数理解析研究所講究録 第2090巻 2018年 102-116

OPTIMAL CONTROL OF THE COEFFICIENT FOR FRACTIONAL P-LAPLACE EQUATION: APPROXIMATION AND CONVERGENCE

HARBIR ANTIL*

Department of Mathematical Sciences
George Mason University
Fairfax, VA 22030, USA

MAHAMADI WARMA

University of Puerto Rico (Rio Piedras Campus)
College of Natural Sciences, Department of Mathematics
PO Box 70377 San Juan PR 00936-8377, USA

ABSTRACT. In [5] we studied optimal control problems with regional fractional p-Laplace equation, of order $s \in (0,1)$ and $p \in [2,\infty)$, as constraints over a bounded open set with Lipschitz continuous boundary. The control, which fulfills the pointwise box constraints, is given by the coefficient of the regional fractional p-Laplace operator. The purpose of this note is to provide a roadmap on how to apply the results of [5] to the fractional p-Laplace case. The existence and uniqueness of solutions to the state equation and existence of solutions to the optimal control problem follow using similar arguments as in [5]. We prove that the fractional p-Laplacian approaches the standard p-Laplacian as s approaches 1. In this sense, the fractional p-Laplacian can be considered degenerate like the standard p-Laplacian. The remaining steps are similar to the regional fractional p-Laplacian case, i.e., introduce an auxiliary state equation and the corresponding control problem and then conclude with the convergence of regularized solutions.

1. INTRODUCTION

This note is a continuation of our work done in [5] where we considered a similar problem but with the regional fractional p-Laplacian. The purpose of this note is to provide a roadmap on how to apply the results of [5] to the fractional p-Laplacian case.

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with boundary $\partial\Omega$, $s \in (0,1)$ and $p \in [2,\infty)$. In this note we introduce and investigate the existence and approximation of solutions to the following **optimal control problem (OCP)**:

$$\text{Minimize}\left\{\mathbb{I}(\kappa,u) := \frac{1}{2}\int_\Omega |u-\xi|^2\,dx + \int_\Omega |\nabla\kappa|\right\}, \tag{1.1}$$

1991 *Mathematics Subject Classification.* 35R11, 49J20, 49J45, 93C73.

Key words and phrases. Fractional p-Laplace operator, regional fractional p-Laplace operator, non-constant coefficient, quasi-linear nonlocal elliptic boundary value problems, optimal control.

The work of the first author is partially supported by NSF grant DMS-1521590. The work of the second author is partially supported by the Air Force Office of Scientific Research (AFOSR) under the Award No: FA9550-15-1-0027.

subject to the **state constraints** given by the fractional p-Laplace equation

$$\begin{cases} (-\Delta)_p^s(\kappa, u) + u = f & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.2}$$

The control κ in (1.2) fulfills

$$\kappa \in \mathfrak{A}_{ad} := \Big\{ \tilde{\eta} = E\eta \in BV(\mathbb{R}^N) \ \text{ for some } ; \eta \in BV(\Omega), \ \tilde{\eta}|_\Omega = \eta, \\ \xi_1(x) \le \eta(x) \le \xi_2(x) \ \text{ a.e. in } \ \Omega \Big\}. \tag{1.3}$$

The extension mapping $E : BV(\Omega) \to BV(\mathbb{R}^N)$, $\eta \mapsto E\eta =: \tilde{\eta}$ is well-defined for Lipschitz continuous domains (cf. [2, Definition 3.20 and Proposition 3.21]). Such an extension fulfills $\tilde{\eta} = 0$ on $\mathbb{R}^N \setminus \mathcal{O}$ where $\mathcal{O}$ is an open set such that $\overline{\Omega} \subset \mathcal{O}$.

Here the fractional operator is given for $x \in \mathbb{R}^N$ by

$$(-\Delta)_p^s(\kappa, u)(x) := C_{N,p,s}\text{P.V.} \int_{\mathbb{R}^N} \kappa(x-y)|u(x) - u(y)|^{p-2} \frac{u(x) - u(y)}{|x-y|^{N+sp}}\, dy. \tag{1.4}$$

Moreover, $\kappa : \mathbb{R}^N \to [0, \infty)$ is a measurable and even function, that is,

$$\kappa(x) = \kappa(-x), \quad \forall\, x \in \mathbb{R}^N. \tag{1.5}$$

In addition, f is a given force and ξ is the given data. The functions ξ_1 and ξ_2 in (1.3) are the **control bounds** and fulfill $0 < \alpha \le \xi_1(x) \le \xi_2(x)$, a.e. $x \in \Omega$, for some constant $\alpha > 0$. The precise regularity requirements for these quantities and the domain Ω will be discussed in Section 3. Notice that the **control** κ appears in the **coefficient** of the quasilinear operator $(-\Delta)_p^s(\kappa, \cdot)$.

The nonlocality of $(-\Delta)_p^s(\kappa, \cdot)$ make both the state equation (1.2) and **OCP** extremely challenging. Indeed the papers [24, 43], where the authors considered $\kappa = 1$, realized that the standard techniques available for the local p-Laplace equation are not directly applicable to the regional fractional p-Laplace equation (cf. [5, Eq. (1.2)]) where the regional fractional p-Laplacian is defined for $x \in \Omega$ by

$$\mathcal{L}_{\Omega,p}^s(\kappa, u)(x) = C_{N,p,s}\text{P.V.} \int_{\Omega} \kappa(x-y)|u(x) - u(y)|^{p-2} \frac{u(x) - u(y)}{|x-y|^{N+sp}}\, dy. \tag{1.6}$$

For **OCP** the additional complication occurs due to the fact that the operator $(-\Delta)_p^s(\kappa, \cdot)$ may degenerate, see Subsection 2.4 for more details. See also [5] for a similar discussion on the regional fractional p-Laplacian.

Recall that the local p-Laplace operator Δ_p is the quasilinear elliptic operator in divergence form given formally by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u). \tag{1.7}$$

Similarly to (1.7), it is well-known that a quasi-linear operator in divergence form with non constant coefficient can be formally defined as follows:

$$\Delta_{p,a} u := \operatorname{div}(a(x)|\nabla u|^{p-2} \nabla u). \tag{1.8}$$

If the coefficient $a(x)$ is smooth, then the latter operator enjoys most of the properties of the p-Laplace operator. A treatment of aforementioned degeneracy for the local operator (1.8) is given in [15, 27]. In [5] inspired by the work in [15] we developed the corresponding results for the regional fractional p-Laplacian. We noticed that the techniques used in [15] or [27] do not extend to the nonlocal case due to the delicate functional analytic framework that nonlocal problems entail. On the other hand we can confidently say that some (or maybe all) of the results obtained in [15, 27] can be obtained from our results by passing to the limit as $s \uparrow 1$.

Next we list some of the challenges associated with our nonlocal problems. Clearly the definition of the fractional p-Laplacian (and the regional one) involves singular integral operators which are in general difficult to manipulate. The classical weak derivative usually used in the standard case (1.8) cannot be used in the fractional case since the involved functions in general do not have enough regularity. For example the standard integration-by-parts formulas which hold for the classical case (1.8) does not immediately carry over to the fractional case (cf. Section 2.4). In the former case, one has to use additional properties of singular integral operators. Even the very classical weak-convergence argument which is obvious in $W^{1,p}(\Omega)$, requires an extreme care in case of the singular integral operators. Thirdly, a classical Lebesgue measure on Ω theory has been used in [15] to obtain some key results. In our previous work in [5] the corresponding results can be obtained only by using a weighted measure theory on $\Omega \times \Omega$, that is, a Lebesgue measure with a suitable weight on the set $\Omega \times \Omega$. In the present note one needs to further extend these results to $\mathbb{R}^N \times \mathbb{R}^N$. These are only some of the examples of novelties of our current work (also [5]) and how our approach differs from the ones given in [15].

In addition as we have noticed above, none of the results discussed in [15] can be immediately applied to our problem. This is not surprising, since (as we have mentioned above) the classical p-Laplace operator (1.7) can be obtained as the limit of the the fractional p-Laplace operator in the sense of Section 2.4 (see also [5] for the regional fractional p-Laplacian case).

The purpose of this note is to provide a roadmap on how to apply our results from [5] to the fractional p-Laplacian case. Notice that the fractional p-Laplacian is defined over $\mathbb{R}^N$ (cf. (1.4)) and the regional fractional p-Laplacian is defined over Ω (cf (1.6)). These two operators differ by a lower order potential term which in practice is difficult to manipulate. This can be easily seen from the definition of the regional fractional p-Laplacian when compared with the fractional p-Laplacian in (1.4).

We also mention that the problem to search for coefficients in case of linear elliptic problems is classical, we refer (but not limited) to [32, 33, 34, 37] and their references. However, our previous work [5] is the first work which provides a mechanism to search for the coefficients in case of a quasilinear, possibly degenerate, regional fractional nonlocal problems. The present work further shows how to apply the results of [5] to the fractional case. From a numerical point of view an added attraction of our theory is the fact that it is Hilbert space L^2-based instead of L^p-based theory.

In order to tackle the aforementioned degeneracy in $(-\Delta)^s_p(\kappa,\cdot)$ (see [5] for the regional case $\mathcal{L}^s_{\Omega,p}(\kappa,\cdot)$) we introduce a **regularized optimal control problem (ROCP)** and following the arguments from [5] we conclude the convergence of solutions of the regularized problem. In the present note we will provide the essential ingredients to apply the results of [5] to the fractional case. Notice that due to the possible degeneracy in the state equation it is unclear how to derive the first order stationarity system for **OCP**. However, **ROCP** comes to rescue, indeed the latter is built to precisely avoid such degeneracy issues.

Differential equations of fractional order have gained a lot of attraction in recent years due to the fact that several phenomena in the sciences are more accurately modelled by such equations rather than the traditional equations of integer order.

Linear and nonlinear equations have been extensively studied. The applications in industry are numerous and cover almost every area. From the long list of phenomena which are more appropriately modelled by fractional differential equations, we mention: viscoelasticity, anomalous transport and diffusion, hereditary phenomena with long memory, nonlocal electrostatics, the latter being relevant to drug design, and Lévy motions which appear in important models in both applied mathematics and applied probability, image processing and phase field models, as well as in models in biology and ecology. We refer to [3, 4, 31, 35, 38] and their references for more details on this topic. For the motivations that lead to the study of nonlinear nonlocal operators that include both fractional p-Laplace operators, we refer the reader to the contribution by Caffarelli [11]. A more complete list of possible applications of such operators related to fractional order Sobolev spaces is contained in [18]. It is well known that the standard p-Laplacian $\Delta_{p,a}$ appears in many applications such as fluid dynamics [19], quantum physics [6], optimal mass transport [22], image and data processing [21], electrorheological fluids [36] and many others. Besides the above mentioned motivation, since by Section 2.4 below, the fractional p-Laplacian approaches the standard p-Laplacian as s approaches 1, it is also a natural problem to consider optimal control or finding the unknown coefficients appearing in the definition of the fractional p-Laplace operator in the same spirit as what it has already been done for the p-Laplacian in [15] and their references. That is the main concern of the present paper (fractional case) and our previous work [5] (regional case).

Precisely following [5] we can show the well-posedness (existence, uniqueness, and continuous dependence on data) of our state equation (1.2) and the regularized state equation (3.8).

The rest of the paper is organized as follows: In Section 2.1 we introduce the fractional order Sobolev spaces that are mostly useful for the regional fractional p-Laplacian. We also discuss the extensions of these spaces to the fractional p-Laplacian case in Section 2.2. This is followed by a discussion on BV-functions in Section 2.3. We provide a precise definition of the fractional p-Laplacian in Section 2.4. Here we also prove that the fractional p-Laplacian approaches the local p-Laplacian as s approaches 1. In Section 3 we discuss how to extend the results for **(OCP)** with regional fractional p-Laplacian discussed in [5] to our fractional p-Laplacian case.

2. Notation and Preliminaries

Here we introduce the function spaces needed to investigate our problem, give a rigorous definition of the operators involved and also provide some intermediate results relevant to the paper. The discussion in (sub)-section 2.1 taken from [5, Section 2] is mostly relevant for the regional fractional p-Laplacian. On the other hand the discussion in (sub)-section 2.2 is for the fractional p-Laplacian where we discuss the relevant modifications to the spaces used for the regional fractional p-Laplacian. The results stated in this section are valid for any $0 < s < 1$.

2.1. The fractional order Sobolev spaces for the regional fractional p-Laplacian. In this (sub)section, we recall some well-known results on fractional order Sobolev spaces that are relevant to the paper and we also introduce some notations.

Let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded open set. If $0 < \tau \leq 1$, then we denote by $C^{0,\tau}(\overline{\Omega})$ the space of all Hölder continuous (Lipschitz continuous if $\tau = 1$) functions of order τ on $\overline{\Omega}$. For $k \in \mathbb{N} \cup \{0\}$, we denote by $C_c^k(\Omega)$ the space of all real functions continuously differentiable in Ω up to the order k and with compact support. If $k = \infty$, then $C_c^\infty(\Omega)$ is denoted by $\mathcal{D}(\Omega)$. We let $C_0^k(\Omega)$ be the closure in the sup-norm of $C_c^k(\Omega)$. If $k = 0$, then we will denote $C_c(\Omega) := C_c^0(\Omega)$ and $C_0(\Omega) := C_0^0(\Omega)$.

For $p \in [1, \infty)$ and $s \in (0,1)$, we denote by

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} dxdy < \infty \right\},$$

the fractional order Sobolev space endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_\Omega |u|^p \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{\frac{1}{p}}.$$

We let

$$W_0^{s,p}(\Omega) := \overline{\mathcal{D}(\Omega)}^{W^{s,p}(\Omega)}.$$

The following result is taken from [25, Theorem 1.4.2.4, p.25] (see also [8, 41]).

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a Lipschitz continuous boundary and $1 < p < \infty$. Then the following assertions hold.*

(a) If $0 < s \leq \frac{1}{p}$, then $W^{s,p}(\Omega) = W_0^{s,p}(\Omega)$.

(b) If $\frac{1}{p} < s < 1$, then $W_0^{s,p}(\Omega)$ is a proper closed subspace of $W^{s,p}(\Omega)$.

A complete characterization of $W_0^{s,p}(\Omega)$ for arbitrary bounded open sets has been given in [41] by using some potential theory. We notice that it follows from Theorem 2.1 that for a bounded open set with a Lipschitz continuous boundary, if $\frac{1}{p} < s < 1$, then

$$\|u\|_{W_0^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{\frac{1}{p}} \tag{2.1}$$

defines an equivalent norm on $W_0^{s,p}(\Omega)$. In that case, we shall always use this norm for the space $W_0^{s,p}(\Omega)$. Let $p^\star$ be given by

$$p^\star = \frac{Np}{N - sp} \text{ if } N > sp \text{ and } p^\star \in [p, \infty) \text{ if } N = sp. \tag{2.2}$$

Then by [18, Theorems 6.7 and 6.10], there is a constant $C = C(N,p,s) > 0$ such that for every $u \in W_0^{s,p}(\Omega)$,

$$\|u\|_{L^q(\Omega)} \leq C \|u\|_{W_0^{s,p}(\Omega)}, \quad \forall \, q \in [1, p^\star]. \tag{2.3}$$

Moreover, the continuous embedding $W_0^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for every $q \in [1, p^\star)$ (see e.g. [18, Corollary 7.2]). If $N < sp$, then one has the continuous embedding $W_0^{s,p}(\Omega) \hookrightarrow C^{0,s-\frac{N}{p}}(\overline{\Omega})$ (see e.g. [18, Theorem 8.2]).

Recall that the first order Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) := \left\{ u \in L^p(\Omega) : \int_\Omega |\nabla u|^p \, dx < \infty \right\} \quad \text{and} \quad W_0^{1,p}(\Omega) := \overline{\mathcal{D}(\Omega)}^{W^{1,p}(\Omega)},$$

where

$$\|u\|_{W^{1,p}(\Omega)} := \left(\int_\Omega |u|^p \, dx + \int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{p}}.$$

Then $\|\nabla u\|_{L^p(\Omega)}$ defines an equivalent norm on $W_0^{1,p}(\Omega)$. It is well-known that for this case $s = 1$, we have the continuous embedding (since Ω is bounded)

$$W_0^{1,q}(\Omega) \hookrightarrow W_0^{1,p}(\Omega), \quad \forall\, q \geq p.$$

More precisely, using the Hölder inequality we get that for every $q \geq p$ and $u \in W_0^{1,q}(\Omega)$,

$$\|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_{L^p(\Omega)} \leq |\Omega|^{\frac{q-p}{pq}} \|\nabla u\|_{L^q(\Omega)} = |\Omega|^{\frac{q-p}{pq}} \|u\|_{W_0^{1,q}(\Omega)}.$$

The situation is different for the fractional order Sobolev spaces $W_0^{s,p}(\Omega)$ $(0 < s < 1)$. In fact we have the following.

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded open set and $p \in [1, \infty)$. Then the following assertions hold.*
(a) If $0 < t \leq s < 1$, then $W_0^{s,p}(\Omega) \hookrightarrow W_0^{t,p}(\Omega)$.
(b) For every $0 < s < 1$, we have that $W_0^{1,p}(\Omega) \hookrightarrow W_0^{s,p}(\Omega)$.
(c) Let $q > p$. If $0 < t < s < 1$, then $W_0^{s,q}(\Omega) \hookrightarrow W_0^{t,p}(\Omega)$.

Proof. The proof of the assertions (a) and (b) is contained in [18, Proposition 2.1] and [41, Proposition 2.3], respectively. We refer to [5, Proposition 2.2] for the proof of the assertion (c). □

2.2. **The fractional order Sobolev spaces for the fractional p-Laplacian.** We recall from [5] the function space needed to study the regional fractional p-Laplacian problem is $W_0^{s,p}(\Omega)$ which we discussed in the previous subsection. In order to study the fractional p-Laplace equation (1.2) we need to consider the following function space

$$W_0^{s,p}(\overline{\Omega}) := \Big\{ u \in W^{s,p}(\mathbb{R}^N) : \; u = 0 \;\text{ on }\; \mathbb{R}^N \setminus \Omega \Big\}.$$

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a Lipschitz continuous boundary, it has been shown in [23, Theorem 6] that $\mathcal{D}(\Omega)$ is dense in $W_0^{s,p}(\overline{\Omega})$. Moreover, for every $0 < s < 1$,

$$\|u\|_{W_0^{s,p}(\overline{\Omega})} := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx dy \right)^{\frac{1}{p}}$$

defines an equivalent norm on $W_0^{s,p}(\overline{\Omega})$. We next state a result for $W_0^{s,p}(\overline{\Omega})$ spaces (recall Theorem 2.1 for $W_0^{s,p}(\Omega)$ spaces).

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a Lipschitz continuous boundary and $1 < p < \infty$. Then the following assertion hold*
- *(a) If $0 < s \leq \frac{1}{p}$, then $W_0^{s,p}(\Omega)$ and $W_0^{s,p}(\overline{\Omega})$ have no obvious relationship or inclusion.*
- *(b) If $\frac{1}{p} < s < 1$ then $W_0^{s,p}(\overline{\Omega}) = W_0^{s,p}(\Omega)$ with equivalent norms.*

Proof. For part (a) it is enough to see that the constant function 1 belongs to $W^{s,p}(\Omega) = W_0^{s,p}(\Omega)$ when $0 < s \leq \frac{1}{p}$, but it does not belong to $W_0^{s,p}(\overline{\Omega})$ (1 is not zero on $\mathbb{R}^N \setminus \Omega$).

Under the assumption that Ω is Lipschitz continuous it has been shown in [25, Formula (1.3.2.12), p.19] that there exist two constants $0 < C_1 \leq C_2$ such that

$$\frac{C_1}{(\operatorname{dist}(x, \partial\Omega))^{ps}} \leq \int_{\mathbb{R}^N \setminus \Omega} \frac{dy}{|x - y|^{N+sp}} \leq \frac{C_2}{(\operatorname{dist}(x, \partial\Omega))^{ps}}. \tag{2.4}$$

Using (2.4) and the *Hardy inequality* for fractional order Sobolev spaces (see e.g. [30, Theorem 1.2] and also [20]), we get that there is a constant $C > 0$ such that for every $u \in \mathcal{D}(\Omega)$,

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy =& \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy\\ &+2\int_\Omega |u(x)|^p\int_{\mathbb{R}^N\setminus\Omega}\frac{dy}{|x-y|^{N+sp}}\,dx\\ \le& \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy + C\int_\Omega \frac{|u(x)|^p}{\left(\operatorname{dist}(x,\partial\Omega)\right)^{ps}}\,dx\\ \le& C\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy,\end{aligned}$$

where in the last step we have used the Hardy inequality. Using the above estimate, we also have that for every $u \in \mathcal{D}(\Omega)$,

$$\begin{aligned}\int_\Omega |u(x)|^p\,dx =& \int_\Omega \left(\operatorname{dist}(x,\partial\Omega)\right)^{ps}\frac{|u(x)|^p}{\left(\operatorname{dist}(x,\partial\Omega)\right)^{ps}}\,dx \le C\int_\Omega \frac{|u(x)|^p}{\left(\operatorname{dist}(x,\partial\Omega)\right)^{ps}}\,dx\\ \le& C\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy.\end{aligned}$$

This completes the proof. □

We also need a result analogous to Proposition 2.2.

Proposition 2.4. *Let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded open set and $p \in [1,\infty)$. Then the following assertions hold.*
(a) If $0 < t \le s < 1$, then $W_0^{s,p}(\overline{\Omega}) \hookrightarrow W_0^{t,p}(\overline{\Omega})$.
(b) For every $0 < s < 1$, we have that $W_0^{1,p}(\overline{\Omega}) \hookrightarrow W_0^{s,p}(\overline{\Omega})$.
(c) Let $q > p$. If $0 < t < s < 1$, then $W_0^{s,q}(\overline{\Omega}) \hookrightarrow W_0^{t,p}(\overline{\Omega})$.

Proof. The proof is analogous to the proof of Proposition 2.2. □

If $0 < s < 1$, $p \in (1,\infty)$ and $p' := \frac{p}{p-1}$, then the space $W^{-s,p'}(\overline{\Omega})$ is defined as usual to be the dual of the reflexive Banach space $W_0^{s,p}(\overline{\Omega})$, that is, $W^{-s,p'}(\overline{\Omega}) := (W_0^{s,p}(\overline{\Omega}))^\star$. For $u \in W_0^{s,p}(\overline{\Omega})$ we shall denote by $U_{(p,s)}$ the function defined on $\mathbb{R}^N \times \mathbb{R}^N$ by

$$U_{(p,s)}(x,y) := \frac{u(x)-u(y)}{|x-y|^{\frac{N}{p}+s}}. \tag{2.5}$$

We will always denote by χ_E the characteristic function of a set $E \subseteq \mathbb{R}^N \times \mathbb{R}^N$.

Remark 2.5. We notice that by definition of $W^{s,p}(\mathbb{R}^N)$ we have that

$$W^{s,p}(\mathbb{R}^N) = \left\{u \in L^p(\mathbb{R}^N) :\ U_{(p,s)} \in L^p(\mathbb{R}^N \times \mathbb{R}^N)\right\}. \tag{2.6}$$

Let $u \in W_0^{s,p}(\overline{\Omega})$, $\{u_n\}_{n\in\mathbb{N}}$ a sequence in $W_0^{s,p}(\overline{\Omega})$ and define $U_{n,(p,s)}$ on $\mathbb{R}^N \times \mathbb{R}^N$ by $U_{n,(p,s)}(x,y) := \dfrac{u_n(x)-u_n(y)}{|x-y|^{\frac{N}{p}+s}}$. Using the above characterization of $W^{s,p}(\mathbb{R}^N)$ and the definition of $W_0^{s,p}(\overline{\Omega})$ we get that the following assertions hold.

(a) If u_n converges weakly to u in $W_0^{s,p}(\overline{\Omega})$ as $n \to \infty$ (that is, $u_n \rightharpoonup u$ in $W_0^{s,p}(\overline{\Omega})$ as $n \to \infty$), then for every $\varphi \in \mathcal{D}(\mathbb{R}^N)$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dxdy$$
$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dxdy.$$

(b) If $u_n \rightharpoonup u$ in $W_0^{s,p}(\overline{\Omega})$ as $n \to \infty$, then $U_{n,(p,s)} \rightharpoonup U_{(p,s)}$ in $L^p(\mathbb{R}^N \times \mathbb{R}^N)$ as $n \to \infty$.

(c) If $u_n \rightharpoonup u$ in $W_0^{s,p}(\overline{\Omega})$ and $U_{n,(p,s)} \to U_{(p,s)}$ in $L^p(\mathbb{R}^N \times \mathbb{R}^N)$ as $n \to \infty$, then $u_n \to u$ in $W_0^{s,p}(\overline{\Omega})$ as $n \to \infty$.

We give an idea of the proof of the above assertions.

Proof. Since $W_0^{s,p}(\overline{\Omega})$ is a closed subspace of $W^{s,p}(\mathbb{R}^N)$, then the assertion (b) follows directly from the characterization of $W^{s,p}(\mathbb{R}^N)$ given in (2.6). It follows from part (b) that if $u_n \rightharpoonup u$ in $W_0^{s,p}(\overline{\Omega})$ as $n \to \infty$, then

$$\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} U_{n,(p,s)}(x,y)\Phi(x,y)\,dxdy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} U_{(p,s)}(x,y)\Phi(x,y)\,dxdy$$

for every $\Phi \in L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$ where $1/p + 1/p' = 1$. Let $\varphi \in \mathcal{D}(\mathbb{R}^N)$. Since the function $\Phi(x,y) := \dfrac{\varphi(x)-\varphi(y)}{|x-y|^{\frac{N}{p'}+s}} \in L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$, it follows that for every $\varphi \in \mathcal{D}(\mathbb{R}^N)$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} U_{n,(p,s)}(x,y)\Phi(x,y)\,dxdy$$
$$= \lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dxdy$$
$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} U_{(p,s)}(x,y)\Phi(x,y)\,dxdy$$
$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dxdy$$

and we have shown part (a). Now assume that $u_n \rightharpoonup u$ in $W_0^{s,p}(\overline{\Omega})$ and $U_{n,(p,s)} \to U_{(p,s)}$ in $L^p(\Omega)$ as $n \to \infty$. Since the embedding $W_0^{s,p}(\overline{\Omega}) \hookrightarrow L^p(\Omega)$ is compact, we have that $u_n \to u$ (strongly) in $L^p(\Omega)$ as $n \to \infty$. Since $u_n = u = 0$ on $\mathbb{R}^N \setminus \Omega$, it follows that $u_n \to u$ in $L^p(\mathbb{R}^N)$ as $n \to \infty$. Since $u_n \to u$ in $L^p(\mathbb{R}^N)$ and $U_{n,(p,s)} \to U_{(p,s)}$ in $L^p(\mathbb{R}^N \times \mathbb{R}^N)$ as $n \to \infty$, then part (c) follows from the characterization (2.6). The proof is finished. □

For more information on fractional order Sobolev spaces we refer the reader to [1, 8, 18, 25, 26, 29, 41] and the references therein.

2.3. **Functions of bounded variation.** Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set. Let

$$BV(\Omega) := \Big\{g \in L^1(\Omega) : \ \|g\|_{BV(\Omega)} < \infty\Big\},$$

be the space of functions of bounded variation, where

$$\|g\|_{BV(\Omega)} := \|g\|_{L^1(\Omega)} + \sup\left\{\int_\Omega g \,\mathrm{div}(\Phi)\,dx : \ \Phi \in C_0^1(\Omega,\mathbb{R}^N),\ |\Phi(x)| \le 1,\ x \in \Omega\right\}.$$

For $g \in BV(\Omega)$, we denote by ∇g the distributional gradient of g. We notice that ∇g belongs to the space of Radon measures $\mathcal{M}(\Omega, \mathbb{R}^N)$.

The following notion of convergence of a sequence in $BV(\Omega)$ is contained in [2, Definition 3.1].

Remark 2.6. Let $g \in BV(\Omega)$ and $\{g_n\}_{n\in\mathbb{N}}$ a sequence in $BV(\Omega)$.

(a) We say that $\{g_n\}_{n\in\mathbb{N}}$ converges weakly* ($\overset{*}{\rightharpoonup}$) to $g \in BV(\Omega)$ as $n \to \infty$, if and only if the following two conditions hold.

(i) $g_n \to g$ (strongly) in $L^1(\Omega)$ as $n \to \infty$, and

(ii) $\nabla g_n \overset{*}{\rightharpoonup} \nabla g$ (weakly*) in $\mathcal{M}(\Omega, \mathbb{R}^N)$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} \int_\Omega \phi \, d\nabla g_n = \int_\Omega \phi \, d\nabla g, \quad \forall \, \phi \in C_0(\Omega).$$

(b) In addition, if g_n converges strongly to some $\tilde{g}$ in $L^1(\Omega)$ as $n \to \infty$ and satisfies $\sup_{n\in\mathbb{N}} \int_\Omega |\nabla g_n| < \infty$, then

$$\tilde{g} \in BV(\Omega), \quad \int_\Omega |\nabla \tilde{g}| \le \liminf_{n\to\infty} \int_\Omega |\nabla g_n| \text{ and } g_n \overset{*}{\rightharpoonup} \tilde{g} \text{ in } BV(\Omega) \text{ as } n \to \infty.$$

We conclude this (sub)section by giving some embedding results for the space $BV(\Omega)$ taken from [2, Theorem 3.49] and [2, Corollary 3.49].

Proposition 2.7. *Let*

$$1^\star := \infty \text{ if } N = 1 \text{ and } 1^\star := \frac{N}{N-1} \text{ if } N > 1.$$

Then the embedding $BV(\mathbb{R}^N) \hookrightarrow L^{1^\star}(\mathbb{R}^N)$ is continuous. If in addition $\Omega \subset \mathbb{R}^N$ is a bounded open set with Lipschitz continuous boundary then the embedding $BV(\Omega) \hookrightarrow L^q(\Omega)$ is compact for every $1 \le q < 1^\star$.

For more details on functions of bounded variation we refer to the monograph [2, Chapter 3].

2.4. **The fractional p-Laplacian.** To introduce the fractional p-Laplace operator, let $0 < s < 1$, $p \in (1, \infty)$ and set

$$\mathbb{L}_s^{p-1}(\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable}, \int_{\mathbb{R}^N} \frac{|u(x)|^{p-1}}{(1+|x|)^{N+ps}} \, dx < \infty \right\}.$$

For $u \in \mathbb{L}_s^{p-1}(\mathbb{R}^N)$, $x \in \mathbb{R}^N$ and $\varepsilon > 0$, we let

$$(-\Delta)^s_{p,\varepsilon} u(x) = C_{N,p,s} \int_{\{y\in\mathbb{R}^N, |y-x|>\varepsilon\}} |u(x) - u(y)|^{p-2} \frac{u(x) - u(y)}{|x-y|^{N+ps}} dy,$$

where the normalized constant

$$C_{N,p,s} := \frac{sp2^{2s-1}\Gamma(\frac{sp+N}{2})}{2\pi^{\frac{N-1}{2}}\Gamma(1-s)\Gamma(\frac{p+1}{2})}, \tag{2.7}$$

and Γ is the usual Gamma function (see, e.g. [7, 12, 13, 14, 18, 10] for the linear case $p = 2$, and [42, 13, 44] for the general case $p \in (1, \infty)$). The constant $C_{N,p,s}$

for $p \neq 2$ has been introduced and justified in [42, Remark 3.10]. The **fractional p-Laplacian** $(-\Delta)_p^s$ is defined for $u \in \mathbb{L}_s^{p-1}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$ by the formula

$$\begin{aligned}(-\Delta)_p^s u(x) =& C_{N,p,s}\text{P.V.} \int_{\mathbb{R}^N} |u(x)-u(y)|^{p-2}\frac{u(x)-u(y)}{|x-y|^{N+ps}}dy \\ =& \lim_{\varepsilon\downarrow 0}(-\Delta)_{p,\varepsilon}^s u(x),\end{aligned} \tag{2.8}$$

provided that the limit exists. We notice that if $0 < s < \frac{p-1}{p}$ and u is smooth (i.e., at least bounded and Lipschitz continuous defined over $\mathbb{R}^N$), then the integral in (2.8) is in fact not really singular near x. We refer to the [5, Section 2.3] for a proof in case of the regional fractional Laplacian.

In addition $\mathbb{L}_s^{p-1}(\mathbb{R}^N)$ is the right space on which $(-\Delta)_{p,\varepsilon}^s u$ exists for every $\varepsilon > 0$ and is continuous where u is continuous (see e.g. [42] for more detail on this topic).

Next we give some known integration by part formulas. Let $1 < p < \infty$ and $0 < s < 1$. For every $u, v \in \mathcal{D}(\mathbb{R}^N)$,

$$\begin{aligned}&\int_{\mathbb{R}^N}(-\Delta)_p^s(\kappa, u)(x)v(x)\,dx \\ =&\frac{C_{N,p,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\kappa(x-y)|u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,dxdy.\end{aligned}$$

and for $u, v \in W^{s,p}(\mathbb{R}^N)$, we have that

$$\begin{aligned}&\langle(-\Delta)_p^s(\kappa, u), v\rangle_{W^{-s,p'}(\mathbb{R}^N)),W^{s,p}(\mathbb{R}^N)} \\ =&\frac{C_{N,p,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\kappa(x-y)|u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,dxdy,\end{aligned} \tag{2.9}$$

where $W^{-s,p'}(\mathbb{R}^N)$ denotes the dual of the reflexive Banach space $W^{s,p}(\mathbb{R}^N)$. For more details on the validity of the above identities we refer to [16, 17, 42] and their references.

Proceeding as in the regional p-Laplacian case [5, Section 2.3], one also has that for every $u \in \mathcal{D}(\Omega)$,

$$\begin{aligned}\lim_{s\uparrow 1}\int_\Omega u(-\Delta)_p^s u\,dx =& \lim_{s\uparrow 1}\int_{\mathbb{R}^N} u(-\Delta)_p^s u\,dx \\ =& \lim_{s\uparrow 1}\frac{C_{N,p,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy \\ =& \int_{\mathbb{R}^N}|\nabla u|^p\,dx = -\int_\Omega u\Delta_p u\,dx.\end{aligned} \tag{2.10}$$

Here the second equality follows from (2.9) with $\kappa = 1$ and the third equality follows from [9] and [10, Proposition 2.2].

Let κ be as in (1.5). For $1 < p < \infty$, $0 < s < 1$ and $u \in \mathbb{L}_s^{p-1}(\mathbb{R}^N)$ we define the operator $(-\Delta)_p^s(\kappa,\cdot)$ as in (1.6). We again call this operator, the **fractional p-Laplace operator**.

For more details on this topic we refer to [18, 42] and their references. We mention that elliptic problems associated with the operator $\mathcal{L}_{\Omega,p}^s(\kappa,\cdot)$ subject to the Dirichlet boundary condition have been investigated in [16, 17, 24, 28] where the authors have obtained some fundamental existence and regularity results. The case of Neumann and Robin type boundary conditions (with $\kappa = 1$) is contained in [41]. We refer to [24, 43] for further results on associated parabolic problems.

3. Optimal control of the fractional p-Laplacian for $0 < s < 1$

Throughout the remainder of the paper we will make the following assumption:

Assumption 3.1. *We shall always assume the following.*

(a) $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ *is a bounded open set with Lipschitz continuous boundary.*
(b) $0 < s < 1$ *and* $2 \leq p < \infty$.
(c) The functions $\xi_1, \xi_2 \in L^\infty(\Omega)$ *and there exists a constant* $\alpha > 0$ *such that*

$$0 < \alpha \leq \xi_1(x) \leq \xi_2(x) \quad a.e. \text{ in } \Omega. \tag{3.1}$$

(d) The measurable function κ *satisfies* (1.5) *and belongs to* $\mathfrak{A}_{ad}$.
(e) $f \in L^2(\Omega)$ *and* $\xi \in L^2(\Omega)$.

Under Assumption 3.1 all our results for the regional fractional p-Laplacian in [5, Sections 3-6] are valid for the fractional p-Laplace operator $(-\Delta)_p^s(\kappa,\cdot)$ given in (1.4). In this case one replaces the state system [5, Eq. (1.2)] by (1.2) and $W_0^{s,p}(\Omega)$ by the space $W_0^{s,p}(\overline{\Omega})$. We state the relevant results in the following (sub-)sections.

With the above setting, we are ready to state the notions of solutions to our state equation (1.2):

Definition 3.2. *A* $u \in W_0^{s,p}(\overline{\Omega})$ *is said to be a weak solution of* (1.2) *if the identity*

$$\frac{C_{N,p,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \kappa(x-y)|u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dxdy$$
$$+\int_\Omega u\varphi\,dx = \int_\Omega f\varphi\,dx \tag{3.2}$$

holds for every $\varphi \in W_0^{s,p}(\overline{\Omega})$.

The following existence result of optimal pair to the **OCP** is our first main result.

Theorem 3.3. *Assume Assumption 3.1. Then the* **OCP** (1.1), (1.2) *and* (1.3) *admits at least one solution* $(\kappa, u) \in BV(\mathbb{R}^N) \times W_0^{s,p}(\overline{\Omega})$.

Remark 3.4. *In* [5] *we needed the assumption* $\frac{1}{2} < s < 1$. *This assumption is not needed in the fractional case, i.e., all the results here hold for every* $0 < s < 1$.

3.1. The regularized optimal control problem.

Let $n \in \mathbb{N}$ and $\mathcal{F}_n : [0,\infty) \to [0,\infty)$ be a function in $C^1([0,\infty))$ satisfying

$$\begin{cases} \mathcal{F}_n(\tau) = \tau & \text{if } 0 \leq \tau \leq n^2, \\ \mathcal{F}_n(\tau) = n^2 + 1 & \text{if } \tau \geq n^2 + 1, \\ \tau \leq \mathcal{F}_n(\tau) \leq \tau + \delta & \text{if } n^2 \leq \tau < n^2 + 1 \text{ for some } \delta \in (0,1). \end{cases} \tag{3.3}$$

Let Δ_p be the p-Laplace operator defined in (1.7). It is well-known that Δ_p is a degenerate operator if $p > 2$. This is also the case for the operator $\Delta_{p,a}$ defined in (1.8). To overcome the degeneracy in the case $p > 2$, an (ε, p)-regularization $\Delta_{\varepsilon,n,p,a}$ of $\Delta_{p,a}$ has been introduced (see e.g. [15]) as follows:

$$\Delta_{\varepsilon,n,p,a} u = \operatorname{div}\Big(a(x)(\varepsilon + \mathcal{F}_n(|\nabla u|^2))^{\frac{p-2}{2}}\nabla u\Big),$$

where $\mathcal{F}_n$ is the function defined in (3.3) for $n \in \mathbb{N}$ and $\varepsilon > 0$ is a small parameter. Using the classical definition of degenerate elliptic operators, one cannot immediately say that $(-\Delta)_p^s$ (or $\mathcal{L}_p^s(\kappa,\cdot)$) is degenerate for $p > 2$. We refer to [39] for a discussion on this topic.

But inspired by the convergence given in (2.10), we introduce the operator $(-\Delta)^s_{p,\varepsilon,n}(\kappa,\cdot)$ defined for $u\in \mathbb{L}^{p-1}_s(\mathbb{R}^N)$ and $x\in\mathbb{R}^N$ by

$$(-\Delta)^s_{p,\varepsilon,n}(\kappa,u):=C_{N,p,s}\text{P.V.}\int_{\mathbb{R}^N}\kappa(x-y)\mathcal{G}_{\varepsilon,n,p}(u)\frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy,$$

where $\mathcal{F}_n$ ($n\in\mathbb{N}$) is the function given in (3.3), $\varepsilon>0$ is a small parameter and $\mathcal{G}_{\varepsilon,n,p}(u)$ is given by

$$\mathcal{G}_{\varepsilon,n,p}(u):=\left[\varepsilon+\mathcal{G}_n(u)\right]^{\frac{p-2}{2}}=\left[\varepsilon+\mathcal{F}_n\left(\frac{|u(x)-u(y)|^2}{|x-y|^{2s}}\right)\right]^{\frac{p-2}{2}}, \tag{3.4}$$

with

$$\mathcal{G}_n(u):=\mathcal{F}_n\left(\frac{|u(x)-u(y)|^2}{|x-y|^{2s}}\right). \tag{3.5}$$

We call $(-\Delta)^s_{p,\varepsilon,n}(\kappa,\cdot)$ an **(ε,p)-regularization** of $(-\Delta)^s_p(\kappa,\cdot)$.

Let $\xi,f\in L^2(\Omega)$ be given functions and $p\in[2,\infty)$. Now we consider our so called regularized optimal control problem **(ROCP)**:

$$\text{Minimize}\left\{\mathbb{I}(\kappa,u):=\frac{1}{2}\int_\Omega|u-\xi|^2\,dx+\int_\Omega|\nabla\kappa|\right\} \tag{3.6}$$

subject to the constraints

$$\kappa\in\mathfrak{A}_{ad}, \tag{3.7}$$

and

$$\begin{cases}(-\Delta)^s_{p,\varepsilon,n}(\kappa,u)+u=f & \text{in }\ \Omega\\ u=0 & \text{on }\ \mathbb{R}^N\setminus\Omega,\end{cases} \tag{3.8}$$

The following is our notion of weak solutions to the system (3.8).

Definition 3.5. *Let $n\in\mathbb{N}$, $\varepsilon>0$, $\kappa\in\mathfrak{A}_{ad}$ and $f\in L^2(\Omega)$. A $u\in W^{s,2}_0(\overline{\Omega})$ is said to be a weak solution to the system (3.8) if the equality*

$$\widetilde{\mathbb{F}}^\kappa_{\varepsilon,n,p}(u,\varphi)=\int_\Omega f\varphi\,dx \tag{3.9}$$

holds for every $\varphi\in W^{s,2}_0(\overline{\Omega})$, where for $u,\varphi\in W^{s,p}_0(\overline{\Omega})$ we have set

$$\widetilde{\mathbb{F}}^\kappa_{\varepsilon,n,p}(u,\varphi):=\int_\Omega u\varphi\,dx$$
$$\frac{C_{N,p,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\kappa(x-y)\mathcal{G}_{\varepsilon,n,p}(u)\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dxdy.$$

The following result for **(ROCP)** holds with very minor changes in the proofs in [5]. Besides the aforementioned changes one also need to replace the expression of $\mathcal{E}^\kappa_{p,s}$ in [5, Eq. (4.2)] by

$$\widetilde{\mathcal{E}}^\kappa_{p,s}(u,\varphi):=\int_\Omega u\varphi\,dx$$
$$+\frac{C_{N,p,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\kappa(x-y)|u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dxdy,$$

for $u, \varphi \in W_0^{s,p}(\overline{\Omega})$. We next state our second main result.

Theorem 3.6. *Assume Assumption 3.1. Then for every $\varepsilon > 0$ and $n \in \mathbb{N}$, the* **ROCP** (3.6)-(3.8) *has at least one solution $(\kappa_{\varepsilon,n}, u_{\varepsilon,n}) \in BV(\mathbb{R}^N) \times W_0^{s,2}(\overline{\Omega})$.*

We conclude the paper by stating the convergence of solutions of **ROCP** to the solutions of **OCP**.

Theorem 3.7. *Assume Assumption 3.1. Let $0 < t \leq s < 1$ with $t = s$ if $p = 2$. Let $n \in \mathbb{N}$ and $\varepsilon > 0$. Let $\{(\kappa^\star_{\varepsilon,n}, u^\star_{\varepsilon,n})\}_{\varepsilon>0,n\in\mathbb{N}} \subset BV(\mathbb{R}^N) \times W_0^{s,2}(\overline{\Omega})$ be an arbitrary sequence of solutions to the* **ROCP** (3.6)-(3.8). *Then $\{(\kappa^\star_{\varepsilon,n}, u^\star_{\varepsilon,n})\}_{\varepsilon>0,n\in\mathbb{N}}$ is bounded in $BV(\mathbb{R}^N) \times W_0^{t,2}(\overline{\Omega})$ and any cluster point $(\kappa_\star, u_\star)$ with respect to the* (weak*, weak) *topology of $BV(\mathbb{R}^N) \times W_0^{t,2}(\overline{\Omega})$ is a solution to the* **OCP** (1.1), (1.2) *and* (1.3). *In addition, if $\kappa^\star_{\varepsilon,n} \overset{*}{\rightharpoonup} \kappa_\star$ in $BV(\mathbb{R}^N)$ and $u^\star_{\varepsilon,n} \rightharpoonup u_\star$ in $W_0^{t,2}(\overline{\Omega})$, as $\varepsilon \to 0$ and $n \to \infty$* (that is, as $(\varepsilon, n) \to (0, \infty)$), *then the following assertions hold.*

(3.10)

$$\lim_{(\varepsilon,n)\to(0,\infty)} (\kappa^\star_{\varepsilon,n}, u^\star_{\varepsilon,n}) = (\kappa_\star, u_\star) \quad \textit{stongly in} \quad L^1(\mathbb{R}^N) \times W_0^{t,2}(\overline{\Omega}).$$

(3.11)

$$\lim_{(\varepsilon,n)\to(0,\infty)} \int_{\mathbb{R}^N} |\nabla \kappa^\star_{\varepsilon,n}| = \int_{\mathbb{R}^N} |\nabla \kappa_\star|.$$

(3.12)

$$\lim_{(\varepsilon,n)\to(0,\infty)} \chi_{(\mathbb{R}^N\times\mathbb{R}^N)\setminus(\mathbb{R}^N\times\mathbb{R}^N)_n(u^\star_{\varepsilon,n})} U^\star_{\varepsilon,n,(p,s)} = U_{\star,(p,s)} \quad \textit{strongly in} \quad L^p(\mathbb{R}^N \times \mathbb{R}^N).$$

(3.13)

$$\lim_{(\varepsilon,n)\to(0,\infty)} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \kappa^\star_{\varepsilon,n}(x-y) \left[\varepsilon + \mathcal{G}_n\left(u^\star_{\varepsilon,n}\right)\right]^{\frac{p-2}{2}} \frac{|u^\star_{\varepsilon,n}(x) - u^\star_{\varepsilon,n}(y)|^2}{|x-y|^{N+2s}}\, dy$$
$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \kappa_\star(x-y) \frac{|u_\star(x) - u_\star(y)|^p}{|x-y|^{N+sp}}\, dxdy.$$

(3.14)

$$\lim_{(\varepsilon,n)\to(0,\infty)} \mathbb{I}(\kappa^\star_{\varepsilon,n}, u^\star_{\varepsilon,n}) = \mathbb{I}(\kappa_\star, u_\star),$$

where we recall that $\mathcal{G}_n$ is given by (3.5). *Here*

$$(\mathbb{R}^N \times \mathbb{R}^N)_n(u^\star_{\varepsilon,n}) := \left\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u^\star_{\varepsilon,n}(x) - u^\star_{\varepsilon,n}(y)|}{|x-y|^s} > \sqrt{n^2+1}\right\}.$$

References

[1] D.R. Adams and L.I. Hedberg. *Function Spaces and Potential Theory*, volume 314 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1996.

[2] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[3] H. Antil and S. Bartels. Spectral Approximation of Fractional PDEs in Image Processing and Phase Field Modeling. *Comput. Methods Appl. Math.*, 17(4):661–678, 2017.

[4] H. Antil and C.N. Rautenberg. Sobolev spaces with non-Muckenhoupt weights, fractional elliptic operators, and applications. *arXiv preprint arXiv:1803.10350*, 2018.

[5] H. Antil and M. Warma. Optimal control of the coefficient for regional fractional {p}-{L} aplace equations: Approximation and convergence. *To appear: Math. Control Relat. Fields.*, 2018.
[6] V. Benci, P. D'Avenia, D. Fortunato, and L. Pisani. Solitons in several space dimensions: Derrick's problem and infinitely many solutions. *Arch. Ration. Mech. Anal.*, 154(4):297–324, 2000.
[7] C. Bjorland, L. Caffarelli, and A. Figalli. Nonlocal tug-of-war and the infinity fractional Laplacian. *Comm. Pure Appl. Math.*, 65(3):337–380, 2012.
[8] K. Bogdan, K. Burdzy, and Z.-Q. Chen. Censored stable processes. *Probab. Theory Related Fields*, 127(1):89–152, 2003.
[9] J. Bourgain, H. Brezis, and P. Mironescu. Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications. *J. Anal. Math.*, 87:77–101, 2002. Dedicated to the memory of Thomas H. Wolff.
[10] L. Brasco, E. Parini, and M. Squassina. Stability of variational eigenvalues for the fractional p-Laplacian. *Discrete Contin. Dyn. Syst.*, 36(4):1813–1845, 2016.
[11] L. Caffarelli. Non-local diffusions, drifts and game. *Nonlinear Partial Differential Equations, Abel Sym- posia*, 7(1-2):37–52, 2012.
[12] L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7-9):1245–1260, 2007.
[13] L.A. Caffarelli, J.-M. Roquejoffre, and Y. Sire. Variational problems for free boundaries for the fractional Laplacian. *J. Eur. Math. Soc. (JEMS)*, 12(5):1151–1179, 2010.
[14] L.A. Caffarelli, S. Salsa, and L. Silvestre. Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian. *Invent. Math.*, 171(2):425–461, 2008.
[15] E. Casas, P.I. Kogut, and G. Leugering. Approximation of optimal control problems in the coefficient for the p-Laplace equation. I. Convergence result. *SIAM J. Control Optim.*, 54(3):1406–1422, 2016.
[16] Agnese Di C., Tuomo K., and G. Palatucci. Nonlocal Harnack inequalities. *J. Funct. Anal.*, 267(6):1807–1836, 2014.
[17] Agnese Di C., T. Kuusi, and G. Palatucci. Local behavior of fractional p-minimizers. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33(5):1279–1299, 2016.
[18] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.
[19] L. Diening, A. Prohl, and M. Růžička. Semi-implicit Euler scheme for generalized Newtonian fluids. *SIAM J. Numer. Anal.*, 44(3):1172–1190 (electronic), 2006.
[20] B. Dyda. A fractional order Hardy inequality. *Illinois J. Math.*, 48(2):575–588, 2004.
[21] A. Elmoataz, M. Toutain, and D. Tenbrinck. On the p-Laplacian and ∞-Laplacian on graphs with applications in image and data processing. *SIAM J. Imaging Sci.*, 8(4):2412–2451, 2015.
[22] L.C. Evans. Partial differential equations and Monge-Kantorovich mass transfer. In *Current developments in mathematics, 1997 (Cambridge, MA)*, pages 65–126. Int. Press, Boston, MA, 1999.
[23] A. Fiscella, R. Servadei, and E. Valdinoci. Density properties for fractional Sobolev spaces. *Ann. Acad. Sci. Fenn. Math.*, 40(1):235–253, 2015.
[24] C.G. Gal and M. Warma. On some degenerate non-local parabolic equation associated with the fractional p-Laplacian. *Dyn. Partial Differ. Equ.*, 14(1):47–77, 2017.
[25] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics.* Pitman (Advanced Publishing Program), Boston, MA, 1985.
[26] A. Jonsson and H. Wallin. Function spaces on subsets of $\mathbf{R}^n$. *Math. Rep.*, 2(1):xiv+221, 1984.
[27] Olha P. Kupenko and Rosanna Manzo. Approximation of an optimal control problem in coefficient for variational inequality with anisotropic p-Laplacian. *NoDEA Nonlinear Differential Equations Appl.*, 23(3):Art. 35, 18, 2016.
[28] T. Kuusi, G. Mingione, and Y. Sire. Nonlocal equations with measure data. *Comm. Math. Phys.*, 337(3):1317–1368, 2015.
[29] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I.* Springer-Verlag, New York-Heidelberg, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.
[30] M. Loss and C. Sloane. Hardy inequalities for fractional integrals on general domains. *J. Funct. Anal.*, 259(6):1369–1379, 2010.

[31] F. Mainardi. *Fractional calculus and waves in linear viscoelasticity.* Imperial College Press, London, 2010. An introduction to mathematical models.
[32] F. Murat. Un contre-exemple pour le problème du contrôle dans les coefficients. *C. R. Acad. Sci. Paris Sér. A-B*, 273:A708–A711, 1971.
[33] F. Murat. Contre-exemples pour divers problèmes où le contrôle intervient dans les coefficients. *Ann. Mat. Pura Appl. (4)*, 112:49–68, 1977.
[34] F. Murat and L. Tartar. H-convergence. In *Topics in the mathematical modelling of composite materials*, volume 31 of *Progr. Nonlinear Differential Equations Appl.*, pages 21–43. Birkhäuser Boston, Boston, MA, 1997.
[35] I. Pan and S. Das. *Intelligent fractional order systems and control: an introduction*, volume 438. Springer, 2012.
[36] M. Růžička. *Electrorheological fluids: modeling and mathematical theory*, volume 1748 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.
[37] L. Tartar. Problèmes de contrôle des coefficients dans des équations aux dérivées partielles. In *Control theory, numerical methods and computer systems modelling (Internat. Sympos., IRIA LABORIA, Rocquencourt, 1974)*, pages 420–426. Lecture Notes in Econom. and Math. Systems, Vol. 107. Springer, Berlin, 1975.
[38] D. Valério and J. Sá da Costa. *An introduction to fractional control*, volume 91 of *IET Control Engineering Series*. Institution of Engineering and Technology (IET), London, 2013.
[39] J.L. Vázquez. The Dirichlet problem for the fractional p-Laplacian evolution equation. *J. Differential Equations*, 260(7):6038–6056, 2016.
[40] M. Warma. A fractional Dirichlet-to-Neumann operator on bounded Lipschitz domains. *Commun. Pure Appl. Anal.*, 14(5):2043–2067, 2015.
[41] M. Warma. The fractional relative capacity and the fractional Laplacian with Neumann and Robin boundary conditions on open sets. *Potential Anal.*, 42(2):499–547, 2015.
[42] M. Warma. The fractional Neumann and Robin type boundary conditions for the regional fractional p-Laplacian. *NoDEA Nonlinear Differential Equations Appl.*, 23(1):Art. 1, 46, 2016.
[43] M. Warma. Local Lipschitz continuity of the inverse of the fractional p-Laplacian, Hölder type continuity and continuous dependence of solutions to associated parabolic equations on bounded domains. *Nonlinear Anal.*, 135:129–157, 2016.
[44] M. Warma. On a fractional (s, p)-Dirichlet-to-Neumann operator on bounded lipschitz domains. *J. Elliptic and Parabol. Equ.*, (2018) DOI: 10.1007/s41808-018-0017-2.

E-mail address: `hantil@gmu.edu`
E-mail address: `mahamadi.warma1@upr.edu, mjwarma@gmail.com`