma^k(N, \mathbb{R}^\ell) = \bigcup_{\lambda \in \Lambda} \Phi_\lambda^{-1}\left(\varphi_\lambda(U_\lambda) \times \mathbb{R}^\ell \times \Sigma^k\right),$$

where Φ_λ and φ_λ are as defined in Section 2. Then, the set $\Sigma^k(N, \mathbb{R}^\ell)$ is a subfiber-bundle of $J^1(N, \mathbb{R}^\ell)$ with the fiber Σ^k satisfying

$$\begin{aligned} \text{codim } \Sigma^k(N, \mathbb{R}^\ell) &= \dim J^1(N, \mathbb{R}^\ell) - \dim \Sigma^k(N, \mathbb{R}^\ell) \\ &= (n - v + k)(\ell - v + k), \end{aligned}$$

where $v = \min\{n, \ell\}$. (For details on Σ^k and $\Sigma^k(N, \mathbb{R}^\ell)$, see for instance [1], pp. 60–61).

As applications of Theorem 1, we get the following Proposition 1, Corollaries 1, 2, 3 and 4.

Proposition 1 ([2]). *Let f be an immersion of N into an open subset U of $\mathbb{R}^m$, where N is a manifold of dimension n. Let $F : U \to \mathbb{R}^\ell$ be a mapping. Then, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to $\Sigma^k(N, \mathbb{R}^\ell)$ for any positive integer k satisfying $1 \leq k \leq v$. Especially, in the case of $\ell \geq 2$, we get $k_0 + 1 \leq v$ and it follows that $j^1(F_\pi \circ f)$ satisfies that $j^1(F_\pi \circ f)(N) \bigcap \Sigma^k(N, \mathbb{R}^\ell) = \emptyset$ for any positive integer k ($k_0 + 1 \leq k \leq v$), where k_0 is the maximum integer satisfying $(n - v + k_0)(\ell - v + k_0) \leq n$ ($v = \min\{n, \ell\}$).*

Remark 1. (1) In Proposition 1, by $(n-v+k_0)(\ell-v+k_0) \leq n$, it is not hard to see that $k_0 \geq 0$.

(2) In Proposition 1, in the case of $\ell = 1$, we get $k_0 + 1 > v$. Indeed, in the case, by $v = 1$, we have $(n-1+k_0)k_0 \leq n$. Thus, it follows that $k_0 = 1$.

A mapping $g : N \to \mathbb{R}$ is called a *Morse function* if all of the singularities of g are nondegenerate (for details on Morse functions, see for instance, [1], p. 63). In the case of $(n, \ell) = (n, 1)$, we get the following.

Corollary 1 ([2]). *Let f be an immersion of N into an open subset U of $\mathbb{R}^m$, where N is a manifold of dimension n. Let $F : U \to \mathbb{R}$ be a mapping. Then, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R})$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}$ is a Morse function.*

For a given mapping $g : N \to \mathbb{R}^{2n-1}$ $(n \geq 2)$, a singular point $q \in N$ is called a *singular point of Whitney umbrella* if there exist two germs of diffeomorphisms $H : (\mathbb{R}^{2n-1}, g(q)) \to (\mathbb{R}^{2n-1}, 0)$ and $h : (N, q) \to (\mathbb{R}^n, 0)$ satisfying $H \circ g \circ h^{-1}(x_1, x_2, \ldots, x_n) = (x_1^2, x_1x_2, \ldots, x_1x_n, x_2, \ldots, x_n)$, where $(x_1, x_2, \ldots, x_n)$ is a local coordinate around the point $h(q) = 0 \in \mathbb{R}^n$. In the case of $(n, \ell) = (n, 2n-1)$ $(n \geq 2)$, we get the following.

Corollary 2 ([2]). *Let f be an immersion of N into an open subset U of $\mathbb{R}^m$, where N is a manifold of dimension n $(n \geq 2)$. Let $F : U \to \mathbb{R}^{2n-1}$ be a mapping. Then, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{2n-1})$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{2n-1}) - \Sigma$, any singular point of the mapping $F_\pi \circ f : N \to \mathbb{R}^{2n-1}$ is a singular point of Whitney umbrella.*

In the case of $\ell \geq 2n$, the immersion property of a given mapping $f : N \to U$ is preserved by composing generic linearly perturbed mappings as follows:

Corollary 3 ([2]). *Let f be an immersion of N into an open subset U of $\mathbb{R}^m$, where N is a manifold of dimension n. Let $F : U \to \mathbb{R}^\ell$ be a mapping $(\ell \geq 2n)$. Then, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is an immersion.*

A mapping $g : N \to \mathbb{R}^\ell$ *has corank at most k singular points* if

$$\sup \{\text{corank } dg_q \mid q \in N\} \leq k,$$

where corank $dg_q = \min\{n, \ell\} - \text{rank } dg_q$. From Proposition 1, we have the following.

Corollary 4 ([2]). *Let f be an immersion of N into an open subset U of $\mathbb{R}^m$, where N is a manifold of dimension n. Let $F : U \to \mathbb{R}^\ell$ be a mapping. Let k_0 be the maximum integer satisfying $(n-v+k_0)(\ell-v+k_0) \leq n$ $(v = \min\{n, \ell\})$. Then, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ has corank at most k_0 singular points.*

3.2. Applications of Theorem 2.

Proposition 2 ([2]). *Let f be an injection of N into an open subset U of $\mathbb{R}^m$, where N is a manifold of dimension n. Let $F : U \to \mathbb{R}^\ell$ be a mapping. If $(s_f - 1)\ell > ns_f$, then there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is a mapping with normal crossings satisfying $(F_\pi \circ f)^{(s_f)}(N^{(s_f)}) \bigcap \Delta_{s_f} = \emptyset$.*

In the case of $\ell > 2n$, the injection property of a given mapping $f : N \to U$ is preserved by composing generic linearly perturbed mappings as follows:

Corollary 5 ([2]). *Let f be an injection of N into an open subset U of $\mathbb{R}^m$, where N is a manifold of dimension n. Let $F : U \to \mathbb{R}^\ell$ be a mapping. If $\ell > 2n$, then there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is injective.*

By combining Corollaries 3 and 5, we get the following.

Corollary 6 ([2]). *Let f be an injective immersion of N into an open subset U of $\mathbb{R}^m$, where N is a manifold of dimension n. Let $F : U \to \mathbb{R}^\ell$ be a mapping. If $\ell > 2n$, then there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, $F_\pi \circ f : N \to \mathbb{R}^\ell$ is an injective immersion.*

In Corollary 6, suppose that the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is proper. Then, an injective immersion $F_\pi \circ f$ is necessarily an embedding (see [1], p. 11). Hence, we have the following.

Corollary 7 ([2]). *Let f be an embedding of N into an open subset U of $\mathbb{R}^m$, where N is a compact manifold of dimension n. Let $F : U \to \mathbb{R}^\ell$ be a mapping. If $\ell > 2n$, then there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is an embedding.*

4. Composing generic linearly perturbed mappings and embeddings

Let $C^\infty(N, P)$ be the set consisting of all C^∞ mappings of N into P, and the topology on $C^\infty(N, P)$ is the Whitney C^∞ topology (for the definition of Whitney C^∞ topology, see for instance [1]). Then, we say that g is *stable* if the $\mathcal{A}$-equivalence class of g is open in $C^\infty(N, P)$.

Let ${}_sJ^r(N, P)$ be the space consisting of elements $(j^r g(q_1), \ldots, j^r g(q_s)) \in J^r(N, P)^s$ satisfying $(q_1, \ldots, q_s) \in N^{(s)}$, where s is a positive integer. Since $N^{(s)}$ is an open submanifold of N^s, the space ${}_sJ^r(N, P)$ is also an open submanifold of $J^r(N, P)^s$. For a given mapping $g : N \to P$, ${}_sj^r g : N^{(s)} \to {}_sJ^r(N, P)$ is defined by $(q_1, \ldots, q_s) \mapsto (j^r g(q_1), \ldots, j^r g(q_s))$.

Let W be a submanifold of ${}_sJ^r(N, P)$. We say that a mapping $g : N \to P$ is *transverse with respect to* W if ${}_sj^r g : N^{(s)} \to {}_sJ^r(N, P)$ is transverse to W.

Following Mather ([4]), we can partition P^s as follows. For any partition π of $\{1, \ldots, s\}$, let P^π be the set of s-tuples $(y_1, \ldots, y_s) \in P^s$ such that $y_i = y_j$ if and only if two positive integers i and j are in the same member of the partition π.

Let Diff N be the group of diffeomorphisms of N. We have a natural action of Diff $N \times$ Diff P on ${}_sJ^r(N, P)$ such that for a mapping $g : N \to P$, the equality $(h, H) \cdot {}_sj^r g(q) = {}_sj^r(H \circ g \circ h^{-1})(q')$ holds, where $q = (q_1, \ldots, q_s)$ and $q' = (h(q_1), \ldots, h(q_s))$. We say that a subset $W \subset {}_sJ^r(N, P)$ is *invariant* if it is invariant under this action.

We recall the following identification $(*)$ from [4]. Let $q = (q_1, \ldots, q_s) \in N^{(s)}$, let $g : U \to P$ be a mapping defined in a neighborhood U of $\{q_1, \ldots, q_s\}$ in N, and let $z = {}_sj^r g(q)$, $q' = (g(q_1), \ldots, g(q_s))$. Let ${}_sJ^r(N, P)_q$ and ${}_sJ^r(N, P)_{q,q'}$ be the fibers of ${}_sJ^r(N, P)$ over q and over (q, q') respectively. Let $J^r(N)_q$ be the $\mathbb{R}$-algebra of r-jets at q of functions on N. Namely, we have

$$J^r(N)_q = {}_sJ^r(N, \mathbb{R})_q.$$

Set $g^*TP = \bigcup_{\widetilde{q}\in U} T_{g(\widetilde{q})}P$, where TP is the tangent bundle of P. Let $J^r(g^*TP)_q$ denote the $J^r(N)_q$-module of r-jets at q of sections of the bundle g^*TP. Let $\mathfrak{m}_q$ be the ideal in $J^r(N)_q$ consisting of jets of functions which vanish at the point q. Namely, we have

$$\mathfrak{m}_q = \{{}_sj^r h(q) \in {}_sJ^r(N,\mathbb{R})_q \mid h(q_1) = \cdots = h(q_s) = 0\}.$$

Let $\mathfrak{m}_q J^r(g^*TP)_q$ denote the set consisting of finite sums of products of an element of $\mathfrak{m}_q$ and an element of $J^r(g^*TP)_q$. Namely, we have

$$\mathfrak{m}_q J^r(g^*TP)_q = J^r(g^*TP)_q \cap \{{}_sj^r\xi(q) \in {}_sJ^r(N,TP)_q \mid \xi(q_1) = \cdots = \xi(q_s) = 0\}.$$

Then, the following canonical identification of $\mathbb{R}$ vector spaces $(*)$ holds.

$$(*) \qquad T({}_sJ^r(N,P)_{q,q'})_z = \mathfrak{m}_q J^r(g^*TP)_q.$$

Now, let W be a non-empty submanifold of ${}_sJ^r(N,P)$. Choose $q = (q_1,\ldots,q_s) \in N^{(s)}$ and $g : N \to P$, and let $z = {}_sj^r g(q)$ and $q' = (g(q_1),\ldots,g(q_s))$. Suppose that $z \in W$. Set $W_{q,q'} = \widetilde{\pi}^{-1}(q,q')$, where $\widetilde{\pi} : W \to N^{(s)} \times P^s$ is defined by $\widetilde{\pi}({}_sj^r\widetilde{g}(\widetilde{q})) = (\widetilde{q}, (\widetilde{g}(\widetilde{q}_1),\ldots,\widetilde{g}(\widetilde{q}_s)))$ and $\widetilde{q} = (\widetilde{q}_1,\ldots,\widetilde{q}_s) \in N^{(s)}$. Suppose that $W_{q,q'}$ is a submanifold of ${}_sJ^r(N,P)$. Then, from $(*)$, the tangent space $T(W_{q,q'})_z$ can be identified with a vector subspace of $\mathfrak{m}_q J^r(g^*TP)_q$. By $E(g,q,W)$, we denote this vector subspace.

Definition 3. A submanifold W of ${}_sJ^r(N,P)$ is said to be *modular* if conditions (α) and (β) below are satisfied:

- (α) The set W is an invariant submanifold of ${}_sJ^r(N,P)$, and lies over P^π for some partition π of $\{1,\ldots,s\}$.
- (β) For any $q \in N^{(s)}$ and any mapping $g : N \to P$ satisfying ${}_sj^r g(q) \in W$, the subspace $E(g,q,W)$ is a $J^r(N)_q$-submodule.

Now, suppose that $P = \mathbb{R}^\ell$. The main theorem of [4] is the following.

Theorem 3 ([4]). *Let f be an embedding of N into $\mathbb{R}^m$, where N is a manifold of dimension n. If W is a modular submanifold of ${}_sJ^r(N,\mathbb{R}^\ell)$ and $m > \ell$, then there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell) - \Sigma$, $\pi \circ f : N \to \mathbb{R}^\ell$ is transverse with respect to W.*

Theorem 4 ([3]). *Let f be an embedding of N into an open subset U of $\mathbb{R}^m$, where N is a manifold of dimension n. Let $F : U \to \mathbb{R}^\ell$ be a mapping. If W is a modular submanifold of ${}_sJ^r(N,\mathbb{R}^\ell)$, then there exists a subset Σ with Lebesgue measure zero of $\mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell)$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is transverse with respect to W.*

By the same way as in the proof of Theorem 3 of [4], we get the following as a corollary of Theorem 4.

Corollary 8 ([3]). *Let f be an embedding of N into an open subset U of $\mathbb{R}^m$, where N is a compact manifold of dimension n. Let $F : U \to \mathbb{R}^\ell$ be a mapping. If a dimension pair (n,ℓ) is in the nice dimensions, then there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell) - \Sigma$, the composition $F_\pi \circ f : N \to \mathbb{R}^\ell$ is stable.*

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