Positivity and fusion of unitary modules for unitary vertex operator algebras

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Abstract

In this expository article, we describe several conjectures arising in the tensor product theory of unitary modules over unitary vertex operator algebras. These conjectures are motivated by the belief that the category of unitary modules over a suitably nice unitary VOA should be a unitary modular tensor category. Of particular interest is the 'positivity conjecture,' which generalizes to non-rational VOAs and provides a candidate for tensor products of modules in this context. This article was prepared for submission to RIMS Kôkyûroku.

Contents

1 Unitarity and positivity conjectures for rational unitary VOAs

Let $\mathcal{V}$ be a simple unitary VOA which is rational and C_2-cofinite. It is widely believed that $\mathrm{Rep}^u(\mathcal{V})$, the category of unitary $\mathcal{V}$-modules, should naturally have the structure of a unitary modular tensor category. This result has recently been confirmed in the examples of type A_n, D_n and G_2 in the work of Bin Gui [Gui17a, Gui17b, Gui]. It is also strongly motivated by the corresponding result for conformal nets [KLM01]. In order to address the general case, however, there are several fundamental questions which need to be addressed first. The following is widely believed, and is necessary to make $\mathrm{Rep}^u(\mathcal{V})$ into a tensor category (using the same tensor product as $\mathrm{Rep}(\mathcal{V})$).

Conjecture 1.1 (Unitary closure conjecture). *Let $\mathcal{V}$ be a simple unitary vertex operator algebra which is rational and C_2-cofinite, and let M and N be unitary $\mathcal{V}$-modules. Then the $P(z)$-tensor product $M \boxtimes_{P(z)} N$ admits a unitary structure.*

In fact, a stronger result may hold.

Conjecture 1.2 (Strong unitarity conjecture)**.** *Let $\mathcal{V}$ be a simple unitary vertex operator algebra which is rational. Then every $\mathcal{V}$-module admits a unitary structure.*

The term *strongly unitary* is sometimes used to describe unitary VOAs for which every module admits a unitary structure. Examples of strongly unitary VOAs include WZW models at positive integral level and Virasoro minimal models. The strong unitarity conjecture asserts that every rational unitary VOA is strongly unitary. Without the assumption of rationality this statement is false, with counterexamples such as Virasoro VOAs with $c \geq 1$.

Of course, to make $\mathrm{Rep}^u(\mathcal{V})$ into a unitary category, it is not enough for the unitary closure conjecture to hold; one must give a specific unitary structure on $M \boxtimes_{P(z)} N$.

There is a natural candidate for such an inner product, which we describe (up to scalar multiple) in the following conjecture.

Conjecture 1.3 (Unitary structure conjecture)**.** *Let $\mathcal{V}$ be a simple unitary vertex operator algebra which is rational and C_2-cofinite, and let M and N be irreducible unitary $\mathcal{V}$-modules. Suppose that $1 > |z| > 2^{-1/2}$. Then there is an inner product on $M \boxtimes_{P(z)} N$ making it into a unitary module which satisfies*

$$\left\langle a_1 \boxtimes_{P(z)} b_1, a_2 \boxtimes_{P(z)} b_2 \right\rangle_{M \boxtimes_{P(z)} N} = \left\langle Y^N(\mathcal{Y}(\tilde{a}_2, \overline{z}^{-1} - z)a_1, z)b_1, b_2 \right\rangle_N, \qquad (1)$$

for some $\mathcal{Y} \in I\binom{\mathcal{V}}{M' M}$, where $\tilde{a}_2 = e^{\overline{z}L_1}(-\overline{z}^{-2})^{L_0}\theta_M a_2$ and $\theta_M : M \to M'$ is the antiunitary isomorphism induced by the inner product.

Note that the condition $|z| < 1$ guarantees that $a \boxtimes_{P(z)} b$ lies in the Hilbert space completion $\mathcal{H}_{M\boxtimes_{P(z)}N}$, using Huang's convergence of products of intertwining operators [Hua05]. The assumption that $|z|$ is not too small ensures that the double sum defining the right-hand side of (1) converges, again by the work of Huang. One could formulate a version of the conjecture for arbitrary $z \in \mathbb{C}^\times$ by using analytic continuation to interpret the right-hand side of (1), and by using the grading to decompose the left-hand side. Also note that since M and N are irreducible, $I\binom{\mathcal{V}}{M' M}$ is one-dimensional, and so a positive answer to the conjecture would specify an $\mathbb{R}^+$ torsor of invariant inner products. To define the unitary structure on $\mathrm{Rep}^u(\mathcal{V})$, one would need to select compatible unitary structures from these torsors for every pair of modules M and N.

We isolate an important consequence of the unitary structure conjecture which is internal to the modules M and N (and does not need to mention $P(z)$-tensor products).

Conjecture 1.4 (Positivity conjecture, rational version)**.** *Let $\mathcal{V}$ be a simple unitary vertex operator algebra which is rational and C_2-cofinite, and let M and N be irreducible unitary $\mathcal{V}$-modules. Suppose that $1 > |z| > 2^{-1/2}$. Then for some non-zero $\mathcal{Y} \in I\binom{\mathcal{V}}{M' M}$ the sesquilinear form on $M \otimes N$ given by*

$$[a_1 \otimes b_1, a_2 \otimes b_2] := \left\langle Y^N(\mathcal{Y}(\tilde{a}_2, \overline{z}^{-1} - z)a_1, z)b_1, b_2 \right\rangle_N, \qquad (2)$$

where $\tilde{a}_2 = e^{\overline{z}L_1}(-\overline{z}^{-2})^{L_0}\theta_M a_2$, is positive semi-definite.

The following result, from our forthcoming article [Ten], highlights the importance of the positivity conjecture.

Theorem 1.5. *Let $\mathcal{V}$ be a simple unitary vertex operator algebra which is rational and C_2-cofinite, and let M and N be irreducible $\mathcal{V}$-modules. Suppose that $\mathcal{V}$ satisfies the unitary closure conjecture, and suppose that $1 > |z| > 2^{-1/2}$. Then there is an invariant, non-degenerate sesquilinear form on $M \boxtimes_{P(z)} N$ such that*

$$\left\langle a_1 \boxtimes_{P(z)} b_1, a_2 \boxtimes_{P(z)} b_2 \right\rangle_{M \boxtimes_{P(z)} N} = \left\langle Y^N(\mathcal{Y}(\tilde{a}_2, \overline{z}^{-1} - z)a_1, z)b_1, b_2 \right\rangle_N,$$

for $\mathcal{Y}$ and $\tilde{a}_2$ as above.

Thus given the unitary closure conjecture and the positivity conjecture, we have a natural construction of a particular invariant inner product (or, at least, a one-dimensional family of inner products). We expect that one may drop the unitary closure conjecture from the hypothesis of this theorem, and it would be desirable to have a proof of such a result. This would show that the positivity conjecture and the unitary closure conjecture are closely related. On the one hand, the improved version of the theorem and the positivity conjecture would imply the unitary closure conjecture. On the other hand, the improved version of the theorem and a counterexample to the positivity conjecture would provide a good candidate for a counterexample to the unitary closure conjecture. Indeed, if the unitary closure closure conjecture fails, it seems like this would not be for the lack of an invariant, non-degenerate sesquilinear form on $M \boxtimes_{P(z)} N$, but only for the lack of positivity of those forms.

A first, essential appearance of the the positivity conjecture is in the work of Wassermann [Was98] on type A WZW models, and later in work of Toledano-Laredo [TL97] and Loke [Lok94] on type D WZW models and Virasoro minimal models, respectively. While the language they used was different than our present discussion, their motivation was the same: to establish the positivity of a certain sesquilinear form. More recently, and in language much closer to ours, Bin Gui established that the positivity conjecture holds for these models, and developed tools for proving positivity in more general examples [Gui17a, Gui17b].

In forthcoming work, we will provide further tools for proving positivity [Ten]. The result is stated in terms of an analytic condition on VOAs called 'bounded localized vertex operators.' At present, this class has only been proven to contain (not necessarily conformal) sub-VOAs of some number of free fermions, which includes many lattice models, Virasoro models, and WZW models. In future work we hope to expand this class, in particular to include all WZW models. The result is as follows.

Theorem 1.6. *Let $\mathcal{W}$ be a simple unitary vertex operator algebra with bounded localized vertex operators, and let $\mathcal{V}$ be a unitary subalgebra of $\mathcal{W}$. Suppose that M and N are $\mathcal{V}$-submodules of $\mathcal{W}$. Then the positivity conjecture holds for the triple $(\mathcal{V}, M, N)$.*

The theorem applies more broadly than previous results, with the most significant difference being that $\mathcal{V}$ is not assumed to be rational. In this case, the convergence of the double series defining the sesquilinear form (2) is part of the statement of the theorem, as it is no longer covered by the work of Huang. In Section 2, we will describe how the positivity conjecture yields a candidate procedure for producing unitary $P(z)$-tensor products of modules when $\mathcal{V}$ is rational, and then in Section 3 we will go on to consider the non-rational case.

2 Unitary construction of $P(z)$-tensor products

Let $\mathcal{V}$ be simple unitary VOA which is rational and C_2-cofinite (or regular, which is equivalent to the two preceding conditions in this context by [ABD04, Thm. 4.5]), and let M and N be unitary $\mathcal{V}$-modules. In this section we will outline a construction of a unitary $\mathcal{V}$-module $M \boxtimes_{P(z)} N$ which relies at various places on unproven conjectures. We do *not*, however, assume the unitary closure conjecture, and this proposed construction can be understood as a strategy to proving the unitary closure conjecture. At the conclusion of the section, we will discuss examples for which all of the relevant conjectures that the construction relies upon have been established, and in Section 3 we will discuss how these ideas suggest a construction of tensor product modules when $\mathcal{V}$ is not necessarily rational.

First, however, we will briefly describe Huang and Lepowsky's construction of $P(z)$-tensor products $M \boxtimes_{P(z)} N$. In fact, Huang and Lepowsky give two constructions. The first is a 'tautological' one [HL95, §12] in which the multiplicity space of irreducible submodules of $M \boxtimes_{P(z)} N$ is defined to be the dual of the space of $P(z)$-intertwining maps of the appropriate type. This construction proves existence of $P(z)$-tensor products, but is difficult to work with.

The second construction of $P(z)$-tensor products proceeds by 'working backwards' and first constructing a pre-dual of $M \boxtimes_{P(z)} N$ [HL95, §13]. One begins by defining an action of $\mathcal{V}$ on the algebraic dual $(M \otimes N)^*$, but this action does not satisfy the necessary axioms to make this space into a $\mathcal{V}$-module. Instead, one considers the smallest subspace $M \boxbackslash_{P(z)} N$ containing all subspaces of $(M \otimes N)^*$ for which the restriction of the $\mathcal{V}$-action makes that subspace into a module. A priori $M \boxbackslash_{P(z)} N$ could be too large to be a $\mathcal{V}$-module, and might only be a generalized module, but the regularity assumptions on $\mathcal{V}$ ensure that $M \boxbackslash_{P(z)} N$ is indeed a $\mathcal{V}$-module. Now the contragredient $(M \boxbackslash_{P(z)} N)'$ is a $P(z)$-tensor product.

One reason to consider the dual $(M \otimes N)^*$ is that it is quite large, since the algebraic tensor product $M \otimes N$ is so small. This provides enough room to find the predual $M \boxbackslash_{P(z)} N$ inside $(M \otimes N)^*$, which is itself too large to be $M \boxbackslash_{P(z)} N$. Our construction below, however, does not'work backwards' and instead works at the level of Hilbert spaces, a comfortable intermediate between small spaces, like VOA modules, and large spaces, like their algebraic completions.

Constructing the Hilbert space

The first step is to construct the Hilbert space which will be the completion of $M \boxtimes_{P(z)} N$, using the inner product suggested by the positivity conjecture. Fix z with $1 > |z| > 2^{-1/2}$, and suppose that $\mathcal{V}$ satisfies the positivity conjecture. Then we define a semidefinite inner product on $M \otimes N$ by

$$[a_1 \otimes b_1, a_2 \otimes b_2] := \left\langle Y^N(\mathcal{Y}(\tilde{a}_2, \overline{z}^{-1} - z)a_1, z)b_1, b_2 \right\rangle_N, \tag{3}$$

for an appropriate $\mathcal{Y} \in I\binom{\mathcal{V}}{M' M}$ and where $\tilde{a}_2 = e^{\overline{z}L_1}(-\overline{z}^{-2})^{L_0}\theta_M a_2$. Note that the series expansions of Y^N and $\mathcal{Y}$ only contain integral powers, and the double sum defining the right-hand side of (3) converges by Huang's work when $1 > |z| > 2^{-1/2}$.

The Hilbert space $\mathcal{H}_\boxtimes$ is defined to be the completion of $M \otimes N$ with respect to this inner product, possibly after quotienting by null vectors. We denote by $\boxtimes_{P(z)} : M \otimes N \to \mathcal{H}_\boxtimes$ the natural inclusion (or, more precisely, quotient) map.

Defining the action and recovering the grading

Next, we discuss how $\mathcal{V}$ acts on $\mathcal{H}_{\boxtimes}$. As in the work of Huang and Lepowsky, there is a canonical candidate to define the action of $\mathcal{V}$ on the range of $\boxtimes_{P(z)}$ in such a way as to make $\boxtimes_{P(z)}$ a $P(z)$-intertwining map. This is given explicitly by

$$v_{(n)}(a \boxtimes_{P(z)} b) := \left(\sum_{m \geq 0} \binom{n}{m} z^{n-m} v_{(m)} a \boxtimes_{P(z)} b \right) + a \boxtimes_{P(z)} v_{(n)} b. \tag{4}$$

Note that the sum has only finitely many non-zero terms, but there is still a subtle question of well-definedness, owing to the potential that $a \boxtimes_{P(z)} b = a' \boxtimes_{P(z)} b'$. We assume that this action is well-defined.

To find the finite energy vectors $M \boxtimes_{P(z)} N$ inside $\mathcal{H}_{\boxtimes}$, we turn to the action of L_0 on $\mathcal{H}_{\boxtimes}$ (or, more precisely, on the image of $\boxtimes_{P(z)}$). One hopes that this operator is essentially self-adjoint, and diagonalizable, in which case the finite energy vectors are given by finite linear combinations of eigenvalues. Next, one must show that these eigenvalues lie in the domains of the closures of the modes $v_{(n)}$ defined above, and that this produces a $\mathcal{V}$-module (in fact, a $P(z)$-tensor product), and that the inner product from $\mathcal{H}_{\boxtimes}$ is invariant.

Examples

For many examples, we can verify that this procedure works [Ten].

Theorem 2.1. *Let $\mathcal{W}$ be a simple unitary VOA with bounded localized vertex operators, let $\mathcal{V}$ be a unitary subalgebra which is rational and C_2-cofinite, and suppose that $\mathcal{V}$ satisfies the unitary closure conjecture. Let M and N be irreducible $\mathcal{V}$-submodules of $\mathcal{W}$. Then the above construction produces a unitary $\mathcal{V}$-module isomorphic to $M \boxtimes_{P(z)} N$.*

For simplicity, one may assume that $\mathcal{W}$ is a tensor product of free fermion VOAs. This allows a broad range of possibilities for $\mathcal{V}$, such as many WZW models, Virasoro minimal models, and many lattice models. In future work we hope to extend this to allow tensor products of free fermion VOAs and copies of E_8 level 1 for $\mathcal{W}$, which would allow $\mathcal{V}$ to be any WZW model. We conjecture that the conclusion of the theorem holds for arbitrary unitary, rational, C_2-cofinite $\mathcal{V}$.

The proof of this theorem relies heavily on the work of Huang and Lepowsky, particularly the rigidity of Rep($\mathcal{V}$) [Hua08]. As a consequence of the proof and further results from that article, we will show that in many cases the inner product constructed on $\mathcal{H}_{M \boxtimes_{P(z)} N}$ agrees up to a scalar with the one arising from Connes' fusion (e.g. in the work of Wassermann [Was98]).

3 Extension to non-rational unitary VOAs

While most of the conjectures from Section 1 cannot be readily generalized outside the context of rational VOAs, the positivity conjecture needs no modification.

Conjecture 3.1 (Non-rational positivity conjecture). *Let $\mathcal{V}$ be a simple unitary vertex operator algebra, and let M and N be irreducible unitary $\mathcal{V}$-modules. Suppose* *that* $1 > |z| > 2^{-1/2}$. *Then for some non-zero* $\mathcal{Y} \in I\binom{\mathcal{V}}{M' M}$ *the sesquilinear form on* $M \otimes N$ *given by*

$$[a_1 \otimes b_1, a_2 \otimes b_2] := \left\langle Y^N(\mathcal{Y}(\tilde{a}_2, \overline{z}^{-1} - z)a_1, z)b_1, b_2 \right\rangle_N,$$

where $\tilde{a}_2 = e^{\overline{z}L_1}(-\overline{z}^{-2})^{L_0}\theta_M a_2$, *is positive semi-definite. Here,* $\theta_M : M \to M'$ *is the antiunitary isomorphism induced by the inner product.*

The convergence of the right-hand side is now a part of the conjecture. Recall from Theorem 1.6 that this conjecture has been established in a broad class of examples.

Given the positivity conjecture, we may attempt to repeat the construction of Section 2 in the non-rational case. Just as before, one defines a positive semidefinite form on $M \otimes N$, and via quotient and completion one obtains a Hilbert space $\mathcal{H}_\boxtimes$ equipped with a linear map $\boxtimes : M \otimes N \to \mathcal{H}_\boxtimes$. There is a unique way to define a $\mathcal{V}$-action on the image of $\boxtimes$ so as to make it a $P(z)$-intertwining map, and we conjecture that L_0 is essentially self-adjoint.

What will certainly change, however, is that L_0 should fail to be diagonalizable in some cases. In examples such as Virasoro with $c \geq 1$, one expects to obtain as tensor products not just direct sums of irreducible modules, but direct integrals. What is required is a species of VOA module for which this construction may be performed, and which produces another module of the same type, in the hopes of defining a tensor category (analogous to the situation with conformal nets, for which rationality plays no role in the definition of tensor category). This will necessarily be somewhere in between ordinary (strong) modules and more general notions like weak modules which may not possess any sort of grading.

If M is a unitary weak $\mathcal{V}$-module, then L_0 defines an unbounded operator on the Hilbert space completion $\mathcal{H}_M$. If L_0 is essentially self-adjoint, we define $\mathcal{H}_M^{\leq k}$ to be the range of the spectral projection for L_0 corresponding to the interval $[0, k]$ (note that L_0 is automatically positive). The compactly supported vectors $\bigcup_{k \geq 0} \mathcal{H}_M^{\leq k}$ are denoted by $\mathcal{H}_M^0$.

Definition 3.2. Let $\mathcal{V}$ be a unitary VOA, and let M be a unitary weak $\mathcal{V}$-module. Then M is called L_0-complete if L_0 is essentially self-adjoint on M and $\mathcal{H}_M^0$ is a core for the closure of $v_{(n)}$, for all $v \in V$ and $n \in \mathbb{Z}$.

In particular, if M is L_0-complete then the closure of $v_{(n)}$ is defined on the Hilbert spaces $\mathcal{H}_M^{\leq k}$, and for general reasons (e.g. the closed graph theorem) the restriction of $v_{(n)}$ to $\mathcal{H}_M^{\leq k}$ will be a bounded map. We hope that the boundedness of these maps will ease any analytic difficulties encountered due to the loss of finite dimensionality and/or semisimplicity.

Returning now to our construction, assuming $\mathcal{V}$ satisfies the positivity conjecture, we have a Hilbert space $\mathcal{H}_\boxtimes$ with actions of $v_{(n)}$. We conjecture that this action of L_0 is essentially self-adjoint, and that $\mathcal{H}_\boxtimes^0$ lies in the domains of the closures of all $v_{(n)}$, and that this is a L_0-complete module.

Conjecture 3.3. *Let* $\mathcal{V}$ *be a simple unitary VOA, and let* M *and* N *be* L_0*-complete unitary* $\mathcal{V}$*-modules. Then the action of* L_0 *on* $\mathcal{H}_\boxtimes$ *is essentially self-adjoint, the domains of the closures of* $v_{(n)}$ *contain* $\mathcal{H}_\boxtimes^0$*, and they make* $\mathcal{H}_\boxtimes^0$ *into a weak* $\mathcal{V}$*-module. Hence* $\mathcal{H}_\boxtimes^0$ *is again a* L_0*-complete unitary* $\mathcal{V}$*-module.*

This is a potential first step to defining a tensor category of $\mathcal{V}$-modules in the spirit of Huang and Lepowsky for such $\mathcal{V}$.

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