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On planar portraits of manifolds associated with graphs of block decompositions of manifolds

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Abstract

For a smooth closed manifold M and its stable map f into the plane, the pair $\mathcal{P}_f$ of the image $f(M)$ and the discriminant set of f is called the planar portrait of M through f. In this article, we note an association of $\mathcal{P}_f$ with a graph representing adjacency of building blocks of M, through examples.

1 Introdution

For a smooth closed n-manifold M of $n = \dim M \geq 1$, the discriminant set D_f of its stable map $f\colon M \to \mathbb{R}^2$, or the set of singular values of f, is a collection of smooth loops which may have cusped points and normal crossings. It can be regarded a pictorial representation of M through f, as one may imagine from the shape detection of surfaces embedded in $\mathbb{R}^3$ by their apparent contours produced by the projections into $\mathbb{R}^2$. In this context, the pair $\mathcal{P}_f = (f(M), D_f)$ up to diffeomorphism of $\mathbb{R}^2$ is referred to as the *planar portrait* of M through f ([K1]). However, the precise relation between $\mathcal{P}_f$ and M is yet to be cleared and we will expose an idea that may help us toward the appreciation of the relation. Henceforth manifolds and maps are assumed to be smooth.

First we consider a block decomposition of a manifold as follows. By D^n we denote the closed n-disc, and for an array of a finite number of positive integers $L = (a_1, a_2, \cdots, a_k)$ of $a_1 + a_2 + \cdots + a_k = n$, we use the notations D^L and $\partial_i D^L$ $(i = 1, 2, \cdots, k)$ as

$$\begin{aligned}
D^L &= D^{a_1} \times \cdots \times D^{a_k} \\
\partial_1 D^L &= \partial D^{a_1} \times \cdots \times D^{a_k} \\
\partial_2 D^L &= D^{a_1} \times \partial D^{a_2} \times \cdots \times D^{a_k} \\
&\ \ \vdots \\
\partial_k D^L &= D^{a_1} \times \cdots \times \partial D^{a_k},
\end{aligned}$$

where $\partial_i D^L$ is referred to as the i-th boundary of D^L.

A *block decomposition* of a closed n-manifold M of type L, or into copies of D^L, is a finite decomposition $M = \cup_{r=1}^{l} H_r$ of M into *blocks* H_r such that

1. Each H_r is diffeomorphic to D^L.
2. For distinct r and s, $H_r \cap H_s \subset \partial H_r \cap \partial H_s$.
3. Under the identification of H_r with D^L, the above intersection is the union of the i-th boundary $\partial_i D^L$ for some i.

A simple example of a block decomposition is the one for the sphere S^n into two copies of D^n. The sphere also enjoys another one into two copies of D^L for an arbitrary L of $a_1 + a_2 + \cdots + a_k = n$, since each hemisphere is diffeomorphic to D^L. One more example is the following decomposition of the total space M of an S^a-bundle over S^b (a, $b \geq 1$) with a cross-section. In actual, M is decomposed into two copies of $S^a \times D^b$ by local triviality so that a tubular neighbourhood of the cross-section is decomposed into two copies of $D^a \times D^b$. The rest of the tubular neighbourhood is also decomposed into two copies of $D^a \times D^b$, and these four pieces become the blocks of a block decomposition of M.

For a block decomposition of a manifold $M = \cup H_r$ of type $L = (a_1, a_2, \cdots, a_k)$, an adjacency graph can be considered, where each node represents a block H_r and each link represents an i-th part $\partial_i D^L$ of the intersection $H_r \cap H_s$ of an adjacent pair of blocks. Since each block has k parts $\partial_1 D^L$, $\partial_2 D^L$, $\cdots$, $\partial_k D^L$ in its boundary, the degree of a node of this graph is the constant k, or it is a k-valency graph. It may enjoy multi-links. This graph is referred to as the *network of blocks*.

For the previous block decomposition of S^n into two copies of D^n, the network of blocks is the two nodes linked by a single edge, or it is represented by a line segment (Throughout we denote this network by I, after the unit interval). The network of blocks of the second decomposition of S^n into two copies of $D^a \times D^{n-a}$ is the facet of a 2-gon, or the two nodes multi-linked by two arcs. The network for the finally mentioned decomposition of an S^a-bundle over S^b of a certain kind is the facet of a rectangle.

In the following part, we consider block decompositions of types $L = (a)$, (a, b), or (a, b, c), and make observations on the association of a planar portrait with the network of blocks.

2 Association of $\mathcal{P}_f$ with a network of blocks

Let N be the network of blocks in a block decomposition of a manifold M. We limit our attention to the case where the valency k of N, or the degree of each node, is at most three, as mentioned. For simplicity we consider N represented by a single line segment I ($k = 1$), or by the facet of a convex polygon, including the 2-gon ($k = 2$), if k is 1 or 2. For $k = 3$, we assume that N is represented by a set of distinct points and embedded arcs connecting them so that no arc contains the nodes except its boundary points and that any pair of arcs are either disjoint or have a finite number of normal crossings. A representation of N of $k \geq 2$ has the *compact side*, which is the closure of the union of the bounded components of its compliment in the plane.

We say that a discriminant set D_f of a stable map $f\colon M \to \mathbb{R}^2$ is *associated with* a representation of N if D_f agrees with the curve in $\mathbb{R}^2$ obtained from the representation by the following process, up to diffeomorphism.

1. Take small 2-disc neighbourhood U_i of each node v_i of N so that links abuting v_i is transverse to ∂U_i.
2. Outside $\cup_{i=1}^{l} U_i$, duplicate each link l_j of N so that the result is two disjoint arcs in a thin tubular neighbourhood of l_j.
3. Produce arcs in U_i as illustrated Fig. 1 (a)-(d), according to the valency k, and then connect them to the duplicated links in 2. Namely, the duplicated link is put a cap in U_i at a degree 1 node (Fig. 1(a)), one regular arc and one cusped arc are produced at a degree 2 node (Fig. 1(b)), where the shade in the figure indicates the compact side of the representation of N, and at a degree 3 node, either one regular and two cusped arcs are produced (Fig. 1(c), in case the node is on the boundary of the compact side), or three regular arcs in mutual crossing are produced (Fig. 1(d), otherwise).

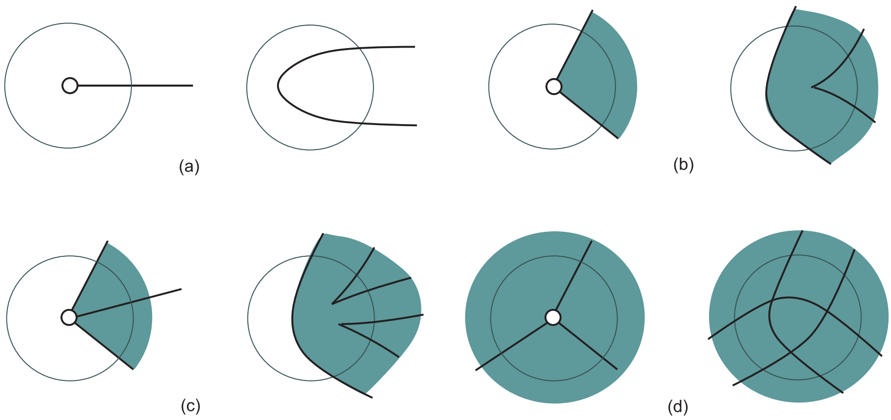

Figure 1: Association at a node: N (left) and D_f (right)

In case the valency $k \geq 2$, the process in Fig. 1 also converts the compact side of N to a compact region bounded by a part of D_f. In case $k = 1$, the representation of N we consider is I, and hence the associated D_f is a regular loop. Therefore one can consider the compact region bounded by D_f also in this case. We say that a planar portrait $\mathcal{P}_f$ is associated to a representation of N if this compact region agrees with $f(M)$ (This formulation of *association* is specialized to the cases considered in this article. One needs a more detailed device than the compact side to specify a compact region that matches $f(M)$, to deal with more general kinds of planar portraits).

Example 2.1. *The network I of blocks of the decomposition of S^n into two copies of D^n admits an associated planar portrait $\mathcal{P}_f = (D^2, \partial D^2)$ of S^n* (Fig. 2 (a))*. That of the decomposition of S^n into two copies of $D^a \times D^{n-a}$ (a is an arbitrary integer of $1 \leq a < n$) admits an associated planar portrait $\mathcal{P}_f$ of S^n* (Fig. 2 (b), [K1, Theorem 2]).

In the following sections, we make observations on samples each of which shows:

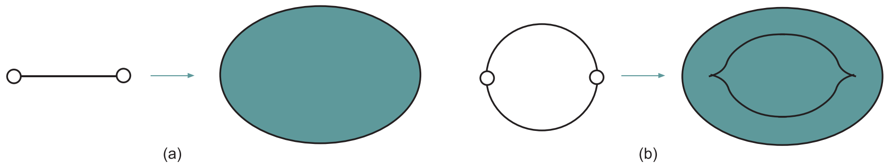

Figure 2: Associations of a network N to a planar portrait $\mathcal{P}_f$

> *A graph manipulation to a network N_0 of blocks of a manifold M_0 implies a planar portrait of a manifold M so that it is associated with a network of blocks of M.*

Generalizations of them will provide a way to obtain new planar portraits recursively, which will support our study of revealing the relevance of the planar portraits to the manifold topology.

3 Observation 1, cone

For a planar representation of a complete graph with r nodes K_r, one obtains a representation of K_{k+1} by adding a new point and linking it to each node of K_r by an edge. Such a representation of K_{r+1} is referred to as a *cone over* K_r.

Some examples show that by taking a cone over K_r one obtains a planar portrait of the projective space P^r which is associated with the network of blocks in a decomposition of P^r into $r+1$ blocks.

Example 3.1. (*Refer to* Fig. 3) *Both $\mathbb{R}P^1$ and $\mathbb{C}P^1$ enjoy the network I for block decompositions of them into two copies of D^1 and D^2, respectively. The triangle facet Δ is a cone over I. We see that Δ is the network of blocks of a decomposition of $M = \mathbb{R}P^2$ or $\mathbb{C}P^2$ into three copies of $D^1 \times D^1$ or $D^2 \times D^2$ which is induced from the decomposition*

$$kP^2 = B_{-1} \cup D^{2i},$$

where $i = 1$ ($k = \mathbb{R}$) or $i = 2$ ($k = \mathbb{C}$) and B_{-1} is the orthogonal D^i-bundle over kP^1 of Euler number -1 (Fig. 4 (a)). *The same block decomposition is obtained also from the decomposition of $S^{3i-1} = \partial(D^i \times D^i \times D^i)$ into three copies of $\partial D^i \times D^i \times D^i$* (Fig. 4 (b)). *One can construct a stable map $f\colon kP^2 \to \mathbb{R}^2$, $k = \mathbb{R}$ or $\mathbb{C}$ so that the planar portrait $\mathcal{P}_f$ is associated with the cone Δ* [K1, Theorem 3]).

Example 3.2. (*Refer to* Fig. 5) *The same observation works for $\mathbb{R}P^3$. Namely, one can consider cones N_1 and N_2 over the network Δ for the block decomposition of $\mathbb{R}P^2$ in Example 3.1. Both N_1 and N_2 are representing the network of four blocks $D^1 \times D^1 \times D^1$ of a block decomposition of $\mathbb{R}P^3$ which is similar to that of $\mathbb{R}P^2$ (Namely, the one induced from the decomposition of $S^3 = \partial(D^1 \times D^1 \times D^1 \times D^1)$ into four copies of $S^0 \times D^1 \times D^1 \times D^1$). We can construct a stable map $f_i\colon \mathbb{R}P^3 \to \mathbb{R}^2$ so that the planar portrait $\mathcal{P}_{f_i}$ is associated with the cone N_i for $i = 1, 2$* ([K2]).

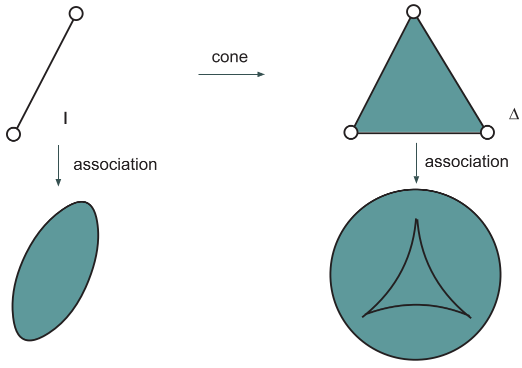

Figure 3: Cone over I implies a planar portrait of kP^2 associated with the network Δ of blocks of kP^2

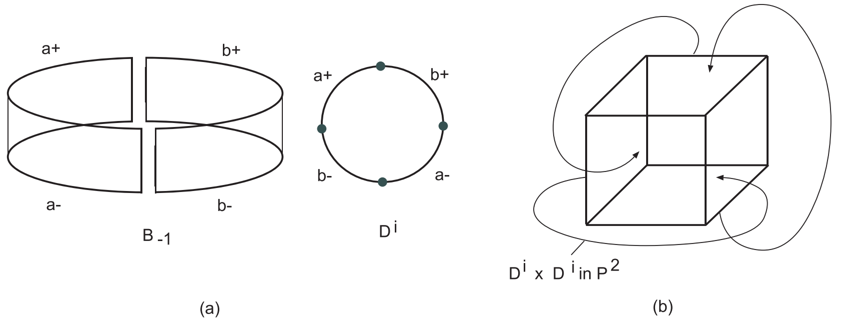

Figure 4: Block decomposition of kP^2 into three $D^i \times D^i$

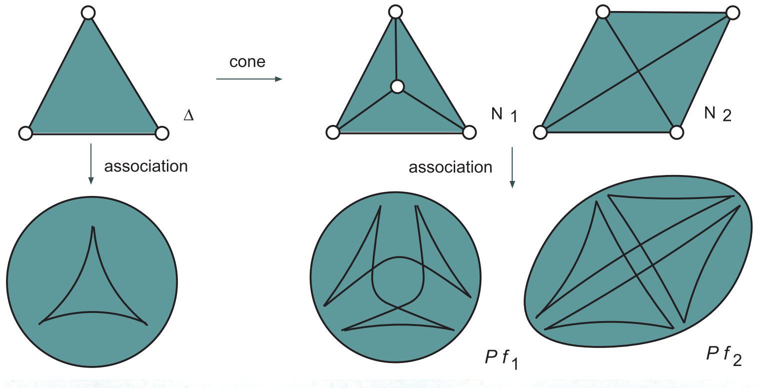

Figure 5: Cone over Δ implies planar portraits of $\mathbb{R}P^3$ associated with the two representations N_1, N_2 of the network K_4 of blocks of $\mathbb{R}P^3$

4 Observation 2, I-product

We continue to denote by I a representation of a graph by a line segment and its two boundary points. For a representation N_0 of a k-valency graph, a representation of a graph is an *I-product* of N_0 if it is a representation of a $k+1$-valency graph obtained from two copies of N_0 by linking each pair of corresponding nodes by an edge so that each pair of corresponding links of N_0 and the links between their boundary points enclose a quadrilateral (Fig. 6).

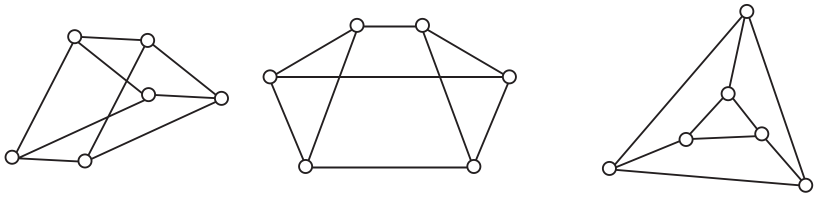

Figure 6: Various I-products of Δ

We give an example that an I-product of a network of blocks of a manifold yields another manifold M and its stable map f into the plane so that the I-product is a network of blocks of M to which the planar portrait $\mathcal{P}_f$ is associated with.

Example 4.1. *Let N be the rectangle facet, which is an obvious I-product of $N_0 = I$. As mentioned, N is the network of blocks of an S^a-bundle over S^b enjoying a cross-section $(a,\, b \geq 1)$ into four $D^a \times D^b$'s, which is common for any such bundle and for arbitrary a and b. One can construct $f\colon M \to \mathbb{R}^2$, where M is the total space of the bundle, so that the portrait $\mathcal{P}_f$ is associated with N* ([K1, Corollary 2]).

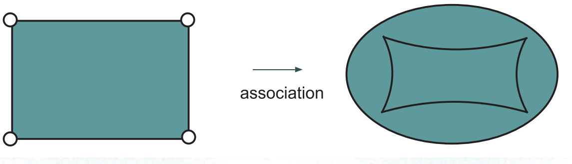

Figure 7: An I-product of I and its associated planar portrait $\mathcal{P}_f$ of S^a-bundle over S^b.

5 Observation 3, I-slide

For a representation N_0 of a k-valency graph, a representation N of a graph is an *I-slide* of N_0 if it is obtained from an I-product of N_0 and a fixed link A in N_0 by collapsing the quadrilateral enclosed by two copies of A and two links between copies of ∂A to the single link A (Fig. 8).

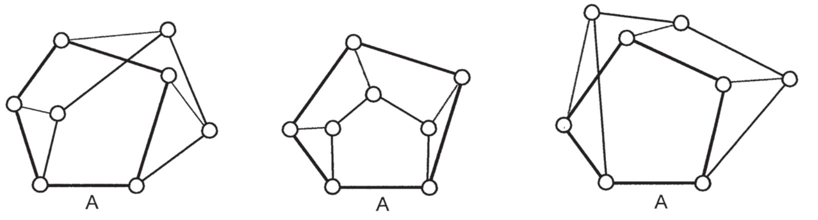

Figure 8: Various I-moves of a pentagon facet (bold)

We give two examples that I-slides of a network of blocks of a manifold yields another manifold M and its stable map f into the plane so that the I-slide is a network of blocks of M to which the planar portrait $\mathcal{P}_f$ is associated with.

Example 5.1. *The triangle facet Δ is the network of the three blocks of $\mathbb{R}P^2$ as mentioned. The two representations of graphs N_1 and N_2 of K_4 in Example 3.2 are both I-slides of Δ* (Fig. 9). *They are both representations of the network K_4 of the four blocks of $\mathbb{R}P^3$ and there exist stable maps $f_i \colon \mathbb{R}P^3 \to \mathbb{R}^2$, $i = 1, 2$ such that the planar portraits $\mathcal{P}_{f_i}$ are associated with N_i, as mentioned.*

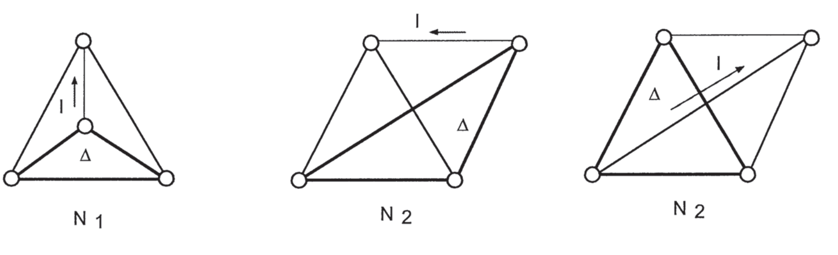

Figure 9: I-moves of Δ

References

[K1] M. Kobayashi, *On the cusped fan in a planar portrait of a manifold*, Geom. Dedicata **162** (2013), 25–43.

[K2] M. Kobayashi and M. Yamamoto, *Views of real projective 3 space by stable maps into the plane* , Experim. Math. **26** (2017) Issue 2, 138–152.