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Construction of wave operators for Hartree equations with a critical Hardy potential

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1. Introduction

In this article we consider the following Hartree equations with a Hardy potential:

$$\textbf{(HE)}_a \qquad \begin{cases} i\,\dfrac{\partial u}{\partial t} = \Big(-\Delta + \dfrac{a}{|x|^2}\Big)u + u\,(K * |u|^2) & \text{in } \mathbb{R}\times\mathbb{R}^N, \\ u(0) = u_0, \end{cases}$$

where $i = \sqrt{-1}$, $N \geq 3$ and

$$a \geq a(N) := -\frac{(N-2)^2}{4}.$$

$K * |u|^2$ is the usual convolution

$$(K * |u|^2)(x) := \int_{\mathbb{R}^N} K(x-y)|u(y)|^2\,dy.$$

We suppose some conditions for K for analyzing $\textbf{(HE)}_a$:

(K1) K is real and even function, that is, $K(-x) = K(x) \in \mathbb{R}$ a.a. $x \in \mathbb{R}^N$;

(K2) $K \in L^\infty(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ with $q > N/4$ and $q \geq 1$;

(K2a) $K \in L^{q_1}(\mathbb{R}^N) + L^{q_2}(\mathbb{R}^N)$ with $q_1 \geq 1$ and $N/4 < q_1 < q_2 \leq N/2$;

(K3) $K_- := \max\{-K, 0\} \in L^\infty(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ with $q > N/2$.

(K3a) $K \geq 0$ and $\widetilde{K} := 2K + x\cdot\nabla K \leq 0$;

(K4) $\widetilde{K} \in L^\infty(\mathbb{R}^N) + L^{\widetilde{q}}(\mathbb{R}^N)$ with $\widetilde{q} > N/4$ and $\widetilde{q} \geq 1$;

(K4a) $\widetilde{K} \in L^{\widetilde{q}_1}(\mathbb{R}^N) + L^{\widetilde{q}_2}(\mathbb{R}^N)$ with $\widetilde{q}_1 \geq 1$ and $N/4 < \widetilde{q}_1 < \widetilde{q}_2 < N/2$.

Note that **(K2a)**, **(K3a)**, and **(K4a)** imply **(K2)**, **(K3)**, and **(K4)**, respectively.

In general, semilinear (or nonlinear) Schrödinger equations is described strongly dispersive effects of waves, for example, propagation of signals in optical fibers. Especially, $\textbf{(HE)}_a$ (without a linear potential term $a|x|^{-2}u$) represents nonlocal interaction, for example, Hartree–Fock theory and WKB approximation for multi-body Schrödinger equation. On the other hand, the linear operator $P_a := -\Delta + a|x|^{-2}$ arises from both physics and mathematics. In physical sides, P_a is concerned with quantum mechanics (Calogero–Moser system), wave propagation on conic manifolds, and combustion theory. On the one hand, in mathematical sides, P_a is concerned with scaling symmetry and the presense of threshold for the nonnegativity and selfadjointness. Since $P_a[u(\lambda x)] = \lambda^2(P_a u)(\lambda x)$, we see that $\textbf{(HE)}_a$ can not be reduced to the case with $|a|$ and $\|u_0\|_{H^1}$ small enough. This implies that the term $a|x|^{-2}$ represents non-negligible effect. Moreover, the restriction of a is affected by *the Hardy inequality*

$$\frac{(N-2)^2}{4}\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^2}\,dx \leq \int_{\mathbb{R}^N}|\nabla u(x)|^2\,dx.$$

Here the coefficient $(N-2)^2/4$ is optimal.

Here one of keys for analyzing $(\mathbf{HE})_a$ is the energy class $\mathcal{D} := D((1+P_a)^{1/2})$. If $a > a(N)$, then $\mathcal{D}$ is just equal to the usual Sobolev space $H^1(\mathbb{R}^N)$. If $a = a(N)$, then $\mathcal{D}$ is a wider space than $H^1(\mathbb{R}^N)$. Thus we denote $X^1(\mathbb{R}^N)$ as $\mathcal{D}$ with $a = a(N)$. Note that $H^1(\mathbb{R}^N) \subsetneq X^1(\mathbb{R}^N) \subsetneq H^s(\mathbb{R}^N)$ $(s<1)$. In fact,

$$\|(-\Delta)^{s/2} f\|_{L^2} \le \frac{\Gamma\big((N+2s)/4\big)\,\Gamma\big((1-s)/2\big)}{\Gamma\big((N-2s)/4\big)\,\Gamma\big((1+s)/2\big)} \|P_{a(N)}^{s/2} f\|_{L^2},$$

where Γ denotes the gamma functions (see Suzuki [**11**, Theorem 3.2]). We also denote $\mathcal{D}^*$ as the conjugate space of $\mathcal{D}$. Thus $H^1(\mathbb{R}^N)^* = H^{-1}(\mathbb{R}^N)$ and $X^1(\mathbb{R}^N)^* = X^{-1}(\mathbb{R}^N)$.

Local and global well-posedness for $(\mathbf{HE})_a$ is proved in [**9**] for $a > a(N)$ and [**11**] for $a = a(N)$.

Proposition 1.1. *Let $a \ge a(N)$. Assume* **(K1)** *and* **(K2)***. Then for any $u_0 \in \mathcal{D}$ there uniquely exists a local weak solution $u \in C([-T,T];\mathcal{D}) \cap C^1([-T,T];\mathcal{D}^*)$ to* $(\mathbf{HE})_a$*. Moreover, u satisfies conservation laws:*

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad E(u(t)) = E(u_0) \quad \forall\, t \in [-T,T], \tag{1.1}$$

where

$$\begin{aligned} E(\varphi) &:= \frac{1}{2}\|P_a^{1/2}\varphi\|_{L^2}^2 + \frac{1}{4}G[K](\varphi) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} \Big[|\nabla\varphi|^2 + \frac{a}{|x|^2}|\varphi|^2\Big]\,dx + \frac{1}{4}\iint_{\mathbb{R}^{N+N}} K(x-y)|\varphi(x)|^2|\varphi(y)|^2\,dxdy. \end{aligned}$$

Furthermore, **(K3)** *yields that the local weak solution of* $(\mathbf{HE})_a$ *can be extended to the global weak solution $u \in C(\mathbb{R};\mathcal{D}) \cap C^1(\mathbb{R};\mathcal{D}^*)$.*

If $u_0 \in \mathcal{D}$ belongs also to $D(|x|)$, that is, $|x|u_0 \in L^2(\mathbb{R}^N)$, then the local weak solution $u \in C([-T,T];\mathcal{D}) \cap C^1([-T,T];\mathcal{D}^*)$ to $(\mathbf{HE})_a$ also belongs to $C([-T,T];D(|x|))$. In fact, we see by a simple calculation that

$$\frac{d}{dt}\|xu(t)\|_{L^2}^2 = 4\,\mathrm{Im}\int_{\mathbb{R}^N} \overline{x\,u(t,x)} \cdot \nabla u(t,x)\,dx.$$

Here the evaluation of $\|xu(t)\|_{L^2}$ is available by assuming further **(K4)**. Actually, we call the identity about $\|xu(t)\|_{L^2}$ *the virial identity.* The identity plays important roles in global analysis for $(\mathbf{HE})_a$:

$$\begin{aligned} \frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 &= 8\|P_a^{1/2}u(t)\|_{L^2}^2 \\ &\quad - 2\iint_{\mathbb{R}^N} (x-y)\cdot\nabla K(x-y)|u(t,x)|^2|u(t,y)|^2\,dxdy \end{aligned} \tag{1.2}$$

(see [**10**, Section 3] for $a > a(N)$ and [**13**, Section 3] for $a = a(N)$). Applying (1.1) we obtain

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = 16\,E(u_0) - 2\iint_{\mathbb{R}^N} \widetilde{K}(x-y)|u(t,x)|^2|u(t,y)|^2\,dxdy. \tag{1.3}$$

We can prove the finite time blowing up for $(\mathbf{HE})_a$ via the virial identity (see [**10**, Theorem 1.2] for $a > a(N)$ and [**13**, Theorem 4.1] for $a = a(N)$).

Proposition 1.2. *Let $a \geq a(N)$. Assume* **(K1)**, **(K2)**, **(K4)**, *and $\widetilde{K} \geq 0$. Then for any $u_0 \in \Sigma := \mathcal{D} \cap D(|x|)$ with $E(u_0) < 0$ the local weak solution $u \in C([-T,T];\Sigma) \cap C^1([-T,T];\mathcal{D}^*)$ to* **(HE)**$_a$ *cannot be extended globally in time $t \in \mathbb{R}$. More precisely, there exist $T_1, T_2 > 0$ such that*

$$\lim_{t\to T_1-0} \|P_a^{1/2}u(t)\|_{L^2} = \infty, \qquad \lim_{t\to -T_2+0} \|P_a^{1/2}u(t)\|_{L^2} = \infty.$$

We are interested in the asymptotic behavior of the global solutions to **(HE)**$_a$. Note that the solutions are oscillating owing to the presence of i in the evolution equation **(HE)**$_a$ and the conservation laws (1.1). Thus we consider the existence of the following limits

$$u_\pm = \lim_{t\to\pm\infty} \exp(itP_a)u(t).$$

We say that **(HE)**$_a$ is *asymptotically free* in Σ if the limits exist for any solution to **(HE)**$_a$ with initial data belonging to Σ. The inverse mappings $W_\pm : u_\pm \mapsto u_0$ may be also considered. The maps $W_\pm$ are called *the wave operators* for **(HE)**$_a$. To construct $W_\pm$ we need to solve the following *final value* problems associated to **(HE)**$_a$:

$$\textbf{(FVP)} \qquad \begin{cases} i\,u_t = P_a u + u\,(K * |u|^2) & \text{in } (0,\infty), \\ \lim\limits_{t\to\infty} \exp(itP_a)u(t) = u_+ & \text{strongly in } \Sigma. \end{cases}$$

In a way similar to Hayashi–Tsutsumi [4] we can apply the *pseudo-conformal transform* also to **(HE)**$_a$ and **(FVP)**:

$$u(t,x) := (\mathcal{C}v)(t,x) = (it)^{-N/2}\exp\Big(\frac{i|x|^2}{4t}\Big)\overline{v\Big(\frac{1}{t},\frac{x}{t}\Big)}.$$

By simple calculations we see that

$$\begin{aligned} &\exp(-itP_a)D_\nu = D_\nu \exp(-\nu^2 tP_a) \quad \forall\, t\in\mathbb{R}, \nu > 0, \\ &\exp(-itP_a)M_b = M_{b/(1+bt)}D_{1/(1+bt)}\exp\Big(-\frac{it}{1+bt}P_a\Big) \quad \forall\, t\in\mathbb{R}, b\in\mathbb{R} \text{ with } 1+bt>0, \end{aligned}$$

where $(D_\nu u)(x) := \nu^{N/2}u(\nu x)$ and $(M_b u)(x) := \exp(ib|x|^2/4)u(x)$. Thus we can rewrite $(\mathcal{C}v)(t,x) = i^{-N/2}M_{1/t}D_{1/t}\overline{v(t^{-1},x)}$. Note that we need to set Σ not as $\mathcal{D}$ but as $\mathcal{D}\cap D(|x|)$ (weighted energy space) so that the transform $\mathcal{C}$ works well. In fact,

$$\|\nabla u(t)\|_{L^2} = \Big\|\Big(\frac{x}{2}+\frac{i}{t}\nabla\Big)v(t^{-1})\Big\|_{L^2}, \quad \big\||x|^\omega u(t)\big\|_{L^2} = |t|^\omega\big\||x|^\omega v(t^{-1})\big\|_{L^2} \quad t\in\mathbb{R}.$$

By applying D_ν and M_b, we have

$$\exp\big(-i(1-t)P_a\big)u(t,x) = i^{-N/2}D_1M_1\overline{\exp\big(-i(1-t^{-1})P_a\big)v(t^{-1},x)}.$$

Letting $t\to\infty$ we see

$$\exp(-iP_a)u_+ = i^{-N/2}M_1\overline{\exp(-iP_a)v(0,x)}. \tag{1.4}$$

Thus **(FVP)** is converted into the following *initial value* problems:

$$\textbf{(IVP)}\qquad \begin{cases} i\, v_t = P_a v + t^{-2}\, v\,(D_{1/t}K * |v|^2), & \text{in } (0,\infty),\\ v(0) = v_+ := i^{-N/2}\exp(iP_a)e^{i|x|^2/4}\exp(iP_a)\overline{u_+} & \text{in } \Sigma. \end{cases}$$

If we solve **(IVP)**, we can also solve **(FVP)**. Thus we can construct the wave operators for $\mathbf{(HE)}_a$. Suzuki [**12**] proved the scattering problems for $\mathbf{(HE)}_a$ with the specified case by applying the contraction methods.

Proposition 1.3. *Let* $K(x) := |x|^{-\gamma}$.
(i) *Assume that* $a \geq a(N)$ *and* $1 < \gamma < \min\{N, 4\}$. *Then for every global solution* $u \in C(\mathbb{R};\Sigma)\cap C^1(\mathbb{R};\mathcal{D}^*)$ *to* $\mathbf{(HE)}_a$ *there exists* $u_+ \in L^2(\mathbb{R}^N)$ *such that* $\exp(itP_a)u(t) \to u_+$ $(t \to \infty)$ *strongly in* $L^2(\mathbb{R}^N)$;
(ii) *Assume either* $a > a(N)$ *and* $1 < \gamma \leq 2$, *or* $a > (\gamma-2)^2/4 + a(N)$ *and* $2 < \gamma < \min\{N, 4\}$. *Then for every* $u_+ \in \Sigma$ *there exists a global solution* $u \in C(\mathbb{R};\Sigma)\cap C^1(\mathbb{R};\mathcal{D}^*)$ *to* $\mathbf{(HE)}_a$ *such that* $\exp(itP_a)u(t) \to u_+$ $(t\to\infty)$ *strongly in* Σ.

In this article we prove the scattering problems for $\mathbf{(HE)}_a$ under more generalized cases via the energy methods.

Theorem 1.4. *Let* $a \geq a(N)$. *Assume* **(K1)**, **(K2a)**, **(K3)**, *and* **(K4a)**. *Then for any* $u_+ \in \Sigma$ *there uniquely exists a solution* $u \in C(\mathbb{R};\Sigma)\cap C^1(\mathbb{R};\mathcal{D}^*)$ *to* **(FVP)**. *Thus the wave operator* $W_+ : u_+ \mapsto u(0)$ *is well-defined in* Σ.

On the contrary, we can show the asymptotic free in Σ of $\mathbf{(HE)}_a$ in an almost similar way to Theorem 1.4.

Theorem 1.5. *Let* $a \geq a(N)$. *Assume that* **(K1)**, **(K2a)**, **(K3a)**, *and* **(K4a)**. *Then for any global solution* $u \in C(\mathbb{R};\Sigma)\cap C^1(\mathbb{R};\mathcal{D}^*)$ *to* $\mathbf{(HE)}_a$ *there exist the following limits*

$$\lim_{t\to\pm\infty} \exp(itP_a)u(t) = u_\pm \quad \textit{strongly in } \Sigma.$$

Thus Theorems 1.4 and 1.5 imply that $\mathbf{(HE)}_a$ is *asymptotically complete* in Σ, that is, $W_\pm$ are bijective in Σ and the scattering operator $\mathcal{S} := W_+^{-1}\circ W_-$ is well-defined.

This article is divided into 4 sections. In Section 2, we give the abstract theory related to **(IVP)**. In Section 3, we show Theorems 1.4 and 1.5 via the energy methods proposed in Section 2. In Section 4, we remark some comments about scattering problems for $\mathbf{(HE)}_a$.

2. Abstract theory for nonlinear Schrödinger equations

Let S be a nonnegative selfadjoint operator in a complex Hilbert space X. Put $X_S := D((1+S)^{1/2})$. Then we have the usual triplet: $X_S \subset X = X^* \subset X_S^*$. Under this setting S can be extended to a nonnegative selfadjoint operator in X_S^* with domain X_S. Now we consider the abstract nonautonomous semilinear Schrödinger equations:

$$\textbf{(ACP)}\qquad \begin{cases} i\dfrac{du}{dt} = Su + g(t,u) \quad t \in (-T,T),\\ u(0) = u_0 \in X_S. \end{cases}$$

$g(t,u)$ is a nonlinearity mapping from $[-T,T]\times X_S$ to X_S^* under the following conditions. For simple notation we denote $B_M := \{u \in X_S; \|u\|_{X_S} \le M\}$. Moreover, $\varphi \in L^p(-T,T)$ $(p>1)$ is a nonnegative function.

(A1) Existence of energy functional of g: for all $t \in [-T,T]$, $u \in X_S$, and $\varepsilon > 0$ there exists $\delta = \delta(u,\varepsilon) > 0$ such that

$$|G(t,u+v) - G(t,v) - \mathrm{Re}\,\langle g(t,u), v\rangle_{X_S^*,X_S}| \le \varepsilon \|v\|_{X_S} \quad \forall\, v \in B_\delta;$$

(A2) Local Lipschitz continuity of u-variable:

$$\|g(t,u) - g(t,v)\|_{X_S^*} \le C(M)\|u-v\|_{X_S} \quad \forall\, t \in [-T,T],\ \forall\, u,\, v \in B_M;$$

(A3) Hölder continuity of t-variable:

$$\|g(t,u) - g(s,u)\|_{X_S^*} \le C(M)\Big|\int_s^t \varphi(\sigma)d\sigma\Big| \quad \forall\, t,\, s \in [-T,T],\ \forall\, u \in B_M;$$

(A4) Hölder-like continuity of energy functional:

$$|G(t,u) - G(t,v)| \le \delta + C_\delta(M)\|u-v\|_X \quad \forall\, \delta > 0,\ \forall\, t \in [-T,T],\ \forall\, u,\, v \in B_M;$$

(A5) Partial differentiability of energy functional and Hölder-like continuity of u-variable:

$$|G_t(t,u) - G_t(t,v)| \le \varphi(t)\,[\delta + C_\delta(M)\|u-v\|_X] \quad \text{a.a. } t \in (-T,T),\ \forall\, u,\, v \in B_M;$$

(A6) Gauge type condition for the conservation of charge:

$$\mathrm{Re}\,\langle g(t,u), iu\rangle_{X_S^*,X_S} = 0 \quad \forall\, t \in [-T,T],\ \forall\, u \in X_S;$$

(A7) Weak closedness condition: let $I \subset (-T,T)$ be an open interval and $\{w_n\}_n \subset L^\infty(I;X_S)$. Then

$$\begin{cases} w_n(t) \to w(t)\ (n\to\infty) & \text{weakly in } X_S \text{ a.a. } t \in I, \\ g(t,w_n(t)) \to f(t)\ (n\to\infty) & \text{weakly* in } L^\infty(I;X_S^*) \end{cases}$$

$$\Rightarrow \int_I \mathrm{Re}\,\langle f(t), i\,w(t)\rangle_{X_S^*,X_S}\,dt = \lim_{n\to\infty}\int_I \mathrm{Re}\,\langle g(t,w_n(t)), i\,w_n(t)\rangle_{X_S^*,X_S}\,dt. \tag{2.1}$$

Moreover, if $w_n(t) \to w(t)$ $(n\to\infty)$ strongly in X a.a. $t\in I$, then $f(t) = g(t,w(t))$;

(A8) Boundedness from below of G: there exists $\varepsilon > 0$ such that

$$G(t,u) \ge -[(1-\varepsilon)/2]\,\|S^{1/2}u\|_X^2 - C(\|u\|_X) \quad \forall t \in [-T,T],\ \forall\, u \in X_S;$$

(A9) Boundedness from below of G_t: there exists $\psi \in L^1(-T,T)$ with $\psi(t) \ge 0$ such that

$$\mathrm{sgn}(t)\,G_t(t,u) \le \psi(t)[\|S^{1/2}u\|_X^2 + C(\|u\|_X)] \quad \text{a.a. } t \in (-T,T),\ \forall\, u \in X_S.$$

If g maps unilaterally, from $[0,T]\times X_S$ to X_S^*, then we consider the even extension:

$$g(t,u) := g(|t|,u), \quad G(t,u) := G(|t|,u) \quad \forall\, t \in [-T,T].$$

Theorem 2.1 (Energy methods). *Assume* **(A1)–(A7)**. *Then for any* $u_0 \in X_S$ *there exists a local solution* $u \in C_{\mathrm{w}}([-T_0, T_0]; X_S) \cap W^{1,\infty}(-T_0, T_0; X_S^*)$ *to* **(ACP)** *with the following conservation laws*

$$\|u(t)\|_X = \|u_0\|_X, \quad E(t, u(t)) - E(0, u_0) \le \int_0^t G_t(s, u(s))\, ds \quad \forall\, t \in [-T_0, T_0],$$

where $E(t,u) := (1/2)\,\|S^{1/2}u\|_X^2 + G(t, u(t))$. *Moreover, assume further* **(A8)** *and* **(A9)**. *Then the solution* u *can be extended globally in time* $t \in [-T, T]$.

Remark 2.1. We need to prove uniqueness for **(ACP)** by another method. In fact, we verify the uniqueness for $\mathbf{(HE)}_a$ and **(IVP)** by applying the Strichartz estimates (see Lemma 3.2). Here the uniqueness yields that the energy inequality of E is just an equality. Hence the solution u is strongly continuous: $u \in C([-T_0, T_0]; X_S) \cap C^1([-T_0, T_0]; X_S^*)$.

One of the keys for proving of Theorem 2.1 is the theory of nonautonomous semilinear evolution equation. Let X be a (complex-valued) Hilbert Banach space and A be a linear maximal monotone operator in X, that is, $R(1 + A) = X$ and $\mathrm{Re}\,\langle Au, u\rangle_X \ge 0$. Then $-A$ generates contraction C_0-semigroups $\{e^{-tA}; t \ge 0\} \subset \mathcal{B}(X)$, the family of bounded linear operators on X. Now we consider

$$\begin{cases} \dfrac{du}{dt} + Au + g_0(t, u) = 0 & \text{in } [0, T] \times X, \\ u(0) = u_0. \end{cases} \tag{2.2}$$

Assume that g_0 satisfies

(H1) Lipschitz continuity of g_0 in u: for all $t \in [0, T]$, and for any u, $v \in X$ with $\|u\|_X \le M$ and $\|v\|_X \le M$

$$\|g_0(t, u) - g_0(t, v)\|_X \le C(M)\|u - v\|_X;$$

(H2) Hölder-like continuity of g_0 in t: there exists $\varphi \in L^p(0, T)$ $(p > 1)$ with $\varphi(t) \ge 0$ such that for all t, $s \in [0, T]$ and for any $u \in X$ with $\|u\|_X \le M$

$$\|g_0(t, u) - g_0(s, u)\|_X \le C(M)\left|\int_s^t \varphi(\sigma) d\sigma\right|.$$

In a way similar to Cazenave–Haraux [**3**, Propositions 4.3.2 and 4.3.9] we can show the unique existence of solution to (2.2):

Lemma 2.2. *Assume* **(H1)** *and* **(H2)**. *Let* $u_0 \in D(A)$. *Then there uniquely exists* $u \in C([0, T_0]; D(A)) \cap C^1([0, T_0]; X)$ *such that* u *is the local solution to* (2.2). *Here* $T_0 \in (0, T]$ *is determined by* $\|u_0\|_X$.

Proof. Unique existence of local solutions $u \in C([0, T_0]; X)$ to (2.2) are followed by **(H1)** with a standard contraction argument for the integral equation related to (2.2):

$$u(t) = \Phi[u](t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} g_0(s, u(s))\, ds.$$

It remains to show that the regularity of solution. Thus let $u_0 \in D(A)$ and $u \in C([0,T_0];X)$ be a local unique solution to the above integral equation. Set $h > 0$ sufficiently small and $t \in [0, T_0 - h]$. We divide $u(t+h) - u(t) = \Phi[u](t+h) - \Phi[u](t)$ into I_0, I_1, I_2, and I_3 as follows:

$$
\begin{aligned}
I_0 &:= e^{-(t+h)A}u_0 - e^{-tA}u_0, \\
I_1 &:= \int_0^t e^{-sA}[g(t+h-s, u(t+h-s)) - g(t+h-s, u(t-s))]\, ds, \\
I_2 &:= \int_0^t e^{-sA}[g(t+h-s, u(t-s)) - g(t-s, u(t-s))]\, ds, \\
I_3 &:= \int_0^h e^{-(t+h-s)A} g(s, u(s))\, ds.
\end{aligned}
$$

First we see for I_0 as a standard evaluation:

$$\|e^{-(t+h)A}u_0 - e^{-tA}u_0\|_X \leq h\|Au_0\|_X.$$

We can evaluate the norm of I_1 by applying **(H1)**:

$$
\begin{aligned}
\|I_1\|_X &\leq \int_0^t \|g(t+h-s, u(t+h-s)) - g(t+h-s, u(t-s))\|_X\, ds \\
&\leq \int_0^t C(M)\|u(t+h-s) - u(t-s)\|_X\, ds = C(M)\int_0^t \|u(s+h) - u(s)\|_X\, ds.
\end{aligned}
$$

Next we consider I_2. Applying **(H2)** we have

$$
\begin{aligned}
\|I_2\|_X &\leq \int_0^t \|g(t+h-s, u(t-s)) - g(t-s, u(t-s))\|_X\, ds \\
&\leq \int_0^t C(M)\Big[\int_{t-s}^{t+h-s} \varphi(\sigma)\, d\sigma\Big]\, ds.
\end{aligned}
$$

Here the last integral is estimated by changing the order of integration (see Figure 1).

$$
\begin{aligned}
&\int_0^t \Big[\int_{t-s}^{t+h-s} \varphi(\sigma)\, d\sigma\Big]\, ds \\
&= \int_0^h \Big[\int_{t-\sigma}^{t} \varphi(\sigma)\, ds\Big]\, d\sigma + \int_h^t \Big[\int_{t-\sigma}^{t-\sigma+h} \varphi(\sigma)\, ds\Big]\, d\sigma + \int_t^{t+h} \Big[\int_0^{t-\sigma+h} \varphi(\sigma)\, ds\Big]\, d\sigma \\
&= \int_0^h \sigma\varphi(\sigma)\, d\sigma + \int_h^t h\varphi(\sigma)\, d\sigma + \int_t^{t+h} (t-\sigma+h)\varphi(\sigma)\, d\sigma \\
&\leq \int_0^h h\varphi(\sigma)\, d\sigma + \int_h^t h\varphi(\sigma)\, d\sigma + \int_t^{t+h} h\varphi(\sigma)\, d\sigma \leq h\int_0^T \varphi(\sigma)\, d\sigma.
\end{aligned}
$$

Thus we obtain

$$\|I_2\|_X \leq C(M)h\int_0^T \varphi(\sigma)\, d\sigma.$$

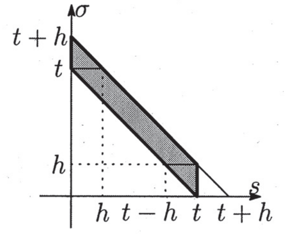

Figure 1: integration on I_2

Next we evaluate I_3 as follows:

$$\begin{aligned}\|I_3\|_X &\le \int_0^h [\|g_0(s,u(s)) - g_0(s,0)\|_X + \|g_0(s,0)\|_X]\,ds \\ &\le h[C(M)M + \|g(\cdot,0)\|_{C([0,T];X)}].\end{aligned}$$

Combining the evaluation for I_j $(j = 0,1,2,3)$, we obtain

$$\|u(t+h) - u(t)\|_X \le C'(M)h + C(M)\int_0^t \|u(s+h) - u(s)\|_X\,ds,$$

where

$$C'(M) := C(M)M + \|g(\cdot,0)\|_{C([0,T];X)} + \int_0^T \varphi(\sigma)\,d\sigma.$$

The Gronwall lemma implies

$$\|u(t+h) - u(t)\|_X \le C'(M)\,h\,e^{C(M)t}.$$

Since u is globally Lipschitz continuous in $[0,T_0]$, $u \in W^{1,\infty}(0,T_0;X)$.

Next we show $u \in C([0,T_0];D(A)) \cap C^1([0,T_0];X)$. To derive this, it sufficient to show that the nonlinear term $g(t,u(t))$ belongs to $W^{1,p}(0,T;X)$ $(p > 1)$:

$$\begin{aligned}&\|g(t+h,u(t+h)) - g(t,u(t))\|_X \\ &\le \|g(t+h,u(t+h)) - g(t+h,u(t))\|_X + \|g(t+h,u(t)) - g(t,u(t))\|_X \\ &\le C(M)\Big|\int_t^{t+h}\varphi(\sigma)\,d\sigma\Big| + C(M)\|u(t+h) - u(t)\|_X \\ &\le \Big|\int_t^{t+h}[C(M)\varphi(\sigma) + C(M)C'(M)e^{C(M)T}]d\sigma\Big|.\end{aligned}$$

By virtue of Cazenave–Haraux [**3**, Proposition 4.1.6], we have proved the regularity of (local) weak solution to (2.2): $u \in C([0,T];D(A)) \cap C^1([0,T];X)$. ∎

Note that semilinear Schrödinger evolution equations can be solved backward and forward. Now we consider

$$\text{(2.3)}\qquad \begin{cases} i\dfrac{du}{dt} = Su + g_0(t,u) & \text{in } [-T,T] \times X, \\ u(0) = u_0 \end{cases}$$

Assume that g_0 satisfies **(H1)**, **(H2)** (with replacing $[0,T]$ by $[-T,T]$), and

(H3) Existence of energy functional: there exists $G_0 \in C([-T,T] \times X; \mathbb{R})$ such that for all $t \in [-T,T]$, $u \in X$, and $\varepsilon > 0$ there exists $\delta = \delta(u,\varepsilon) > 0$ such that

$$|G_0(t,u+v) - G_0(t,u) - \operatorname{Re}\langle g_0(t,u), v\rangle_X| \leq \varepsilon \|v\|_X \quad \forall\, v \in X \text{ with } \|v\|_X \leq \delta;$$

(H4) Hölder-like continuity of G_{0t}: $G_0(t,u)$ is partially differentiable in t for any $u \in X$. Moreover, for any $u \in X$ with $\|u\|_X \leq M$

$$|G_{0t}(t,u) - G_{0t}(t,v)| \leq \varphi(t)[\delta + C_\delta(M)\|u-v\|_X] \quad \text{a.a. } t \in (-T,T);$$

(H5) Gauge type condition:

$$\operatorname{Re}\langle g_0(t,u), i\,u\rangle_X = 0 \quad \forall\, t \in [-T,T],\ \forall\, u \in X.$$

Apply Lemma 2.2 with letting $A := \pm iS$ and replacing $g_0(t,u)$ by $\pm i\, g_0(\pm t, u)$ (double-sign corresponds). Thus **(H1)** and **(H2)** yield the unique existence of local solution $u \in C([-T_0,T_0]; D(S)) \cap C^1([-T_0,T_0]; X)$ to (2.3). **(H3)**–**(H5)** imply the conservation laws:

$$\|u(t)\|_X = \|u_0\|_X, \quad E_0(t,u(t)) = E_0(0,u_0) + \int_0^t G_{0t}(s,u(s))\,ds, \tag{2.4}$$

where $E_0(t,u) := (1/2)\,\|S^{1/2}u\|_X^2 + G_0(t,u)$. More precisely, **(H5)** implies the charge conservation (the former of (2.4)); **(H3)** and **(H4)** imply the energy conservation (the latter of (2.4)); By virtue of the conservation laws (2.4), the local solution can be extended globally in time t: $u \in C([-T,T]; D(S)) \cap C^1([-T,T]; X)$. Finally, arguments of denseness (see [**2**, Theorem 3.3.1]) follow the assertion.

Lemma 2.3. *Assume* **(H1)**–**(H5)**. *Then for any $u_0 \in X_S$ there uniquely exists the global solution of* (2.3) *$u \in C([-T,T]; X_S) \cap C^1([-T,T]; X_S^*)$. Moreover, u satisfies the conservation laws* (2.4).

2.1. Outline of proof Theorem 2.1

Theorem 2.1 is proved in [**14**]. Now we give the outline of proof. In a way similar to [**7**] we divide into 5 steps as follows:

Step 1. Construct a global and approximated solution of **(ACP)**:

$$\textbf{(ACP)}_\varepsilon \qquad \begin{cases} i\dfrac{du_\varepsilon}{dt} = Su_\varepsilon + g_\varepsilon(t,u_\varepsilon) & t \in (-T,T), \\ u(0) = u_0 \in X_S, \end{cases}$$

where $g_\varepsilon(t,u) := (1+\varepsilon S)^{-1} g(t,(1+\varepsilon S)^{-1}u)$. Since g_ε maps from $[-T,T] \times X$ to X, we can apply Lemma 2.3. **(H1)**–**(H5)** are verified by **(A2)**, **(A3)**, **(A1)**, **(A5)** and **(A6)**, respectively. Here $u_\varepsilon \in C([-T,T]; X_S) \cap C^1([-T,T]; X_S^*)$ satisfies the following conservation laws:

$$\|u_\varepsilon(t)\|_X = \|u_0\|_X, \quad E_\varepsilon(t,u_\varepsilon(t)) = E_\varepsilon(0,u_0) + \int_0^t \partial_t G_\varepsilon(s,u_\varepsilon(s))\,ds,$$

where $G_\varepsilon(t,u) := G(t,(1+\varepsilon S)^{-1}u)$ and

$$E_\varepsilon(t,u) := \frac{1}{2}\|(1+S)^{1/2}u\|_X^2 + G_\varepsilon(t,u).$$

Step 2. Evaluate $\|(1+S)^{1/2}u_\varepsilon(t)\|_X$ uniformly in $t \in [-T_M, T_M]$ and in $\varepsilon > 0$. This is the same way to [7]. To end this, we need to assume further **(A4)**.

Step 3. Confirm the weak convergence of **(ACP)**$_\varepsilon$ to **(ACP)**. By virtue of Step 2, there exists the limit function u of u_ε, which satisfies

$$\begin{cases} i\dfrac{du}{dt} = Su + f(t) \quad t \in (-T_M, T_M), \\ u(0) = u_0 \in X_S. \end{cases}$$

Here $f(t)$ is the weak* limit of $g_\varepsilon(t, u_\varepsilon(t))$ in $L^\infty(-T_M, T_M; X_S^*)$.

Step 4. Check the charge conservation and make a solution. By virtue of former half of **(A7)**, we obtain that

$$\mathrm{Re} \int_{-T_M}^{T_M} \langle f(t), i\,u(t)\rangle_{X_S^*, X_S}\, dt = 0.$$

This yields the charge conservation $\|u(t)\|_X = \|u_0\|_X$. Next, the charge conservation implies the strong convergence of u_ε in X. By virtue of latter half of **(A7)**, we see $f(t) = g(t, u(t))$. Hence we can show that the limit function $u(t)$ is a just solution to **(ACP)**.

Step 5. Verify the energy pseudo-conservation. Weak convergence of $u_\varepsilon(t)$ to $u(t)$ in X_S and strong convergence of $u_\varepsilon(t)$ to $u(t)$ in X yield the energy pseudo-conservation.

3. Verifications of asymptotic completeness

3.1. Proof of Theorem 1.4 (exisitence of wave operators)

To show Theorem 1.4, we prove the following assertion.

Proposition 3.1. *Let $a \ge a(N)$. Assume* **(K1)**, **(K2a)**, *and* **(K4a)**. *Then for any $v_+ \in \mathcal{D}$ there uniquely exists a local weak solution $v \in C([-T,T];\mathcal{D}) \cap C^1([-T,T];\mathcal{D}^*)$ to* **(IVP)**. *Moreover, v satisfies*

$$\|v(t)\|_{L^2} = \|v_+\|_{L^2},$$

$$E(t, v(t)) = E(0, v_+) - \int_0^t \frac{1}{4s^3}\Big[\iint_{\mathbb{R}^{N+N}} \widetilde{K}\Big(\frac{x-y}{|s|}\Big)|v(s,x)|^2|v(s,y)|^2\, dxdy\Big]\, ds,$$

where

$$E(t,u) := \frac{1}{2}\|P_a^{1/2}u\|_{L^2}^2 + \frac{1}{4t^2}\iint_{\mathbb{R}^{N+N}} K\Big(\frac{x-y}{|t|}\Big)|u(x)|^2|u(y)|^2\, dxdy.$$

Furthermore, if $v_+ \in \Sigma = \mathcal{D} \cap D(|x|)$, then v belongs also to $C([-T,T];\Sigma)$.

To confirm Proposition 3.1 we check the uniqueness and the conditions **(A1)**–**(A7)** and apply Theorem 2.1. We define $X := L^2(\mathbb{R}^N)$, $S := P_a$, $X_S := \mathcal{D}$,

$$g(t,v) := t^{-2}v\,(D_{1/|t|}K * |v|^2) = t^{-2}v \int_{\mathbb{R}^N} K\Big(\frac{x-y}{|t|}\Big)|v(y)|^2\, dy, \tag{3.1}$$

$$G(t,v) := \frac{1}{4}G[t^{-2}D_{1/|t|}K](v) = \frac{1}{4t^2}\iint_{\mathbb{R}^{N+N}} K\Big(\frac{x-y}{|t|}\Big)|v(x)|^2|v(y)|^2\, dxdy. \tag{3.2}$$

$\mathcal{D}$ is the energy space related to P_a:

$$\|u\|_{\mathcal{D}} := \left[\int_{\mathbb{R}^N}\left(|u|^2+|\nabla u|^2+\frac{a}{|x|^2}|u|^2\right)dx\right]^{1/2}, \quad \mathcal{D}=\begin{cases}H^1(\mathbb{R}^N) & a>a(N),\\ X^1(\mathbb{R}^N) & a>a(N).\end{cases}$$

The Sobolev type embeddings are available: $\mathcal{D}\subset L^r(\mathbb{R}^N)$ $(2\le r<2N/(N-2))$, more precisely,

$$\|u\|_{L^r}\le c(r)\|u\|_{L^2}^{1-\theta}\|u\|_{\mathcal{D}}^{\theta}\le c(r)\|u\|_{\mathcal{D}} \quad \forall\, u\in L^{r'}(\mathbb{R}^N), \tag{3.3}$$

where

$$\frac{1}{r}=\frac{1}{2}-\frac{\theta}{N},\ 0<\theta<1$$

(see Suzuki [**11**, Section 4] for $a=a(N)$). Here we denote

$$\|u\|_{L^p(\Omega)} := \begin{cases}\left(\int_\Omega |u(x)|^p\,dx\right)^{1/p} & 1\le p<\infty,\\ \operatorname{ess.\,sup}|u(x)| & p=\infty.\end{cases}$$

If $\Omega=\mathbb{R}^N$, then we omit to denote $\mathbb{R}^N$: $\|u\|_{L^p}:=\|u\|_{L^p(\mathbb{R}^N)}$. Moreover, if $\Omega\subset\mathbb{R}^N$ is a bounded open set with smooth boundary, then $\mathcal{D}\subset L^r(\Omega)$ $(2\le r<2N/(N-2))$ is compact (The Rellich compactness lemma). On the oner hand, since

$$\langle f,u\rangle_{\mathcal{D}^*,\mathcal{D}}=\int_{\mathbb{R}^N}\overline{f(x)}u(x)\,dx,$$

we see that

$$\|u\|_{\mathcal{D}^*}\le c(r)\|u\|_{L^{r'}}, \quad \forall\, u\in L^{r'}(\mathbb{R}^N), \tag{3.4}$$

where r' is a Hölder conjugate of $r\in[2,2N/(N-2))$: $r'=r/(r-1)$.

Also we see that

$$\|t^{-2}D_{1/|t|}K\|_{L^q}=|t|^{-2+N/q}\|K\|_{L^q}, \quad \partial_t[t^{-2}D_{1/|t|}K]=-t^{-3}D_{1/|t|}\widetilde{K}.$$

We divide K and $\widetilde{K}$ into K_1+K_2 and $\widetilde{K}_1+\widetilde{K}_2$ so that $K_j\in L^{q_j}(\mathbb{R}^N)$ and $\widetilde{K}_j\in L^{\widetilde{q}_j}(\mathbb{R}^N)$ $(j=1,2)$. Note that **(K2a)** implies $N/4<q_1<q_2\le N/2$ and **(K4a)** implies $N/4<\widetilde{q}_1<\widetilde{q}_2<N/2$. The Young and the Hölder inequalities imply that

$$\left\|u_2(x)\int_{\mathbb{R}^N}K(x-y)u_3(y)u_4(y)\,dy\right\|_{L^{r'}}\le\|K\|_{L^q}\|u_2\|_{L^r}\|u_3\|_{L^r}\|u_4\|_{L^r}, \tag{3.5}$$

$$\begin{aligned}&\left|\iint_{\mathbb{R}^{N+N}}K(x-y)u_1(x)u_2(x)u_3(y)u_4(y)\,dxdy\right|\\ &\le\|K\|_{L^q}\|u_1\|_{L^r}\|u_2\|_{L^r}\|u_3\|_{L^r}\|u_4\|_{L^r},\end{aligned} \tag{3.6}$$

where $r=4q/(2q-1)$ and $r'=4q/(2q+1)$.

Verification of (A1). Let $u,\, v \in \mathcal{D}$. Then we see from **(K1)** that

$$(3.7)\qquad \begin{aligned} &G(t,u+v)-G(t,u)-\mathrm{Re}\,\langle g(t,u),v\rangle_{\mathcal{D}^*,\mathcal{D}} \\ &= \frac{1}{4t^2}\iint_{\mathbb{R}^{N+N}} K\Big(\frac{x-y}{|t|}\Big)[|(u+v)(x)|^2|(u+v)(y)|^2-|u(x)|^2|u(y)|^2]\,dxdy \\ &-\frac{1}{4t^2}\iint_{\mathbb{R}^{N+N}} K\Big(\frac{x-y}{|t|}\Big)[2\,\mathrm{Re}\,(v(x)\overline{u(x)})|u(y)|^2+2\,\mathrm{Re}\,(v(y)\overline{u(y)})|u(x)|^2]\,dxdy. \end{aligned}$$

Now let $\alpha,\, \beta,\, \xi,\, \eta \in \mathbb{C}$. Then we see that

$$(3.8)\qquad \begin{aligned} &|\alpha+\xi|^2|\beta+\eta|^2-|\alpha|^2|\beta|^2-2\,|\beta|^2\mathrm{Re}\,(\overline{\alpha}\xi)-2\,|\alpha|^2\mathrm{Re}\,(\overline{\beta}\eta) \\ &= 4\,\mathrm{Re}\,(\overline{\alpha}\xi)\mathrm{Re}\,(\overline{\beta}\eta)+|\xi|^2[|\beta|^2+2\mathrm{Re}\,(\overline{\beta}\eta)]+|\eta|^2[|\alpha|^2+2\mathrm{Re}\,(\overline{\alpha}\xi)]+|\xi|^2|\eta|^2. \end{aligned}$$

Put $\alpha := u(x)$, $\beta := u(y)$, $\xi := v(x)$, $\eta := v(y)$ in (3.8). It follows from (3.7) that

$$(3.9)\qquad G(t,u+v)-G(t,u)-\mathrm{Re}\,\langle g(t,u),v\rangle_{\mathcal{D}^*,\mathcal{D}} = I_1+I_2+I_3,$$

where

$$\begin{aligned} I_1 &:= \frac{1}{t^2}\iint_{\mathbb{R}^{N+N}} K\Big(\frac{x-y}{|t|}\Big)\mathrm{Re}\,(\overline{u(x)}v(x))\,\mathrm{Re}\,(\overline{u(y)}v(y))\,dxdy, \\ I_2 &:= \frac{1}{2t^2}\iint_{\mathbb{R}^{N+N}} K\Big(\frac{x-y}{|t|}\Big)|v(x)|^2[|u(y)|^2+2\mathrm{Re}\,(\overline{u(y)}v(y))]\,dxdy, \\ I_3 &:= \frac{1}{4t^2}\iint_{\mathbb{R}^{N+N}} K\Big(\frac{x-y}{|t|}\Big)|v(x)|^2|v(y)|^2\,dxdy. \end{aligned}$$

We see from (3.6) and (3.3) that

$$(3.10)\qquad \begin{aligned} |I_1| &\le \sum_{j=1}^{2}\Big|\frac{1}{t^2}\iint_{\mathbb{R}^{N+N}} K_j\Big(\frac{x-y}{|t|}\Big)\mathrm{Re}\,(\overline{u(x)}v(x))\,\mathrm{Re}\,(\overline{u(y)}v(y))\,dxdy\Big| \\ &\le \sum_{j=1}^{2} t^{-2+N/q_j}\|K_j\|_{L^{q_j}}\|u\|_{L^{r_j}}^2\|v\|_{L^{r_j}}^2 \le \sum_{j=1}^{2} c(r_j)^4 t^{-2+N/q_j}\|K_j\|_{L^{q_j}}\|u\|_{\mathcal{D}}^2\|v\|_{\mathcal{D}}^2, \end{aligned}$$

where $r_j = 4q_j/(2q_j-1)$. Note that $2 < 2N/(N-1) \le r_2 < r_1 < 2N/(N-2)$ by **(K2a)**. In a way similar to I_1, we obtain

$$(3.11)\qquad |I_2| \le d_K(t)\,\|v\|_{\mathcal{D}}^2\,[\|u\|_{\mathcal{D}}^2+2\|u\|_{\mathcal{D}}\|v\|_{\mathcal{D}}],$$

$$(3.12)\qquad |I_3| \le d_K(t)\,\|v\|_{\mathcal{D}}^4,$$

where

$$d_K(t) := \sum_{j=1}^{2} c(r_j)^4|t|^{-2+N/q_j}\|K_j\|_{L^{q_j}}.$$

Since $-2+N/q_1 > -2+N/q_2 \ge 0$ by **(K2a)**, we see $d_K(t) \le d_K(T)$ for $t \in [-T,T]$. (3.10), (3.11), and (3.12) imply that

$$\begin{aligned} &|G(t,u+v)-G(t,u)-\mathrm{Re}\,\langle g(t,u),v\rangle_{\mathcal{D}^*,\mathcal{D}}| \\ &\le d_K(T)\,\|v\|_{\mathcal{D}}^2\,[6\|u\|_{\mathcal{D}}^2+4\|u\|_{\mathcal{D}}\|v\|_{\mathcal{D}}+\|v\|_{\mathcal{D}}^2] \quad \forall\, t \in [-T,T],\ \forall\, u,\, v \in \mathcal{D}. \end{aligned}$$

Let $M > 0$ and $\varepsilon > 0$. Then we see that

$$|G(t,u+v) - G(t,u) - \mathrm{Re}\,\langle g(t,u), v\rangle_{\mathcal{D}^*,\mathcal{D}}| \le d_K(T)\,(6M^2+4M+1)\,\|v\|_{\mathcal{D}}^2$$
$$\forall\, t \in [-T,T],\ \forall\, u,\ v \in \mathcal{D} \text{ with } \|u\|_{\mathcal{D}} \le M,\ \|v\|_{\mathcal{D}} \le 1.$$

Hence by setting $\delta > 0$ as

$$\delta = \delta(u,\varepsilon) = 1 \wedge \frac{\varepsilon}{d_K(T)\,(6M^2+4M+1)},$$

we conclude **(A1)**:

$$|G(t,u+v) - G(t,u) - \mathrm{Re}\,\langle g(t,u), v\rangle_{\mathcal{D}^*,\mathcal{D}}| \le \varepsilon\|v\|_{\mathcal{D}} \quad \forall\, v \in \mathcal{D} \text{ with } \|v\|_{\mathcal{D}} \le \delta.$$

Verification of (A2). First we define

$$g_j(t,u) := t^{-2}u \int_{\mathbb{R}^N} K_j\Big(\frac{x-y}{|t|}\Big)|u(y)|^2\,dy$$

for $j = 1,2$. Note that $g(t,u) = g_1(t,u) + g_2(t,u)$. Let $u,\ v \in \mathcal{D}$. Then we see that

$$g_j(t,u) - g_j(t,v) = u\,(t^{-2}D_{1/t}K_j) * [|u|^2 - |v|^2] + (u-v)\,(t^{-2}D_{1/t}K_j) * |v|^2.$$

Applying (3.5), we can calculate

$$\begin{aligned}
(3.13)\qquad & \|g_j(t,u) - g_j(t,v)\|_{L^{r_j'}} \\
&\le \|u\,(t^{-2}D_{1/t}K_j) * [|u|^2 - |v|^2]\|_{L^{r_j'}} + \|(u-v)\,(t^{-2}D_{1/t}K_j) * |v|^2\|_{L^{r_j'}} \\
&\le |t|^{-2+N/q_j}\|K_j\|_{L^{q_j}}\,\|u\|_{L^{r_j}}\,\|u+v\|_{L^{r_j}}\,\|u-v\|_{L^{r_j}} \\
&+ |t|^{-2+N/q_j}\|K_j\|_{L^{q_j}}\,\|u-v\|_{L^{r_j}}\,\|v\|_{L^{r_j}}^2 \\
&\le |t|^{-2+N/q_j}\|K_j\|_{L^{q_j}}\,[\|u\|_{L^{r_j}}^2 + \|u\|_{L^{r_j}}\|v\|_{L^{r_j}} + \|v\|_{L^{r_j}}^2]\,\|u-v\|_{L^{r_j}}.
\end{aligned}$$

Thus (3.3) and (3.4) yield that

$$\begin{aligned}
(3.14)\qquad & \|g(t,u) - g(t,v)\|_{\mathcal{D}^*} \le \sum_{j=1}^{2} c(r_j)\,\|g_j(t,u) - g_j(t,v)\|_{L^{r_j'}} \\
&\le \sum_{j=1}^{2} c(r_j)|t|^{-2+N/q_j}\|K_j\|_{L^{q_j}}\,[\|u\|_{L^{r_j}}^2 + \|u\|_{L^{r_j}}\|v\|_{L^{r_j}} + \|v\|_{L^{r_j}}^2]\,\|u-v\|_{L^{r_j}} \\
&\le d_K(t)\,[\|u\|_{\mathcal{D}}^2 + \|u\|_{\mathcal{D}}\|v\|_{\mathcal{D}} + \|v\|_{\mathcal{D}}^2]\,\|u-v\|_{\mathcal{D}}.
\end{aligned}$$

This implies **(A2)**.

Verification of (A3). We see

$$\begin{aligned}
& g(t,u) - g(s,u) = u\,[t^{-2}D_{1/|t|}K - s^{-2}D_{1/|s|}K] * |u|^2 \\
&= u\left[\int_s^t \frac{\partial}{\partial\sigma}(\sigma^{-2}D_{1/|\sigma|}K)\,d\sigma\right] * |u|^2 = u\left[\int_s^t -\sigma^{-3}D_{1/|\sigma|}\widetilde{K}\,d\sigma\right] * |u|^2 \\
&= -\sum_{j=1}^{2} u\left[\int_s^t \sigma^{-3}D_{1/|\sigma|}\widetilde{K}_j\,d\sigma\right] * |u|^2.
\end{aligned}$$

By virtue of (3.5) and (3.3), we obtain

$$\Big\|u\Big[\int_s^t \sigma^{-3}D_{1/|\sigma|}\widetilde{K}_j\,d\sigma\Big]*|u|^2\Big\|_{L^{\widetilde{r}_j'}} \le \Big|\int_s^t \|\sigma^{-3}D_{1/|\sigma|}\widetilde{K}_j\|_{L^{\widetilde{q}_j}}\,d\sigma\Big|\|u\|_{L^{\widetilde{r}_j}}^3$$
$$\le \Big|\int_s^t |\sigma|^{-3+N/\widetilde{q}_j}\|\widetilde{K}_j\|_{L^{\widetilde{q}_j}}\,d\sigma\Big|\|u\|_{L^{\widetilde{r}_j}}^3 \le c(\widetilde{r}_j)^3\|\widetilde{K}_j\|_{L^{\widetilde{q}_j}}\Big|\int_s^t |\sigma|^{-3+N/\widetilde{q}_j}\,d\sigma\Big|\|u\|_{\mathcal{D}}^3,$$

where $\widetilde{r}_j := 4\widetilde{q}_j/(2\widetilde{q}_j-1)\in(2,2N/(N-2))$ by **(K4a)** and $\widetilde{r}_j' := 4\widetilde{q}_j/(2\widetilde{q}_j+1)$. Thus we see from (3.4) that

$$\|g(t,u)-g(s,u)\|_{\mathcal{D}^*} \le \|u\|_{\mathcal{D}}^3\sum_{j=1}^2 c(\widetilde{r}_j)^3\|\widetilde{K}_j\|_{L^{\widetilde{q}_j}}\Big|\int_s^t |\sigma|^{-3+N/\widetilde{q}_j}\,d\sigma\Big|.$$

Since **(K4a)** implies $-3+N/\widetilde{q}_1 > -3+N/\widetilde{q}_2 > -1$, the integrands belong to $L^p(-T,T)$ for some $p>1$. This concludes **(A3)**.

Verification of (A4). First we define

$$G_j(t,u) := \frac{1}{4t^2}\iint_{\mathbb{R}^{N+N}} K_j\Big(\frac{x-y}{|t|}\Big)|u(x)|^2|u(y)|^2\,dxdy \quad (j=1,2).$$

Let $u,\ v\in\mathcal{D}$. Then we see from **(K1)** that

$$G_j(t,u)-G_j(t,v) = \frac{1}{4t^2}\iint_{\mathbb{R}^{N+N}} K_j\Big(\frac{x-y}{|t|}\Big)[|u(y)|^2-|v(y)|^2]\,[|u(x)|^2+|v(x)|^2]\,dxdy.$$

Applying (3.6) we have

$$|G_j(t,u)-G_j(t,v)| \le t^{-2+N/q_j}\|K_j\|_{L^{q_j}}[\|u\|_{L^{r_j}}^2+\|v\|_{L^{r_j}}^2][\|u\|_{L^{r_j}}+\|v\|_{L^{r_j}}]\|u-v\|_{L^{r_j}}.$$

(3.3) yields for any $t\in[-T,T]$ and for all $u,v\in\mathcal{D}$ with $\|u\|_{\mathcal{D}}\le M$, $\|v\|_{\mathcal{D}}\le M$

$$\begin{aligned}&|G_j(t,u)-G_j(t,v)|\\ &\le c(r_j)^4|t|^{-2+N/q_j}\|K_j\|_{L^{q_j}}[\|u\|_{\mathcal{D}}^2+\|v\|_{\mathcal{D}}^2][\|u\|_{\mathcal{D}}+\|v\|_{\mathcal{D}}]\|u-v\|_{\mathcal{D}}^{\theta_j}\|u-v\|_{L^2}^{1-\theta_j}\\ &\le c(r_j)^4T^{-2+N/q_j}\|K_j\|_{L^{q_j}}2^{2+\theta_j}M^{3+\theta_j}\|u-v\|_{L^2}^{1-\theta_j},\end{aligned}$$

where $\theta_j = N(2^{-1}-r_j^{-1}) = N/(4q_j)\in(0,1)$ by **(K2a)**. Applying the Young inequality

$$y^{1-\theta}\le\varepsilon+\theta\Big(\frac{1-\theta}{\varepsilon}\Big)^{(1-\theta)/\theta}y,$$

we see

$$|G_j(t,u)-G_j(t,v)|\le\frac12\delta+C_{j,\delta}(M)\|u-v\|_{L^2},$$

where $C_{j,\delta}(M) := \theta_j[\delta^{-1}(1-\theta_j)(2M)^{3+\theta_j}T^{-2+N/q_j}c(r_j)^4\|K_j\|_{L^{q_j}}]^{(1-\theta_j)/\theta_j}$. Since $G(t,u) = G_1(t,u)+G_2(t,u)$, we have confirmed **(A4)**.

Verification of (A5). By a standard argument of weak derivatives, we see that

$$G_t(t,u) = -\frac{1}{4t^3}\iint_{\mathbb{R}^{N+N}}\widetilde{K}\Big(\frac{x-y}{|t|}\Big)|u(x)|^2|u(y)|^2\,dxdy.$$

Define

$$\widetilde{G}_j(t,u) := \frac{1}{4t^3}\iint_{\mathbb{R}^{N+N}} \widetilde{K}_j\Big(\frac{x-y}{|t|}\Big)|u(x)|^2|u(y)|^2\,dxdy \quad (j=1,2).$$

In a way similar to **(A4)** we see that for all $u, v \in \mathcal{D}$ with $\|u\|_{\mathcal{D}} \le M$, $\|v\|_{\mathcal{D}} \le M$

$$\begin{aligned}|\widetilde{G}_j(t,u) - \widetilde{G}_j(t,v)| &\le c(\widetilde{r}_j)^4|t|^{-3+N/\widetilde{q}_j}\|\widetilde{K}_j\|_{L^{\widetilde{q}_j}}2^{2+\widetilde{\theta}_j}M^{3+\widetilde{\theta}_j}\|u-v\|_{L^2}^{1-\widetilde{\theta}_j},\\ &\le |t|^{-3+N/\widetilde{q}_j}\Big[\frac{1}{2}\delta + \widetilde{C}_{j,\delta}(M)\|u-v\|_{L^2}\Big],\end{aligned}$$

where $\widetilde{\theta}_j = N/(4\widetilde{q}_j)$ and $\widetilde{C}_{j,\delta}(M) := \widetilde{\theta}_j[\delta^{-1}(1-\widetilde{\theta}_j)(2M)^{3+\widetilde{\theta}_j}c(\widetilde{r}_j)^4\|\widetilde{K}_j\|_{L^{\widetilde{q}_j}}]^{(1-\widetilde{\theta}_j)/\widetilde{\theta}_j}$. Since $-3+N/\widetilde{q}_1 > -3+N/\widetilde{q}_2 > -1$ by **(K4a)** and $G_t(t,u) = -\widetilde{G}_1(t,u) - \widetilde{G}_2(t,u)$, we obtain **(A5)**.

Verification of (A6). **(K1)** implies **(A6)** by a simple calculation:

$$\begin{aligned}\operatorname{Re}\langle g(t,u), iu\rangle_{\mathcal{D}^*,\mathcal{D}} &= \operatorname{Re}\int_{\mathbb{R}^N}\Big[\int_{\mathbb{R}^N} t^{-2}K\Big(\frac{x-y}{|t|}\Big)|u(y)|^2\overline{u(x)}\,iu(x)\,dy\Big]\,dx\\ &= \operatorname{Re} i\iint_{\mathbb{R}^{N+N}} t^{-2}K\Big(\frac{x-y}{|t|}\Big)|u(y)|^2|u(y)|^2\,dxdy = 0.\end{aligned}$$

Verification of (A7). Let $I \subset \mathbb{R}$ be an open and bounded interval and assume that $\{w_n\}_n$ is a sequence in $L^\infty(I;\mathcal{D})$ satisfying

$$\begin{cases} w_n(t) \to w(t)\ (n\to\infty) & \text{weakly in } \mathcal{D} \quad \text{a.a. } t \in I,\\ g(t,w_n(t)) \to f(t)\ (n\to\infty) & \text{weakly}^* \text{ in } L^\infty(I;\mathcal{D}^*).\end{cases}$$

Since $\{g_1(w_n)\}_n$ and $\{g_2(w_n)\}_n$ are bounded in $L^\infty(I;\mathcal{D}^*)$ and the Sobolev embeddings, there exist a subsequence $\{w_{n(j)}\}_j$ of $\{w_n\}_n$ and f_1, $f_2 \in L^\infty(I;\mathcal{D}^*)$ such that

$$g_j(t,w_{n(m)}(t)) \to f_j(t)\ (m\to\infty) \quad \text{weakly}^* \text{ in } L^\infty(I;L^{r_j'}(\mathbb{R}^N))\ (j=1,\,2). \tag{3.15}$$

To confirm (2.1) let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded open subset with C^1 boundary. Then

$$\begin{aligned}\langle f_j(t), w(t)\rangle_{L^{r_j'}(\Omega),L^{r_j}(\Omega)} &= \langle f_j(t) - g_j(t,w_{n(m)}(t)), w(t)\rangle_{L^{r_j'}(\Omega),L^{r_j}(\Omega)}\\ &\quad + \langle g_j(t,w_{n(m)}(t)), w(t)-w_{n(m)}(t)\rangle_{L^{r_j'}(\Omega),L^{r_j}(\Omega)}\\ &\quad + \langle g_j(t,w_{n(m)}(t)), w_{n(m)}(t)\rangle_{L^{r_j'}(\Omega),L^{r_j}(\Omega)}\\ &=: J_{j1}(t) + J_{j2}(t) + J_{j3}(t) \quad (j=1,\,2).\end{aligned} \tag{3.16}$$

The weak convergence (3.15) asserts that

$$\int_I J_{j1}(t)\,dt \to 0\ (m\to\infty), \quad j=1,\,2. \tag{3.17}$$

Next we consider J_{j2}. The Rellich compactness lemma implies that $w_{n(m)}(t) \to w(t)$ $(m\to\infty)$ strongly in $L^{r_j}(\Omega)$ a.a. $t \in I$. It follows from the boundedness of $\{g_j(w_{n(m)}(t))\}_m$

in $L^{r_j}(\Omega)$ a.a. $t \in I$ that $I_{12}(t) \to 0$ $(m \to \infty)$ for a.a. $t \in I$. We see the boundedness of $\{w_{n(m)}\}_m$ in $L^\infty(-T,T;L^{r_j}(\Omega))$ and $\{g_j(w_{n(m)})\}_m$ in $L^\infty(I;L^{r_j'}(\Omega))$. The dominated convergence lemma yields that

$$\int_I J_{j2}(t)\,dt \to 0\ (m \to \infty). \tag{3.18}$$

(K1) implies that $\operatorname{Im} J_{j3}(t) = 0$ a.a. $t \in I$ $(j = 1,\, 2)$. Integrating (3.16) over I and using (3.17) and (3.18), we obtain

$$\operatorname{Re} \int_I \langle f_j(t), i\, w(t)\rangle_{L^{r_j'}(\Omega), L^{r_j}(\Omega)}\, dt = 0.$$

Since Ω is arbitrary and $f = f_1 + f_2$, **(A6)** implies (2.1):

$$\operatorname{Re} \int_I \langle f(t), i\, w(t)\rangle_{\mathcal{D}^*,\mathcal{D}}\, dt = 0 = \lim_{n\to\infty} \operatorname{Re} \int_I \langle g(t, w_n(t)), i\, w_n(t)\rangle_{\mathcal{D}^*,\mathcal{D}}\, dt.$$

Next we show that $f(t) = g(t, w(t))$ by assuming further that $w_n(t) \to w(t)$ $(n \to \infty)$ in $L^2(\mathbb{R}^N)$ a.a. $t \in I$. Let $M := \sup_n \|w_n\|_{L^\infty(I;\mathcal{D})}$. It follows from (3.13), (3.3), and (3.4) that

$$\begin{aligned}\|g(t, w_n(t)) - g(t, w(t))\|_{\mathcal{D}^*} &\le \sum_{j=1}^{2} c(r_j)^4 6 M^{2+\theta_j} |I|^{-2+N/q_j} \|K_j\|_{L^{q_j}} \|w_n(t) - w(t)\|_{L^2}^{1-\theta_j} \\ &\to 0 \quad (n \to \infty) \quad \text{a.a. } t \in I.\end{aligned}$$

Thus we see that $g(t, w_n(t)) \to g(t, w(t))$ $(n \to \infty)$ strongly in $L^\infty(I;\mathcal{D}^*)$ and **(A7)** is verified.

To show the uniqueness we apply the Strichartz estimates for $\{e^{-itP_a}\}$ established by Burq, Planchon, Stalker and Tahvildar-Zadeh [**1**] (see also [**6**, Theorems 2.3 and 2.5]).

Definition 3.1. The pair (τ, ρ) is called a *Schrödinger admissible pair* if

$$\frac{2}{\tau} + \frac{N}{\rho} = \frac{N}{2}, \qquad \tau > 2,\ \rho \ge 2.$$

Lemma 3.2. *Let $N \ge 3$, $a \ge a(N)$ and (τ, ρ) be a Schrödinger admissible pair. Then the following inequality holds*:

$$\|\exp(-itP_a)\varphi\|_{L^\tau(\mathbb{R};L^\rho)} \le C_\tau \|\varphi\|_{L^2} \quad \forall\, \varphi \in L^2(\mathbb{R}^N). \tag{3.19}$$

Moreover, let (τ_j, ρ_j) $(j = 1,\, 2)$ be Schrödinger admissible pairs. Then for all $\Phi \in L^{\tau_1'}(\mathbb{R}; L^{\rho_1'}(\mathbb{R}^N))$

$$\left\| \int_0^t \exp(-i(t-s)P_a)\Phi(s,x)\, ds \right\|_{L^{\tau_2}(\mathbb{R};L^{\rho_2})} \le C_{\tau_1,\tau_2} \|\Phi\|_{L^{\tau_1'}(\mathbb{R};L^{\rho_1'})}. \tag{3.20}$$

We exclude the *endpoint* $(\tau, \rho) = (2, 2N/(N-2))$ from the Schrödinger admissible pair. Let $a > a(N)$. Burq, Planchon, Stalker, and Tahvildar-Zadeh [**1**, Theorem 3] confirmed (3.19) for the endpoint; Pierfelice [**8**, Theorem 2 in Section 3] confirmed (3.20) for the endpoint. On the other hand, Mizutani [**5**] showed that (3.19) and (3.20) for the endpoint are broken down for $a = a(N)$.

Lemma 3.3. *Let u_j $(j = 1,\, 2)$ be local weak solutions to* **(IVP)** *on $I = (-T_1, T_2) \subset \mathbb{R}$ with initial values $u_j(0) = u_0 \in \mathcal{D}$. Then $u_1(t) = u_2(t)$ on $t \in I$.*

Proof. Let $u_j \in L^\infty(I;\mathcal{D})$ $(j = 1,\, 2)$ be local weak solutions to **(IVP)** on I with initial values $u_j(0) = u_0$. Then u_j $(j = 1,\, 2)$ satisfy the following integral equations:

$$u_j(t) = \exp(-itP_a)u_0 - i\int_0^t \exp\big(-i(t-s)P_a\big)g(s, u_j(s))\,ds.$$

Therefore we see that $v(t) := u_1(t) - u_2(t)$ satisfies

$$v(t) = -i\int_0^t \exp\big(-i(t-s)P_a\big)[g(s, u_1(s)) - g(s, u_2(s))]\,ds.$$

Here $(8q_j/N, r_j)$ $(j = 1, 2)$ are Schrödinger admissible pairs. Applying (3.13) and the Strichartz estimates (3.20), we see that for every Schrödinger admissible pair (τ, ρ),

$$\begin{aligned}(3.21)\qquad &\Big\|\int_0^t \exp\big(-i(t-s)P_a\big)[g_j(u_1(s)) - g_j(u_2(s))]\,ds\Big\|_{L^\tau(I;L^\rho)}\\ &\le C_{8q_j/N,\tau}\|g_j(u_1) - g_j(u_2)\|_{L^{(8q_j/N)'}(I;L^{r_j'})}\\ &\le C_{8q_j/N,\tau}\|t^{-2}D_{1/t}K_j\|_{L^{(4q_j/N)'}(I;L^{q_j})}\\ &\times [\|u_1\|^2_{L_t^\infty L^{r_j}} + \|u_1\|_{L_t^\infty L^{r_j}}\|u_2\|_{L_t^\infty L^{r_j}} + \|u_2\|^2_{L_t^\infty L^{r_j}}]\|v\|_{L^{8q_j/N}(I;L^{r_j})},\end{aligned}$$

where $\|\cdot\|_{L_t^\infty L^p} := \|\cdot\|_{L^\infty(I;L^p)}$. Putting $(\tau,\rho) := (8q_j/N, r_j)$ $(j = 1, 2)$ in (3.21), we see that

$$\begin{aligned}(3.22)\qquad &\|v\|_{L^{8q_1/N}(I;L^{r_1})} + \|v\|_{L^{8q_2/N}(I;L^{r_2})}\\ &\le 3(C_{8q_1/N,8q_1/N} + C_{8q_1/N,8q_2/N} + C_{8q_2/N,8q_1/N} + C_{8q_2/N,8q_2/N})\,M^2\\ &\times [\|t^{-2+N/q_1}\|_{L^{(4q_1/N)'}(I)}\|K_1\|_{L^{q_1}} + \|t^{-2+N/q_1}\|_{L^{(4q_2/N)'}(I)}\|K_2\|_{L^{q_2}}]\\ &\times [\|v\|_{L^{8q_1/N}(I;L^{r_1})} + \|v\|_{L^{8q_2/N}(I;L^{r_2})}],\end{aligned}$$

where

$$M := \max_{j=1,2}\{\|u_j\|_{L^\infty(I;L^{r_1})}, \|u_j\|_{L^\infty(I;L^{r_2})}\}.$$

Since $-2 + N/q_1 > -2 + N/q_2 \ge 0$ by **(K2a)**, (3.22) yields

$$\|v\|_{L^{r(\gamma)}(I;L^{r_1})} + \|v\|_{L^\infty(I;L^{r_2})} \le 0$$

for the interval I sufficiently small. Extending the interval step by step, we conclude the uniqueness on any interval I. ∎

Since **(A1)**–**(A7)** are verified and the uniqueness of local weak solutions for **(IVP)** is proved, Theorem 2.1 yields the unique existence of local weak solutions to **(IVP)**. Moreover, in a way similar to (1.3) (for self-excited system $\mathbf{(HE)}_a$), the virial identity for **(IVP)** can be constructed owing to **(K4a)**:

$$(3.23)\qquad \frac{d}{dt}\|xv(t)\|^2_{L^2} = 4\,\mathrm{Im}\int_{\mathbb{R}^N}\overline{x\,v(t,x)}\cdot\nabla v(t,x)\,dx,$$

$$(3.24)\qquad \frac{d^2}{dt^2}\|xv(t)\|^2_{L^2} = 16\,E(t, v(t)) - \frac{2}{t^2}\iint_{\mathbb{R}^N}\widetilde{K}\Big(\frac{x-y}{|t|}\Big)|u(t,x)|^2|u(t,y)|^2\,dxdy.$$

Note that we do not differentiate the nonlinear term $g(t,u) = u\,(t^{-2}D_{1/t}K * |u|^2)$ by t to deriving the virial identity. Thus Proposition 3.1 is fully proved.

To end this section, it remains to show the well-definedness of W_+. We have constructed the local solution $v \in C([-T,T];\Sigma) \cap C^1([-T,T];\mathcal{D}^*)$ to **(IVP)**. Applying the pseudo-conformal transform $\mathcal{C}$ we can define

$$u_{1/T} := (\mathcal{C}v)(1/T,x) = (-iT)^{N/2} e^{iT|x|^2/4}\overline{v(T,Tx)} \in \Sigma.$$

Proposition 1.1 implies that **(K1)**, **(K2)**, and **(K3)** admit a unique global weak solution $u \in C(\mathbb{R};\Sigma) \cap C^1(\mathbb{R};\mathcal{D}^*)$ to

$$\begin{cases} i\,u_t = P_a u + u\,(K * |u|^2) & \text{in } \mathbb{R} \times \mathbb{R}^N, \\ u(1/T) = u_{1/T} \in \Sigma. \end{cases}$$

The uniqueness of **(IVP)** implies that $u(t,x) = (\mathcal{C}v)(t,x)$ on $(1/T,\infty)$. Thus there exists a unique global weak solution $u \in C(\mathbb{R};\Sigma) \cap C^1(\mathbb{R};\mathcal{D}^*)$ to **(FVP)**. Hence Theorem 1.4 has been fully proved.

3.2. Proof of Theorem 1.5 (asymptotic free)

Now we show the asymptotic free of $\mathbf{(HE)}_a$. To end this, first we consider the global weak solution. Assume **(K1)**, **(K2)**, and **(K3)**. Let $u_0 \in \Sigma$. Then Proposition 1.1 implies that there uniquely exists a global weak solution $u \in C(\mathbb{R};\Sigma) \cap C^1(\mathbb{R};\mathcal{D}^*)$ to $\mathbf{(HE)}_a$. Thus $v = \mathcal{C}^{-1}u$ belongs to $C((0,\infty);\Sigma) \cap C^1((0,\infty);\mathcal{D}^*)$ and satisfies

$$i\,\frac{\partial v}{\partial t} = P_a v + t^{-2}\,v\,(D_{1/t}K * |v|^2) \quad \text{on } (0,\infty).$$

To prove Theorem 1.5, In a view of 1.4, it is sufficient to show that v can be continuously extended to $t = 0$. Now we show the uniform boundedness of $\|P_a^{1/2} v(t)\|_{L^2}$ in $t \in (0,1)$. The energy conservation laws yields that

$$\begin{aligned} &\|(1+P_a)^{1/2}v(t)\|_{L^2}^2 \\ &= \|(1+P_a)^{1/2}v(1)\|_{L^2}^2 + 2G(1,v(1)) - 2G(t,v(t)) + 2\int_1^t G_t(s,v(s))\,ds. \end{aligned}$$

Here $K \geq 0$ by **(K3a)** implies that

$$G(t,u) = \frac{1}{4t^2} \iint_{\mathbb{R}^{N+N}} K\Big(\frac{x-y}{|t|}\Big)|u(x)|^2|u(y)|^2\,dxdy \geq 0.$$

Moreover, $\widetilde{K} \leq 0$ by **(K3a)** implies that

$$G_t(t,u) = -\frac{1}{4t^3} \iint_{\mathbb{R}^{N+N}} \widetilde{K}\Big(\frac{x-y}{|t|}\Big)|u(x)|^2|u(y)|^2\,dxdy \geq 0.$$

Thus we see the uniform boundedness:

$$\|(1+P_a)^{1/2}v(t)\|_{L^2}^2 \leq \|(1+P_a)^{1/2}v(1)\|_{L^2}^2 + 2G(1,v(1)) \quad t \in (0,1).$$

On the other hand, [**13**, Lemma 3.1] implies

$$\left|\mathrm{Im}\int_{\mathbb{R}^N} x\,\overline{u}\cdot\nabla u\,dx\right| \le \|xu\|_{L^2}\|(1+P_a)^{1/2}u\|_{L^2}.$$

(3.23) ensures

$$\left|\frac{d}{dt}\|xv(t)\|_{L^2}\right| \le 2\|(1+P_a)^{1/2}v(t)\|_{L^2}.$$

The uniform boundedness of $\|(1+P_a)^{1/2}v(t)\|_{L^2}$ implies that there exists $v_0 \in \Sigma$ such that $v(t) \to v_0$ $(t \to +0)$ weakly in Σ. Here **(K1)**, **(K2a)**, **(K3)** and **(K4a)** yield the unique solution $\widetilde{v} \in C([0,\infty);\Sigma) \cap C^1([0,\infty);\mathcal{D}^*)$ to

$$\begin{cases} i\dfrac{\partial \widetilde{v}}{\partial t} = P_a\widetilde{v} + t^{-2}\,\widetilde{v}\,(D_{1/t}K * |\widetilde{v}|^2) \quad \text{in } \mathbb{R}\times\mathbb{R}^N, \\ \widetilde{v}(0) = v_0 \in \Sigma. \end{cases}$$

The uniqueness on $(1,\infty)$ implies that $v(t) = \widetilde{v}(t)$. Since $\widetilde{v}$ is continuous in Σ at $t = 0$, v can be continuousely extended to $t = 0$.

Remark 3.1. Since $g(t,u) = t^{-2}u\,D_{1/t}K * |u|^2$ satisfies $\overline{g(t,u)} = g(t,\overline{u})$, the wave operator W_- and the asymptotic free for $t \to -\infty$ can be considered by comming down to W_+ and $t \to \infty$, respectively. In fact, $W_-u_- = \overline{W_+\overline{u_-}}$ and

$$\lim_{t\to-\infty} \exp(itP_a)u(t) = \overline{\lim_{t\to+\infty} \exp(itP_a)\overline{u(-t)}}.$$

Note that if v is a unique solution to

$$\begin{cases} i\,v_t = P_a v + g(t,v) \quad \text{in } \mathbb{R}\times\mathbb{R}^N, \\ v(0) = v_0 \in \Sigma, \end{cases}$$

then $w(t) := \overline{v(-t)}$ satisfies

$$\begin{cases} i\,w_t = P_a w + g(t,w) \quad \text{in } \mathbb{R}\times\mathbb{R}^N, \\ w(0) = \overline{v_0} \in \Sigma. \end{cases}$$

4. Concluding remarks

4.1. Conditions for K

Conditions for the integrability of K can be relaxed. $L^{q_j}(\mathbb{R}^N)$ can be replaced into the Lorentz space (or weak-L^q space) $L^{q,\infty}(\mathbb{R}^N)$:

$$\|K\|_{L^{p,\infty}} := \sup_{z>0} z\,\mu\big(\{x \in \mathbb{R}^N; |K(x)| > z\}\big)^{1/p} < \infty,$$

where μ is the Lebesgue measure. For example, the usulal Hartree kernel $|x|^{-\gamma} \in L^{N/\gamma,\infty}(\mathbb{R}^N)$ $(0 < \gamma < N)$ and the Yukawa-type kernel $e^{-\lambda|x|}|x|^{-\gamma} \in L^{N/\gamma,\infty}(\mathbb{R}^N)$ $(0 < \gamma < N,\ \lambda > 0)$. Thus the scattering problems for usual Hartree equations can be solved.

Corollary 4.1. *Let $a \geq a(N)$ and $K(x) := e^{-\lambda|x|}|x|^{-\gamma}$ $(2 < \gamma < \min\{N, 4\},\ \lambda \geq 0)$. Note that $\widetilde{K} = (-\lambda|x| + 2 - \gamma)K$.*

(i) *For any $u_+ \in \Sigma$ there uniquely exists a solution $u \in C(\mathbb{R}; \Sigma) \cap C^1(\mathbb{R}; \mathcal{D}^*)$ to* **(FVP)**. *Thus the wave operators $W_\pm : u_\pm \mapsto u(0)$ is well-defined in Σ.*

(ii) *For any global solution $u \in C(\mathbb{R}; \Sigma) \cap C^1(\mathbb{R}; \mathcal{D}^*)$ to $\mathbf{(HE)}_a$ with initial value $u_0 \in \Sigma$ there exist the following limits*

$$\lim_{t \to \pm\infty} \exp(itP_a)u(t) = u_\pm \quad \textit{strongly in } \Sigma.$$

On the other hand, nonnegativity of K can be also relaxed:

$$\iint_{\mathbb{R}^{N+N}} K(x-y)\varphi(x)\varphi(y)\,dxdy \geq 0$$

for any measurable and nonnegative function φ. For example, $\mathcal{F}K(\xi) \geq 0$ a.e. on $\mathbb{R}^N$, where $\mathcal{F}$ is the Fourier transform. In fact, it follows from the Plancherel lemma and the Parseval identities that

$$\begin{aligned}
&\iint_{\mathbb{R}^{N+N}} K(x-y)\varphi(x)\varphi(y)\,dxdy = \int_{\mathbb{R}^N} (K * \varphi)(x)\overline{\varphi(x)}\,dx \\
&= \int_{\mathbb{R}^N} \mathcal{F}(K * \varphi)(\xi)\overline{\mathcal{F}\varphi(\xi)}\,d\xi = \int_{\mathbb{R}^N} (2\pi)^{N/2}\mathcal{F}K(\xi)\mathcal{F}\varphi(\xi)\overline{\mathcal{F}\varphi(\xi)}\,d\xi \\
&= \int_{\mathbb{R}^N} (2\pi)^{N/2}\mathcal{F}K(\xi)|\mathcal{F}\varphi(\xi)|^2\,d\xi \geq 0.
\end{aligned}$$

4.2. Abstract theory

Lemma 2.3 can be generalized for applying the systems of nonautonomous semilinear Schrödinger evolution equations. Let $B : X_S^* \to X_S^*$ be a bounded linear operator with the following conditions:

- $BSu = SBu$ for $u \in X_S$;
- B is bounded and symmetric operator in X;
- B is coercive in X: there exists $\varepsilon > 0$ such that $\mathrm{Re}\langle Bu, u\rangle_X \geq \varepsilon\|u\|_X^2$.

By using B, **(H5)** is replaced with **(H5a)**:

$$\mathrm{Re}\,\langle g_0(t,u), iB\,u\rangle_X = 0 \quad \forall\, t \in [-T,T],\ \forall\, u \in X.$$

Lemma 4.2. *Assume* **(H1)**–**(H4)** *and* **(H5a)**. *Then for any $u_0 \in X_S$ there uniquely exists the global solution of* (2.3) *$u \in C([-T,T]; X_S) \cap C^1([-T,T]; X_S^*)$. Moreover, u satisfies the conservation laws*

$$\mathrm{Re}\langle Bu(t), u(t)\rangle_X = \mathrm{Re}\langle Bu_0, u_0\rangle_X, \quad E_0(t, u(t)) = E_0(0, u_0) + \int_0^t G_{0t}(s, u(s))\,ds.$$

Also, we can generalize Theorem 2.1 in a way similar to Section 2.1.

References

[1] N. Burq, F. Planchon, J. Stalker, A. S. Tahvildar-Zadeh, *Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential*, J. Funct. Anal. **203** (2003), 519–549.

[2] T. Cazenave, "Semilinear Schrödinger Equations," Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of Mathematical Sciences, AMS, New York, 2003.

[3] T. Cazenave, A. Haraux, "An Introduction to Semilinear Evolution Equations," Oxford Lecture Series in Mathematics and its Applications, 13. The Clarendon Press, Oxford University Press, New York, 1998.

[4] N. Hayashi, Y. Tsutsumi, *Scattering theory for Hartree type equations*, Ann. Inst. Henri Poincaré **46** (1987), 187–213.

[5] H. Mizutani, *Remarks on endpoint Strichartz estimates for Schrödinger equations with the critical inverse-square potential*. J. Differ. Equ. **263** (2017), 3832–3853.

[6] N. Okazawa, T. Suzuki, T. Yokota, *Cauchy problem for nonlinear Schrödinger equations with inverse-square potentials*, Appl. Anal. **91** (2012), 1605–1629.

[7] N. Okazawa, T. Suzuki, T. Yokota, *Energy methods for abstract nonlinear Schrödinger equations*, Evol. Equ. Control Theory **1** (2012), 337–354.

[8] V. Pierfelice, *Weighted Strichartz estimates for the Schrödinger and wave equations on Damek-Ricci spaces*, Math. Z. **260** (2008), 377–392.

[9] T. Suzuki, *Energy methods for Hartree type equation with inverse-square potentials*, Evol. Equ. Control Theory, **2** (2013), 531–542.

[10] T. Suzuki, *Blowup of nonlinear Schrödinger equations with inverse-square potentials*, Differ. Equ. Appl., **6** (2014), 309–333.

[11] T. Suzuki, *Solvability of nonlinear Schrödinger equations with some critical singular potential via generalized Hardy-Rellich inequalities*, Funkcial. Ekvac., **59** (2016), 1–34.

[12] T. Suzuki, *Scattering theory for Hartree equations with inverse-square potentials*, Appl. Anal., **96** (2017), 2032–2043.

[13] T. Suzuki, *Virial identities for nonlinear Schrödinger equations with an inverse-square potential of critical coefficient*, Differ. Equ. Appl., **9** (2017), 327–352 .

[14] T. Suzuki, *Scattering theory for semilinear Schrödinger equations with an inverse-square potential via energy methods*, preprint.