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Distributions of additive functions on shifted primes

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1 Introduction

A function f defined on positive integers n is called additive, if

$$f(mn) = f(m) + f(n)$$

for every coprime pair $(m, n) = 1$.

In probabilistic number theory, a central problem is to decide when an additive function f renormalised or not possesses a limit distribution. First, we can study the frequency

$$\nu_x(f(n) < u) := \frac{1}{[x]} \#\{n \le x : f(n) < u\}.$$

More generally, for additive function f it is natural to consider when functions $\alpha(x)$ and $\beta(x) > 0$ may be found such that frequencies

$$\nu_x\left(\frac{f(n) - \alpha(x)}{\beta(x)} < u\right)$$

possess a limiting distribution.

Erdős and Wintner [7] solved completely the case when $\alpha(x) = 0$, $\beta(x) = 1$. They obtained that

$$\nu_x(f(n) < u) \Rightarrow F(u)$$

as $x \to \infty$ if and only if three series

$$\sum_{|f(p)|>1} \frac{1}{p}, \quad \sum_{|f(p)|\le 1} \frac{f(p)}{p}, \quad \sum_{|f(p)|\le 1} \frac{f^2(p)}{p} \tag{1}$$

converge.

Here and further $\Rightarrow$ is the sign of a weak convergence and $F(u)$ means some limit distribution function.

The case which corresponds to the choice $\beta(x) = 1$ was completely solved by Elliott and Ryavec [2] and by Levin and Timofeev [17]. They got that there exists a real function $\alpha(x)$ for which

$$\nu_x(f(n) - \alpha(x) < u) \Rightarrow F(u) \tag{2}$$

as $x \to \infty$ if and only if there is a constant c so that $f(n) = c \log n + h(n)$, where the series

$$\sum_{|h(p)|>1} \frac{1}{p}, \quad \sum_{|h(p)|\le 1} \frac{h^2(p)}{p} \tag{3}$$

are convergent.

See also the paper of Erdős [9].

The common case was partially solved by Erdős and Kac [8] and by Kubilius [15]. They proved that for some class of additive functions

$$\nu_x\left(\frac{f(n)-A(x)}{B(x)}<u\right) \Rightarrow F(u) \tag{4}$$

as $x \to \infty$ with

$$A(x)=\sum_{p\le x}\frac{f(p)}{p}, \quad B(x)=\left(\sum_{p\le x}\frac{f^2(p)}{p}\right)^{1/2}.$$

The distributions of additive functions are interesting also on subsets of the set of positive integers. In the literature you can find papers devoted to the behaviour of additive functions on arithmetically interesting subsequences like

$$\{an+b : n\in\mathbb{N}\}, \quad \{G(n) : n\in\mathbb{N},\, G \text{ is a polinomial}\},$$
$$\{[n^\alpha] : n\in\mathbb{N}\}, \quad \{[\alpha n] : n\in\mathbb{N}\}, \quad \{n : n\in\mathbb{N},\, n \text{ is squarefree}\},$$
$$\{n : n\in\mathbb{N},\, n \text{ has no large prime factors}\},$$
$$\{p+a : p\in\mathbb{P}\}, \quad \{ap+b : p\in\mathbb{P}\},$$

and others.

The methods used in the investigation of asymptotic behaviour of additive functions on the set of positive integers can be applied to other sequences which are well distributed in most residue classes to moduli which are not to large.

2 Distributions on shifted primes

In the present paper I will give a survey on distributions of additive functions on the set of shifted prime numbers.

The case of shifted primes was considered by Barban, Vinogradov, Levin [1] Kátai [12, 13], Halberstam, Hildebrand [11], Timofeev [33, 34, 35, 37], Elliott [3], and others.

A. Hildebrand [11] and N.M. Timofeev [34] proved that

$$\nu_x(f(p+1)<u) \Rightarrow F(u)$$

if and only if three series (1) converge.

The conditions for the convergence in this result are the same as in the classical Erdős-Wintner theorem, although the limit distributions may be different. Therefore, the distributions of $f(n)$ and $f(p+1)$ can only converge simultaneously.

The case (2) for shifted primes was solved by Timofeev [34] as well. He got that there exist a real function $\alpha(x)$ for which

$$\nu_x(f(n)-\alpha(x)<u) \Rightarrow F(u)$$

as $x \to \infty$ if and only if there is a constant c so that $f(n) = c \log n + h(n)$ and the series (3) are convergent. In this case $\alpha(x)$ can be chosen

$$\alpha(x) = c \log x + \sum_{\substack{p \le x \\ |h(p)| \le 1}} \frac{h(p)}{p}.$$

Barban, Vinogradov, Levin [1] and Hildebrand [11] considered the analogues of (4). They proved that for some class of additive functions with suitable $\alpha(x)$ and $\beta(x)$

$$\nu_x \left(\frac{f(p+1) - \alpha(x)}{\beta(x)} < u \right) \Rightarrow F(u).$$

Elliott [4] investigated the case

$$\nu_x \left(\frac{f([x] - p) - \alpha(x)}{\beta(x)} < u \right) \Rightarrow F(u).$$

Let $\omega(n)$ mean the number of prime divisors of n. And let $\tau_k(n)$ denote the number of ways of expressing n as the product of k divisors. $\tau_2(n) = \tau(n)$ is the number of divisors of n. From the results above, can be deduced that

$$\nu_x \left(\frac{\omega(p+1) - \log\log x}{\sqrt{\log\log x}} < u \right) \Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-v^2/2} dv =: \Phi(u),$$

$$\frac{1}{\pi(x)} \sum_{p \le x} \omega(p+1) \sim \log\log x,$$

$$\nu_x \left(\frac{\log_k \tau_k(p+1) - \log\log x}{\sqrt{\log\log x}} < u \right) \Rightarrow \Phi(u).$$

Investigations of the limit behaviour of additive functions on shifted primes can be generalized to the sum of additive functions with different shifts. The idea to consider the sums of shifted additive functions is not new. The first result in this direction belongs to LeVeque [16]. More general results later were established by Kubilius [15], Kátai [14], Hildebrand [10], Elliott [3, 4, 5, 6], Timofeev and Usmanov [36], Stepanauskas [28, 29, 30, 31], and others. There the cases when the values of additive functions can be taken on different arithmetic progressions, on shifted primes, and when the number of additive functions as summands may slowly increase together with x were examined. All these results were given by using elementary methods, the method of characteristic functions, or the Kubilius model of probability spaces.

The results (general enough) for shifted primes are given by Stepanauskas [28, 30]:

1. The distributions

$$\nu_x(f_1(a_1 p + b_1) + \cdots + f_s(a_s p + b_s) < u)$$

converge weakly as $x \to \infty$ to some limit distribution if the tree series converge:

$$\sum_{\substack{|f_1(p)| \le 1 \\ \cdots\cdots \\ |f_s(p)| \le 1}} \frac{f_1(p) + \cdots + f_s(p)}{p} < \infty,$$

$$\sum_{|f_i(p)|>1} \frac{1}{p} < \infty, \quad \sum_{|f_i(p)|\le 1} \frac{f_i^2(p)}{p} < \infty.$$

The conditions for the convergence in this result are the same as in the result for arithmetic progressions for positive integers, although the limit distributions may be different. Therefore, the distributions of the sum for $f_i(a_i n + b_i)$ and the sum for $f_i(a_i p + b_i)$ can only converge simultaneously.

2. Let ϕ be the Euler totient function, $\sigma(n)$ be the sum of positive divisors of n. Then

$$\nu_x\left(\frac{\phi(a_1p+b_1)\dots\phi(a_sp+b_s)}{(a_1p+b_1)\dots(a_sp+b_s)} < u\right) \Rightarrow F(u),$$

$$\nu_x\left(\frac{\sigma(a_1p+b_1)\dots\sigma(a_sp+b_s)}{(a_1p+b_1)\dots(a_sp+b_s)} < u\right) \Rightarrow F(u),$$

$$\nu_x\left(\frac{\phi(a_1p+b_1)\dots\phi(a_kp+b_k)\sigma(a_{k+1}p+b_{k+1}\dots\phi(a_sp+b_s)}{(a_1p+b_1)\dots(a_sp+b_s)} < u\right) \Rightarrow F(u)$$

as $x \to \infty$.

3. Let for $i = 1, 2, \dots, s$

$$f_i(p) \to 0, \;\; p \to \infty,$$

$$\sum_{\log\log x < p \le x} \frac{f_i^2(p)}{p} \to 0, \;\; x \to \infty,$$

and for at least one of f_k

$$\sum_{p\le x} \frac{f_k^2(p)}{p} \to \infty, \;\; x \to \infty.$$

Then the sum

$$f_1(a_1p+b_1) + \dots + f_s(a_sp+b_s)$$

is asymptotically uniformly distributed mod 1 on the set of primes.

4. Let the integer-valued additive functions f_i, $i = 1, \dots, s$, be such that the series

$$\sum_{|f_i(p)|\ne 0} \frac{1}{p} < \infty.$$

converge. Then

$$\nu_x(f_1(a_1p+b_1) + \dots + f_s(a_sp+b_s) = k) \to \lambda_k, \quad x \to \infty,$$

for every $k \in \mathbb{Z}$.

3 Distributions of sets of additive functions on shifted primes

Elliott [4] showed that every stable law F can occur as a limit distribution for

$$\nu_x\left(\frac{f(n)-\alpha(x)}{\beta(x)} < u\right) \tag{5}$$

with suitable chosen α and β and that there are uncountably many distributions F which cannot occur as the limit distributions for (5) no matter how they are centred and normalised. The Poisson distribution is among them.

It is clear that the set of possible limit distributions can be expanded if we consider additive functions which values vary together with x, i.e. the sets of additive functions $f_x(n)$, $x \geq 2$. In the books [15, 3, 4] there were considered (at most) additive functions h_x having a special expression:

$$h_x(n) = \frac{f(n)}{\beta(x)},$$

where β is some normalising function. The more common sets were investigated by Šiaulys [18, 19, 20, 21, 22].

The question which appear here is what happens with the case of shifted primes.

In [23, 24] by Šiaulys and Stepanauskas the case of the Poisson limit distribution was considered. I will present here several consequences.

Let $f_x, x \geqslant 2$, be a set of strongly additive functions such that $f_x(p) \in \{0,1\}$ $\forall p \in \mathbb{P}, \forall x \geqslant 2$.

1. For every parameter $\lambda > 0$ the Poisson distribution

$$\Pi(u,\lambda) := \sum_{\substack{k=0,1,\dots\\ k<u}} \frac{\lambda^k}{k!} e^{-k}$$

can occur (accordingly choosing f_x) as limit distribution for

$$\nu_x(f_x(p+1) < u), \quad \nu_x(f_x(p+1) + g_x(p+2) < u).$$

2. Let

$$f_x(p) = \begin{cases} 1 & \text{if } \log x < p \leq (\log x)^\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g_x(p) = \begin{cases} 1 & \text{if } \log\log x < p \leq (\log\log x)^\beta, \\ 0 & \text{otherwise,} \end{cases}$$

with some $\alpha, \beta > 1$. Then

$$\lim_{x\to\infty} \nu_x(f_x(p+1) = k) = \frac{(\log\alpha)^k}{\alpha\, k!},$$

$$\lim_{x\to\infty} \nu_x(f_x(p+1)+g_x(p+2)=k) = \frac{(\log \alpha\beta)^k}{\alpha\beta\, k!}.$$

3. Let ψ and χ be unboundedly increasing functions such that $\log\psi(x)/\log x \to 0$ and $\log\chi(x)/\log x \to 0$ as $x\to\infty$. Then

$$\#\left\{p\le x,\ \begin{matrix} p+1 \text{ has exactly } k \text{ prime factors} \\ \text{from the interval } (\psi(x),\psi^\alpha(x)] \end{matrix}\right\} \sim \frac{(\log\alpha)^k}{\alpha\, k!}\,\frac{x}{\log x},$$

$$\#\left\{p\le x,\ \begin{matrix} (p+1)(p+2) \text{ has exactly } k \text{ prime factors} \\ \text{from the interval } (\psi(x),\psi^\alpha(x)] \end{matrix}\right\} \sim \frac{(2\log\alpha)^k}{\alpha^2\, k!}\,\frac{x}{\log x}.$$

$$\#\left\{p\le x,\ \begin{matrix} (p+1)(p+2) \text{ has exactly } k \text{ prime factors} \\ \text{from } (\psi(x),\psi^\alpha(x)]\bigcup(\chi(x),\chi^\beta(x)],\ \text{where} \\ \text{the prime factors of p+1 are counted from} \\ \text{the interval } (\psi(x),\psi^\alpha(x)] \text{ and the prime} \\ \text{factors of } p+2 \text{ from } (\chi(x),\chi^\beta(x)] \end{matrix}\right\} \sim \frac{(\log\alpha\beta)^k}{\alpha\beta\, k!}\,\frac{x}{\log x}.$$

Discrete uniform distribution

$$\mathbb{U}(u,L) := \frac{1}{L}\sum_{\substack{k=0,1,\dots,L-1\\ k<u}} 1$$

for shifted primes was considered by Šiaulys, Stepanauskas and Žvinytė [32, 27]. I give only some consequences from these investigations.

1. The discrete uniform distribution $\mathbb{U}(u,L)$ can occur as a limit distribution for

$$\nu_x(f_x(p+1)<u),\quad \nu_x(f_x(p+1)+g_x(p+2)<u)$$

with some set of additive functions if $L=2$ and cannot occur if $L=3,4,5,\dots$.

2. Let

$$f_x(p) = \begin{cases} 1 & \text{if } p=3,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\nu_x(f_x(p+1)<u) \Rightarrow \mathcal{U}(u,2).$$

3. Let

$$f_x(p)+g_x(p) = \begin{cases} 1 & \text{if } p=2,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\nu_x(f_x(n)+g_x(n+1)<u) \Rightarrow \mathcal{U}(u,2).$$

4. Let either

$$f_x(p)=g_x(p) = \begin{cases} 1 & \text{if } p=5,\\ 0 & \text{otherwise,}\end{cases}$$

or

$$f_x(p) + g_x(p) = \begin{cases} 1 & \text{if } p = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\nu_x(f_x(p+1) + g_x(p+2) < u) \Rightarrow \mathcal{U}(u, 2).$$

In the proofs of the main results for the Poisson and discrete limit distributions, the authors combined different methods. It was used elementary methods, the method of characteristic functions, and the Kubilius method of probability spaces. But the method of factorial moments played here the crucial role.

It would be interesting to examine the binomial, the Bernoulli, the shifted Poisson, the geometrical, and other distributions for shifted primes. Can they occur or not? It is known [25, 26] that some of them can occur as limit distributions even for $\nu_x(f(n) < u)$, but some of them can not.

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