数理解析研究所講究録 第2089巻 2018年 105-118

Spectral Analysis of Infinite-dimensional Dirac Operators on an Abstract Boson-Fermion Fock Space

Asao Arai
Department of Mathematics, Hokkaido University
Sapporo 060-0810, Japan
E-mail: arai@math.sci.hokudai.ac.jp

Abstract

A review on spectral analysis of infinite dimensional Dirac type operators on an abstract boson-fermion Fock space is presented.

1 Introduction

For each pair $(\mathscr{H}, \mathscr{K})$ of complex Hilbert spaces, the tensor product Hilbert space

$$\mathscr{F}(\mathscr{H}, \mathscr{K}) := \mathscr{F}_{\mathrm{b}}(\mathscr{H}) \otimes \mathscr{F}_{\mathrm{f}}(\mathscr{K})$$

of the boson Fock space

$$\mathscr{F}_{\mathrm{b}}(\mathscr{H}) := \bigoplus_{n=0}^{\infty} \bigotimes_{\mathrm{s}}^{n} \mathscr{H} = \left\{ \psi = \{\psi^{(n)}\}_{n=0}^{\infty} | \psi^{(n)} \in \bigotimes_{\mathrm{s}}^{n} \mathscr{H}, \sum_{n=0}^{\infty} \|\psi^{(n)}\|^2 < \infty \right\}$$

over $\mathscr{H}$ and the fermion Fock space

$$\mathscr{F}_{\mathrm{f}}(\mathscr{K}) := \bigoplus_{p=0}^{\infty} \bigwedge^{p} \mathscr{K} = \left\{ \phi = \{\phi^{(p)}\}_{p=0}^{\infty} | \phi^{(p)} \in \bigwedge^{p} \mathscr{K}, \sum_{p=0}^{\infty} \|\phi^{(p)}\|^2 < \infty \right\}$$

over $\mathscr{K}$ is defined, where $\otimes_{\mathrm{s}}^{n} \mathscr{H}$ denotes the n-fold symmetric tensor product of $\mathscr{H}$ with $\otimes_{\mathrm{s}}^{0} \mathscr{H} := \mathbb{C}$, $\wedge^{p} \mathscr{K}$ denotes the p-fold anti-symmetric tensor product of $\mathscr{K}$ with $\wedge^{0} \mathscr{K} := \mathbb{C}$ and, for a vector Ψ in a Hilbert space, $\|\Psi\|$ denotes the norm of Ψ. We call the Hilbert space $\mathscr{F}(\mathscr{H}, \mathscr{K})$ the abstract boson-fermion Fock space over $(\mathscr{H}, \mathscr{K})$. In a previous paper [2], the author introduced a general class of infinite-dimensional

Dirac operators on $\mathscr{F}(\mathscr{H},\mathscr{K})$ and clarified general mathematical structures behind some supersymmetric quantum field models giving an abstract unification of them. In particular, a path (functional) integral representation of analytical index of an infinite dimensional Dirac operator was derived, which gives a kind of index theorem. But spectral analysis of the infinite dimensional Dirac operators is still missing. Only partial results are available [10]. In the present paper, we review some aspects of spectral analysis of infinite dimensional Dirac operators.

2 Preliminaries

We first recall basic objects and facts associated with Fock spaces. See [11] for more details.

In general, for a linear operator A from a Hilbert space to a Hilbert space, we denote its domain by $D(A)$.

For each vector $f \in \mathscr{H}$, there is a unique densely defined closed linear operator $a(f)$ on $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$ such that its adjoint $a(f)^*$ takes the following form:

$$D(a(f)^*) = \left\{ \psi \in \mathscr{F}_{\mathrm{b}}(\mathscr{H}) | \sum_{n=1}^{\infty} \|\sqrt{n} S_n(f \otimes \psi^{(n-1)})\|^2 < \infty \right\},$$
$$(a(f)^*\psi)^{(0)} = 0, \quad (a(f)^*\psi)^{(n)} = \sqrt{n} S_n(f \otimes \psi^{(n-1)}),\ n \geq 1,\ \psi \in D(a(f)^*),$$

where S_n denotes the symmetrization operator (symmetrizer) on the n-fold tensor product $\otimes^n \mathscr{H}$ of $\mathscr{H}$. The operator $a(f)$ (resp. $a(f)^*$) is called the boson annihilation (resp. creation) operator with test vector f.

There is a distinguished vector

$$\Omega_{\mathrm{b}} := \{1, 0, 0, \cdots\} \in \mathscr{F}_{\mathrm{b}}(\mathscr{H}),$$

called the boson Fock vacuum in $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$, which is vanished by the annihilation operator:

$$a(f)\Omega_{\mathrm{b}} = 0,\ \forall f \in \mathscr{H}.$$

The set $\{a(f), a(f)^* | f \in \mathscr{H}\}$ of boson annihilation operators and boson creation operators obeys the canonical commutation relations (CCR) over $\mathscr{H}$:

$$[a(f), a(g)^*] = \langle f, g \rangle_{\mathscr{H}}, \quad [a(f), a(g)] = 0, \quad f, g \in \mathscr{H}$$

on the bosonic finite particle subspace

$$\mathscr{F}_{\mathrm{b},0}(\mathscr{H}) := \{\psi \in \mathscr{F}_{\mathrm{b}}(\mathscr{H}) | \exists n_0 \in \mathbb{N} \text{ s.t. } \psi^{(n)} = 0,\ \forall n \geq n_0\},$$

where $[X, Y] := XY - YX$ and $\langle\ ,\ \rangle_{\mathscr{H}}$ denotes the inner product of $\mathscr{H}$ (linear in the second variable).

In general, for a subset $\mathscr{E}$ of a vector space, span($\mathscr{E}$) or span$\,\mathscr{E}$ denotes the subspace generated by all the vectors of $\mathscr{E}$.

It is well known that, for each dense subspace $\mathscr{D}$ of $\mathscr{H}$, the subspace

$$\mathscr{F}_{\mathrm{b,fin}}(\mathscr{D}) := \mathrm{span}\{\Omega_{\mathrm{b}},\, a(f_1)^* \cdots a(f_n)^*\Omega_{\mathrm{b}} | n \in \mathbb{N},\, f_j \in \mathscr{D},\, j = 1, \ldots, n\}$$

is dense in $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$. In fact, one has

$$\mathscr{F}_{\mathrm{b,fin}}(\mathscr{D}) = \hat{\otimes}_{\mathrm{s}}^n \mathscr{D},$$

the algebraic n-fold symmetric tensor product of $\mathscr{D}$.

We next move on to the fermion Fock space $\mathscr{F}_{\mathrm{f}}(\mathscr{K})$. For each $u \in \mathscr{K}$, there is a unique bounded linear operator $b(u)$ on $\mathscr{F}_{\mathrm{f}}(\mathscr{K})$ such that $b(u)^*$ is given as follows:

$$(b(u)^*\phi)^{(0)} = 0, \quad (b(u)^*\phi)^{(p)} = \sqrt{p}A_p(f \otimes \phi^{(p-1)}),\, p \geq 1,\, \phi \in \mathscr{F}_{\mathrm{f}}(\mathscr{K}),$$

where A_p is the anti-symmetrization operator (anti-symmetrizer) on $\otimes^p\mathscr{K}$. The operator $b(u)$ (resp. $b(u)^*$) is called the fermion annihilation (resp. creation) operator with test vector u.

The vector

$$\Omega_{\mathrm{f}} := \{1, 0, 0, \cdots\} \in \mathscr{F}_{\mathrm{f}}(\mathscr{K})$$

is called the fermion Fock vacuum in $\mathscr{F}_{\mathrm{f}}(\mathscr{K})$, which is vanished by $b(u)$:

$$b(u)\Omega_{\mathrm{f}} = 0, \quad \forall u \in \mathscr{K}.$$

The set $\{b(u), b(u)^* | u \in \mathscr{K}\}$ obeys the canonical anti-commutation relations (CAR) over $\mathscr{K}$:

$$\{b(u), b(v)^*\} = \langle u, v\rangle_{\mathscr{K}}, \quad \{b(u), b(v)\} = 0, \quad u, v \in \mathscr{K},$$

where $\{X, Y\} := XY + YX$. It follows that

$$\|b(u)\| = \|u\|, \quad \|b(u)^*\| = \|u\|, \quad b(u)^2 = 0, \quad (b(u)^*)^2 = 0, \quad \forall u \in \mathscr{K},$$

where, for a bounded linear operator T on a Hilbert space, $\|T\|$ denotes the operator norm of T.

For each dense subspace $\mathscr{D}$ of $\mathscr{K}$, the subspace

$$\mathscr{F}_{\mathrm{f,fin}}(\mathscr{D}) := \mathrm{span}\{\Omega_{\mathrm{f}},\, b(u_1)^* \cdots b(u_p)^*\Omega_{\mathrm{f}} \Big| p \in \mathbb{N},\, u_k \in \mathscr{D},\, k = 1, \ldots, p\},$$

is dense in $\mathscr{F}_{\mathrm{f}}(\mathscr{K})$.

3 Exterior Differential Operators on the Boson-Fermion Fock Space

For a linear operator L on a Hilbert space, we set

$$C^\infty(L) := \cap_{n=1}^\infty D(L^n),$$

the C^∞-domain of L. If L is self-adjoint, then $C^\infty(L)$ is dense.

Let A be a densely defined closed linear operator from $\mathscr{H}$ to $\mathscr{K}$. Then, by von Neumann's theorem, A^*A and AA^* are non-negative self-adjoint operators on $\mathscr{H}$ and $\mathscr{K}$ respectively and hence $C^\infty(A^*A)$ and $C^\infty(AA^*)$ are dense in $\mathscr{H}$ and $\mathscr{K}$ respectively. Therefore the algebraic tensor product

$$\mathscr{D}_A^\infty := \mathscr{F}_{\mathrm{b,fin}}(C^\infty(A^*A))\hat{\otimes}\mathscr{F}_{\mathrm{f,fin}}(C^\infty(AA^*))$$

is dense in the boson-fermion Fock space $\mathscr{F}(\mathscr{H},\mathscr{K})$.

Proposition 3.1 *There exists a unique densely defined closed linear operator d_A on $\mathscr{F}(\mathscr{H},\mathcal{K})$ such that the following* (i) *and* (ii) *hold:*

(i) *$\mathscr{D}_A^\infty \subset D(d_A)$ and $\mathscr{D}_A^\infty$ is a core of d_A.*

(ii) *For each vector $\Psi \in \mathscr{D}_A^\infty$ of the form*

$$\Psi = a(f_1)^* \cdots a(f_n)^*\Omega_{\mathrm{b}} \otimes b(u_1)^* \cdots b(u_p)^*\Omega_{\mathrm{f}}, \quad n,\, p \geq 0,$$

where $a(f_1)^* \cdots a(f_n)^*\Omega_{\mathrm{b}}$ (resp. $b(u_1)^* \cdots b(u_p)^*\Omega_{\mathrm{f}}$) with $n = 0$ (resp. $p = 0$) should read Ω_{b} (resp. Ω_{f}), d_A acts as

$$d_A\Psi = 0 \quad \text{for } n = 0,$$
$$d_A\Psi = \sum_{j=1}^n a(f_1)^* \cdots \widehat{a(f_j)^*} \cdots a(f_n)^*\Omega_{\mathrm{b}} \otimes b(Af_j)^* b(u_1)^* \cdots b(u_p)^*\Omega_{\mathrm{f}}$$

for $n \geq 1$, where $\widehat{a(f_j)^*}$ indicates the omission of $a(f_j)^*$. In particular, d_A leaves $\mathscr{D}_A^\infty$ invariant.

Moreover, the following (iii)–(v) *hold:*

(iii) *$\mathscr{D}_A^\infty \subset D(d_A^*)$ and $d_A^*\Psi = 0$ for $p = 0$,*

$$d_A^*\Psi = \sum_{k=1}^p (-1)^{k-1} a(A^*u_k)^* a(f_1)^* \cdots a(f_n)^*\Omega_{\mathrm{b}} \otimes b(u_1)^* \cdots \widehat{b(u_k)^*} \cdots b(u_p)^*\Omega_{\mathrm{f}}$$

for $p \geq 1$. In particular, d_A^* leaves $\mathscr{D}_A^\infty$ invariant.

(iv) $D(d_A^2) = D(d_A)$ *and, for all* $\Psi \in D(d_A)$, $d_A^2\Psi = 0$.

(v) *Let* B *be a bounded linear operator from* $\mathscr{H}$ *to* $\mathcal{K}$ *with* $D(B) = \mathscr{H}$. *Then, for all* $\Psi \in \mathscr{D}_A^\infty$ *and* $\alpha, \beta \in \mathbb{C}$,

$$\alpha d_A \Psi + \beta d_B \Psi = d_{\alpha A + \beta B} \Psi.$$

We call the operator d_A the exterior differential operator on $\mathscr{F}(\mathscr{H}, \mathscr{K})$ associated with A.

4 Infinite Dimensional Dirac Operators

The Dirac operator on $\mathscr{F}(\mathscr{H}, \mathscr{K})$ associated with A is defined by

$$Q_A := d_A + d_A^*.$$

Theorem 4.1 *The operator* Q_A *is self-adjoint and unbounded from above and below.*

The Laplace-Beltrami-de Rham operator on $\mathscr{F}(\mathscr{H}, \mathscr{K})$ associated with A is defined by

$$\Delta_A := d_A^* d_A + d_A d_A^*.$$

Theorem 4.2 $\Delta_A = Q_A^2$.

5 Supersymmetric Structure

Let

$$\mathscr{F}_+ := \mathscr{F}_{\mathrm{b}}(\mathscr{H}) \otimes (\oplus_{p=0}^\infty \wedge^{2p} \mathscr{K}) \quad \text{(even forms)},$$
$$\mathscr{F}_- := \mathscr{F}_{\mathrm{b}}(\mathscr{H}) \otimes (\oplus_{p=0}^\infty \wedge^{2p+1} \mathscr{K}) \quad \text{(odd forms)}.$$

Then we have the orthogonal decomposition

$$\mathscr{F}(\mathscr{H}, \mathscr{K}) = \mathscr{F}_+ \oplus \mathscr{F}_-.$$

Let $P_\pm : \mathscr{F}(\mathscr{H}, \mathscr{K}) \to \mathscr{F}_\pm$ be the orthogonal projections. Then the operator

$$\Gamma := P_+ - P_-.$$

is unitary, self-adjoint and the grading operator for the above orthogonal decomposition.

Proposition 5.1 *(anti-commutativity) Operator equality* $Q_A\Gamma = -\Gamma Q_A$ *holds.*

Corollary 5.2 *(spectral symmetry) The spectrum* $\sigma(Q_A)$ *of* Q_A *is reflection symmetric with respect to the origin of* $\mathbb{R}$*:* $\sigma(Q_A) = \sigma(-Q_A)$.

The quadruple $\mathrm{SQFT}_A := (\mathscr{F}(\mathscr{H},\mathscr{K}), Q_A, \Delta_A, \Gamma)$ is a supersymmetric quantum theory in the abstract sense [1], where Q_A is a self-adjoint supercharge, Δ_A is the supersymmetric Hamiltonian and Γ is the state-sign operator. We remark that SQFT_A gives a unification of some supersymmetric free quantum field models [2, 3, 4, 5, 6].

6 Relations with Second Quantization Operators

For each self-adjoint operator S on $\mathscr{H}$, one can define the bosonic second quantization of S by

$$d\Gamma_{\mathrm{b}}(S) := \oplus_{n=0}^{\infty} d\Gamma_{\mathrm{b}}^{(n)}(S)$$

with

$$d\Gamma_{\mathrm{b}}^{(0)}(S) := 0, \quad d\Gamma_{\mathrm{b}}^{(n)}(S) := \overline{\sum_{j=1}^{n} I\otimes\cdots\otimes I\otimes \underbrace{S}_{j\text{th}} \otimes I\otimes\cdots\otimes I}, \quad n\geq 1,$$

where, for a closable operator T on a Hilbert space, $\overline{T}$ denotes the closure of T. It follows that $d\Gamma_{\mathrm{b}}(S)$ is self-adjoint. If $S \geq 0$, then $d\Gamma_{\mathrm{b}}(S) \geq 0$. Moreover,

$$0 \in \sigma_{\mathrm{p}}(d\Gamma_{\mathrm{b}}(S)), \quad \Omega_{\mathrm{b}} \in \ker(d\Gamma_{\mathrm{b}}(S)).$$

Similarly, for each self-adjoint operator T on $\mathscr{K}$, one can define the fermionic second quantization of T by

$$d\Gamma_{\mathrm{f}}(T) := \oplus_{p=0}^{\infty} d\Gamma_{\mathrm{f}}^{(p)}(T)$$

with

$$d\Gamma_{\mathrm{f}}^{(0)}(T) := 0, \quad d\Gamma_{\mathrm{f}}^{(p)}(T) := \overline{\sum_{j=1}^{p} I\otimes\cdots\otimes I\otimes \underbrace{T}_{j\text{th}} \otimes I\otimes\cdots\otimes I}, \quad p\geq 1.$$

It follows that $d\Gamma_{\mathrm{f}}(T)$ is self-adjoint. If $T \geq 0$, then $d\Gamma_{\mathrm{f}}(T) \geq 0$. Moreover,

$$0 \in \sigma_{\mathrm{p}}(d\Gamma_{\mathrm{f}}(T)), \quad \Omega_{\mathrm{f}} \in \ker(d\Gamma_{\mathrm{f}}(T)).$$

As we have already mentioned, the operator A yields the non-negative self-adjoint operators A^*A and AA^*. Therefore A^*A (resp. AA^*) may be a one-particle Hamiltonian for a boson (resp. fermion). Then the Hamiltonian of a non-interacting system consisting of such bosons and fermions is given by

$$H(A) := d\Gamma_{\mathrm{b}}(A^*A) \otimes I + I \otimes d\Gamma_{\mathrm{f}}(AA^*).$$

It follows that $H(A)$ is a non-negative self-adjoint operator acting in $\mathscr{F}(\mathscr{H}, \mathscr{K})$ and

$$0 \in \sigma_{\mathrm{p}}(H(A)), \quad \Omega_{\mathrm{b}} \otimes \Omega_{\mathrm{f}} \in \ker H(A).$$

Theorem 6.1 $H(A) = \Delta_A$. *In particular, $H(A)$ is a supersymmetric Hamiltonian.*

7 Spectra of $H(A)$ and Q_A

In what follows, we assume that *$\mathscr{H}$ and $\mathscr{K}$ are separable.* For a linear operator T from a Hilbert space to a Hilbert space, we set

$$\mathrm{nul}\, T := \dim\ker T \in \{0\} \cup \mathbb{N} \cup \{+\infty\}.$$

Theorem 7.1

$$\begin{aligned} \sigma(H(A)) &= \{0\} \cup \overline{\left(\bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^{n} \lambda_j \,\middle|\, \lambda_j \in \sigma(A^*A) \setminus \{0\}, j = 1, \cdots, n \right\}\right)}, \\ \sigma_{\mathrm{p}}(H(A)) &= \{0\} \cup \left(\bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^{n} \lambda_j \,\middle|\, \lambda_j \in \sigma_{\mathrm{p}}(A^*A) \setminus \{0\}, j = 1, \cdots, n \right\}\right). \end{aligned}$$

Theorem 7.2 *The spectrum $\sigma(Q_A)$ and the point spectrum $\sigma_{\mathrm{p}}(Q_A)$ of Q_A are symmetric with respect to the origin and*

$$\begin{aligned} \sigma(Q_A) &= \{0\} \cup \overline{\left(\bigcup_{n=1}^{\infty} \left\{ \pm\sqrt{\sum_{j=1}^{n} \lambda_j} \,\middle|\, \lambda_j \in \sigma(A^*A) \setminus \{0\}, j = 1, \cdots, n \right\}\right)}, \\ \sigma_{\mathrm{p}}(Q_A) &= \{0\} \cup \left(\bigcup_{n=1}^{\infty} \left\{ \pm\sqrt{\sum_{j=1}^{n} \lambda_j} \,\middle|\, \lambda_j \in \sigma_{\mathrm{p}}(A^*A) \setminus \{0\}, j = 1, \cdots, n \right\}\right) \end{aligned}$$

with

$$\mathrm{nul}\,(Q_A - \lambda) = \mathrm{nul}\,(Q_A + \lambda), \quad \lambda \in \sigma_{\mathrm{p}}(Q_A).$$

8 A Simple Perturbation

In this section, we consider a simple perturbation of Q_A via a perturbation of d_A. Let

$$g \in D(A) \setminus \{0\}, \quad v \in D(A^*) \setminus \{0\}$$

and

$$d(\alpha) := d_A + \alpha\, a(g) \otimes b(v)^*.$$

with a constant $\alpha \in \mathbb{C}$ being a perturbation parameter. It is easy to see that $d(\alpha)$ is densely defined with $D(d(\alpha)) \supset \mathscr{D}_A^\infty$ and

$$d(\alpha)^2 = 0 \quad \text{on } \mathscr{D}_A^\infty.$$

Moreover, $d(\alpha)^*$ is densely defined with $\mathscr{D}_A^\infty \subset D(d(\alpha)^*)$ and

$$d(\alpha)^* = d_A^* + \alpha^* a(g)^* \otimes b(v) \quad \text{on } \mathscr{D}_A^\infty.$$

Hence $d(\alpha)$ is closable. We denote the closure of $d(\alpha) \restriction \mathscr{D}_A^\infty$ by $\bar{d}(\alpha)$.

Lemma 8.1 *For all* $\Psi \in D(\bar{d}(\alpha))$, $\bar{d}(\alpha)\Psi$ *is in* $D(\bar{d}(\alpha))$ *and*

$$\bar{d}(\alpha)^2\Psi = 0.$$

Using the operator $\bar{d}(\alpha)$, one can define a perturbed Dirac operator:

$$Q(\alpha) := \bar{d}(\alpha) + \bar{d}(\alpha)^*.$$

We note that

$$Q(\alpha) = Q_A + V_{g,v}(\alpha) \quad \text{on } \mathscr{D}_A^\infty$$

with

$$V_{g,v}(\alpha) := \alpha\, a(g) \otimes b(v)^* + \alpha^* a(g)^* \otimes b(v).$$

8.1 Self-adjointness of $Q(\alpha)$

Let $T_{g,v} : \mathscr{H} \to \mathscr{K}$ be defined by

$$T_{g,v} f := \langle g, f \rangle\, v, \quad f \in \mathscr{H}.$$

It is obvious that $T_{g,v}$ is a bounded linear operator (a one-rank operator). Hence

$$A(\alpha) := A + \alpha T_{g,v}$$

is a densely defined closed linear operator with $D(A(\alpha)) = D(A)$.

Remark 8.2 *Perturbations of a linear operator by one-rank or two-rank operators have been studied in various contexts. See, e.g. [12, 13] and references therein.*

Lemma 8.3 *(a key lemma) For all $\alpha \in \mathbb{C}$, the following operator equality holds:*

$$\bar{d}(\alpha) = d_{A(\alpha)}.$$

Theorem 8.4

(i) *For all $\alpha \in \mathbb{C}$, $Q(\alpha)$ is self-adjoint and*

$$Q(\alpha) = Q_{A(\alpha)}.$$

(ii) *For all $\alpha \in \mathbb{C}$, $Q(\alpha)$ is essentially self-adjoint on $\mathscr{D}_A^\infty$.*

(iii) *For all $\alpha \in \mathbb{C}$,*

$$Q(\alpha) = \overline{Q_A + V_{g,v}(\alpha)}.$$

(iv) *The operator Γ leaves $D(Q(\alpha))$ invariant and*

$$\Gamma Q(\alpha) + Q(\alpha)\Gamma = 0 \quad \text{on } D(Q(\alpha)).$$

(v) *For all $\Psi \in \mathscr{D}_A^\infty$, the vector-valued function:$\alpha \mapsto Q(\alpha)\Psi$ is strongly continuous on $\mathbb{C}$. Moreover, for all $z \in \mathbb{C} \setminus \mathbb{R}$, $(Q(\alpha) - z)^{-1}$ is strongly continuous in $\alpha \in \mathbb{C}$.*

8.2 Spectra of $Q(\alpha)$

Theorem 8.5 *For all $\alpha \in \mathbb{C}$, $\sigma(Q(\alpha))$ and $\sigma_{\mathrm{p}}(Q(\alpha))$ are symmetric with respect to the origin and*

$$\sigma(Q(\alpha)) = \{0\} \bigcup \overline{\left(\bigcup_{n=1}^{\infty} \left\{ \pm\sqrt{\sum_{j=1}^{n} \lambda_j} \middle| \lambda_j \in \sigma(A(\alpha)^* A(\alpha)) \setminus \{0\}, j = 1, \cdots, n \right\} \right)},$$

$$\sigma_{\mathrm{p}}(Q(\alpha)) = \{0\} \bigcup \left(\bigcup_{n=1}^{\infty} \left\{ \pm\sqrt{\sum_{j=1}^{n} \lambda_j} \middle| \lambda_j \in \sigma_{\mathrm{p}}(A(\alpha)^* A(\alpha)) \setminus \{0\}, j = 1, \cdots, n \right\} \right)$$

with

$$\mathrm{nul}\,(Q(\alpha) - \lambda) = \mathrm{nul}\,(Q(\alpha) + \lambda), \quad \lambda \in \sigma_{\mathrm{p}}(Q(\alpha)).$$

This theorem shows that the spectrum and the point spectrum of $Q(\alpha)$ are completely determined from those of $A(\alpha)^* A(\alpha) \setminus \{0\}$.

8.3 Identification of the domain of $Q(\alpha)$

Recall that $|A| := (A^*A)^{1/2}$ acting in $\mathscr{H}$. It follows that A is injective if and only if $|A|$ is injective.

Theorem 8.6 *Suppose that A is injective and $g \in D(|A|^{-1})$. Then, for all $|\alpha| < 1/(\|v\| \, \| \, |A|^{-1}g\|)$, $Q(\alpha)$ is self-adjoint with $D(Q(\alpha)) = D(Q_A)$ and*

$$Q(\alpha) = Q_A + V_{g,v}(\alpha).$$

Moreover, $Q(\alpha)$ is essentially self-adjoint on any core for Q_A.

Proof. The essential part of the proof is to show that $V_{g,v}(\alpha)$ is Q_A-bounded with a relative upper bound $|\alpha| \, |v| \, \| \, |A|^{-1}g\|$. Then one needs only to apply the Kato-Rellich theorem. For more details, see the proof of [10, Theorem 17]. ∎

9 Kernel of $Q(\alpha)$

We now investigate the kernel of $Q(\alpha)$. We need a classification for conditions on $\{A, g, v\}$:

(C.1) A is injective, $v \in D(A^{-1})$ and $\langle g, A^{-1}v\rangle \neq 0$. In this case we introduce a constant

$$\alpha_0 := -\frac{1}{\langle g, A^{-1}v\rangle}. \tag{9.1}$$

(C.2) A^* is injective, $g \in D(A^{*-1})$ and $\left\langle v, A^{*-1}g\right\rangle \neq 0$. In this case we introduce a constant

$$\beta_0 := -\frac{1}{\left\langle A^{*-1}g, v\right\rangle}.$$

(C.3) (a) A is injective and $v \notin D(A^{-1})$ or (b) A is injective and $v \in D(A^{-1})$ with $\langle g, A^{-1}v\rangle = 0$.

(C.4) (a) A^* is injective and $g \notin D(A^{*-1})$ or (b) A^* is injective $g \in D(A^{*-1})$ with $\left\langle v, A^{*-1}g\right\rangle = 0$.

We first consider the kernel of $A(\alpha)$ and $A(\alpha)^*$.

Lemma 9.1

(i) *Suppose that (C.1) holds. Then*

$$\begin{aligned} \ker A(\alpha) &= \{0\}, \quad \alpha \neq \alpha_0, \\ \ker A(\alpha_0) &= \{cA^{-1}v | c \in \mathbb{C}\}. \end{aligned}$$

(ii) *Suppose that (C.2) holds. Then*

$$\begin{aligned} \ker A(\alpha)^* &= \{0\}, \quad \alpha \neq \beta_0, \\ \ker A(\beta_0)^* &= \{cA^{*-1}g | c \in \mathbb{C}\}. \end{aligned}$$

(iii) *Suppose that (C.3) holds. Then, for all* $\alpha \in \mathbb{C}$,

$$\ker A(\alpha) = \{0\}.$$

(iv) *Suppose that (C.4) holds. Then, for all* $\alpha \in \mathbb{C}$,

$$\ker A(\alpha)^* = \{0\}.$$

Theorem 9.2

(i) *Assume (C.1). Then*

$$\ker Q(\alpha_0) = \oplus_{n,p=0}^{\infty}[(\otimes^n \{zA^{-1}v | z \in \mathbb{C}\}) \otimes \wedge^p(\ker A(\alpha_0)^*)].$$

and hence $\operatorname{nul} Q(\alpha_0) = \infty$.

Moreover, for all $\alpha \neq \alpha_0$,

$$\ker Q(\alpha) = \oplus_{p=0}^{\infty} \mathbb{C} \otimes \wedge^p(\ker A(\alpha)^*).$$

(ii) *Assume (C.2). Then*

$$\begin{aligned} \ker Q(\beta_0) &= \oplus_{n=0}^{\infty} \left\{ [\otimes_{\mathrm{s}}^n \ker(A(\beta_0))] \otimes [\mathbb{C} \oplus \operatorname{span}(\{A^{*-1}g\})] \right\}, \\ \ker Q(\alpha) &= \oplus_{n=0}^{\infty} \left[\otimes_{\mathrm{s}}^n \ker A(\alpha) \otimes \mathbb{C} \right], \quad \alpha \neq \beta_0. \end{aligned}$$

(iii) *Assume (C.3) . Then, for all* $\alpha \in \mathbb{C}$,

$$\ker Q(\alpha) = \oplus_{p=0}^{\infty} \left[\mathbb{C} \otimes \wedge^p(\ker(A(\alpha)^*) \right].$$

(iv) *Assume (C.4). Then, for all* $\alpha \in \mathbb{C}$,

$$\ker Q(\alpha) = \oplus_{n=0}^{\infty} \left[\otimes_{\mathrm{s}}^n \ker A(\alpha) \otimes \mathbb{C} \right].$$

Corollary 9.3

(i) *Assume (C.1) and (C.2). Then*

$$\begin{aligned} \ker Q(\alpha_0) &= \overline{\operatorname{span}\left(\left\{ a(A^{-1}v)^{*n}\Omega_{\mathrm{b}} \otimes b(A^{*-1}g)^{*j}\Omega_{\mathrm{f}} | n \geq 0, j = 0, 1 \right\} \right)}, \\ \ker Q(\alpha) &= \{c\Omega_{\mathrm{b}} \otimes \Omega_{\mathrm{f}} | c \in \mathbb{C}\}, \quad \alpha \neq \alpha_0. \end{aligned}$$

(ii) *Assume (C.1) and (C.4). Then*

$$\ker Q(\alpha_0) = \overline{\text{span}\,(\{a(A^{-1}v)^{*n}\Omega_\text{b} \otimes \Omega_\text{f} | n \geq 0\})},$$
$$\ker Q(\alpha) = \{c\Omega_\text{b} \otimes \Omega_\text{f} | c \in \mathbb{C}\}, \quad \alpha \neq \alpha_0.$$

(iii) *Assume (C.2) and (C.3). Then*

$$\ker Q(\beta_0) = \text{span}(\{\Omega_\text{b} \otimes b(A^{*-1}g)^{*^j}\Omega_\text{f} | j = 0, 1\}).$$
$$\ker Q(\alpha) = \{c\Omega_\text{b} \otimes \Omega_\text{f} | c \in \mathbb{C}\}, \quad \alpha \neq \beta_0.$$

(iv) *Assume (C.3) and (C.4). Then, for all* $\alpha \in \mathbb{C}$,

$$\ker Q(\alpha) = \{c\Omega_\text{b} \otimes \Omega_\text{f} | c \in \mathbb{C}\}.$$

10 Non-zero Eigenvalues of $Q(\alpha)$

Hypothesis (A)

(i) $\mathscr{H} = \mathcal{K}$;

(ii) A is an injective and nonnegative self-adjoint operator;

(iii) $g = v \in D(A^{-1})$.

Under Hypothesis (A), the constant α_0 defined by (9.1) takes the form

$$\alpha_0 = -\frac{1}{\langle v, A^{-1}v\rangle} < 0.$$

Theorem 10.1 *Let Hypothesis (A) be satisfied and* $\alpha < \alpha_0$ (< 0). *Then, there exists a unique constant* $x_0(\alpha) < 0$ *such that* $\alpha \langle v, (x_0(\alpha) - A)^{-1}v\rangle = 1$ *and, for all* $n \in \{0\} \cup \mathbb{N}$,

$$\pm\sqrt{n}x_0(\alpha) \in \sigma_\text{p}(Q(\alpha)).$$

with eigenvectors

$$[Q(\alpha) \pm \sqrt{n}x_0(\alpha)]\left\{a(\phi_\alpha)^{*n-p}\Omega_\text{b} \otimes b(\phi_\alpha)^{*p}\Omega_\text{f}\right\}$$
$$\in \ker(Q(\alpha) \mp \sqrt{n}x_0(\alpha)) \ (n \geq p \geq 0),$$

where

$$\phi_\alpha := (x_0(\alpha) - A)^{-1}v.$$

Moreover, $x_0(\alpha)$, *as a function of* $\alpha < \alpha_0$, *is strictly monotone increasing on* $(-\infty, \alpha_0)$ *with* $\lim_{\alpha\to-\infty} x_0(\alpha) = -\infty$ *and* $\lim_{\alpha\to\alpha_0} x_0(\alpha) = 0$.

Note that Theorem 10.1 holds even if Q_A has no non-zero eigenvalues. This is an interesting phenomenon. Since the condition $\alpha < \alpha_0 < 0$ implies that $|\alpha| > |\alpha_0|$, the phenomenon may be regarded as a strong coupling effect.

Acknowledgement

This work is supported by KAKENHI 15K04888 from JSPS.

References

[1] A. Arai, Supersymmetry and singular perturbations, *J. Funct. Anal.* **60** (1985), 378–393.

[2] A. Arai, A general class of infinite-dimensional Dirac operators and path integral representation of their index, *J. Funct. Anal.* **105** (1992), pp.342–408.

[3] A. Arai, Dirac operators in Boson-Fermion Fock spaces and supersymmetric quantum field theory, *J. Geome. Phys.*, **11** (1993), 465–490.

[4] A. Arai, Supersymmetric extension of quantum scalar field theories, *Quantum and Noncommutative Analysis* (H. Araki et al, eds.), Kluwer Academic Publishers, Dordrecht, 73-90, 1993.

[5] A. Arai, On self-adjointness of Dirac operators in boson-fermion Fock spaces, *Hokkaido Math. J.* **23** (1994), 319–353.

[6] 新井朝雄, 超対称的場の量子論と無限次元解析, 数学 **46** (1994), 1–10.

A. Arai, Supersymmetric quantum field theory and infinite-dimensional analysis [translation of Sugaku 46 (1994), no. 1, 1–10], *Sugaku Expositions* **9** (1996), no. 1, 87–98.

[7] A. Arai, Operator-theoretical analysis of a representation of a supersymmetry algebra in Hilbert space, *J. Math. Phys.* **36** (1995), 613–621.

[8] A. Arai, Strong anti-commutativity of Dirac operators on Boson-Fermion Fock spaces and representations of a supersymmetry algebra, *Math. Nachr.* **207** (1999), 61–77.

[9] A. Arai, Infinite dimensional analysis and analytic number theory, *Acta Applicandae Mathematicae* **63** (1999), 41–78.

[10] A. Arai, A special class of infinite dimensional Dirac operators on the abstract boson-fermion Fock space, *J. Math.* Vol. 2014, Article ID 713690, 13 pages.

[11] 新井朝雄,『フォック空間と量子場　上下 [増補改訂版]』, 日本評論社, 2017.

A. Arai, *Fock Spaces and Quantum Fields I, II* (revised and expanded version), Nippon-hyoron-sha, 2017, in Japanese.

[12] A. Kula, M. Wojtylak and J. Wysoczański, Rank two perturbations of matrices and operators and operator model for t-transformation of probability measures, *J. Funct. Anal.* **272** (2017), no. 3, 1147–1181.

[13] B. Simon, Spectral analysis of rank one perturbations and applications. Mathematical quantum theory II. Schrödinger operators, 109–149, CRM Proc. Lecture Notes, 8, Amer. Math. Soc., Providence, RI, 1995.