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# A survey on the evaluation of the values of Dirichlet $L$-functions and of their logarithmic derivatives at $1+it_0$

Sumaia Saad Eddin

**Abstract**

In this note, we survey certain known results on the evaluation of values of Dirichlet $L$-functions and of their logarithmic derivatives at $1+it_0$ for fixed real number $t_0$.

## 1 Introduction

Let $\chi$ be a Dirichlet character modulo $q$, let $L(s,\chi)$ be the attached Dirichlet $L$-function, and let $L'(s,\chi)$ denote its derivative with respect to the complex variable $s$. The values at 1 of Dirichlet $L$-functions have received considerable attention since long time, due to their algebraical or geometrical interpretation. In 1837, Dirichlet produced finite expansions for $L(1,\chi)$ in the form

$$L(1,\chi)=\sum_{n\geq 1}\frac{\chi(n)}{n}=-\frac{2\tau(\chi)}{q}\begin{cases}2\displaystyle\sum_{1\leq m\leq q/2}\bar{\chi}(m)\log\left|\sin\frac{\pi m}{q}\right| & \text{when } \chi(-1)=+1,\\ i\pi\displaystyle\sum_{1\leq m\leq q/2}\bar{\chi}(m)\left(1-\frac{2m}{q}\right) & \text{when } \chi(-1)=-1.\end{cases}$$

where $\tau(\chi)$ is the Gaussian sum attached to $\chi$. Similar finite expansions for its derivatives form at $s=1$ have been obtained by many authors, such as: Berger [2], de Séguier [29], Selberg and Chowla [30], Gut [8], Deninger [6] and Kanemitsu [12]. In this paper, we shall restrict our attention to the values $L(1,\chi)$ and $(L'/L)(1+it_0,\chi)$ for any fixed real number $t_0$.

One of the important problems in Number Theory is to get good estimates for the size of $L(1,\chi)$. Many mathematicians have been studied upper and lower bounds of $|L(1,\chi)|$. Several of them have obtained upper bounds for this latter via character sums estimates, the functional equation and approximate formulas, or a mix of three. The best bounds known for $|L(1,\chi)|$ are of the form:

$$q^{-\varepsilon}\ll_\varepsilon |L(1,\chi)|\ll\log q.$$

Less is known about logarithmic derivatives $(L'/L)(s, \chi)$ at $s = 1$, through these values are known to be fundamental in studying the distribution of primes.
In this note, we survey certain known results of upper and lower bounds of $|L(1, \chi)|$ and the $2k$-th mean values of the Dirichlet $L$-functions at $s = 1$ and of their logarithmic derivatives at $1 + it_0$ for fixed real number $t_0$ and any positive integer $k$.

## 2 Upper bounds of $|L(1, \chi)|$

The classical result on bounds of $|L(1, \chi)|$ is due to Littlewood, see [15]. Assuming the generalized Riemann hypothesis, he proved that

$$|L(1, \chi)| \leq (2 + o(1))\, e^{\gamma} \log\log q.$$

For infinity many real characters $\chi$, we have

$$|L(1, \chi)| \geq (1 + o(1))\, e^{\gamma} \log\log q.$$

After a long while, Chowla [4] proved that this latter lower bound holds unconditionally. Littlewood bounds give us the correct range of the size of $|L(1, \chi)|$. His upper bound is still unproven unconditionally.

For $q = p$ is a prime number and $\chi$ quadratic characters. Chowla [5] showed that the following upper bound

$$|L(1, \chi)| \leq \left(\frac{1}{4} + o(1)\right) \log p.$$

holds for $\chi$ a real non-principal character modulo $p$. Using an argument of Polya-Vinogradov, Burgess [3] gave an improvement of Chowla's result. No analogous improvements over the Chowla and Burgess bounds were known for complex characters $\chi$. In [33], Stephens gave the following upper bound

$$|L(1, \chi)| \leq \frac{1}{2}\left(1 - \frac{1}{\sqrt{e}} + o(1)\right) \log p,$$

for $p$ sufficiently large. In 1977, Pintz [22] generalized this latter upper bound for every quadratic character, whose modulus is not necessarily prime. Recently, Granville and Soundararajan [7] determined the constant $c$, as small as possible, for which the bound

$$|L(1, \chi)| \leq (c + o(1)) \log q$$

holds. They showed that this constant can be $17/70$ for a non-principal character $\chi$ and when $q$ is cube-free. We point out that all above bounds are asymptotic and that explicit error terms are not known. So, in the next section, we are going to focus on explicit upper bounds of $|L(1, \chi)|$.

# 3 Explicit upper bounds of $|L(1,\chi)|$

We recall that the Dirichlet character $\chi$ is even if $\chi(-1) = 1$, and that it is odd if $\chi(-1) = -1$. The best explicit upper bound known up to date for $|L(1,\chi)|$ is of the form

$$|L(1,\chi)| \leq \frac{1}{2}\log q + C. \tag{1}$$

Concerning the constant $C$, Louboutin [16] and [17] proved that

$$|L(1,\chi)| \leq \begin{cases} \frac{1}{2}\log q + 0.009 & \text{if } \chi(-1) = +1, \\ \frac{1}{2}\log q + 0.716 & \text{if } \chi(-1) = -1. \end{cases}$$

where $\chi$ is a primitive character of conductor $q$. As a spacial case, when the conductor $q$ is even, Louboutin showed that

$$|L(1,\chi)| \leq \begin{cases} \frac{1}{4}\log q + 0.358 & \text{if } \chi(-1) = +1, \\ \frac{1}{4}\log q + 0.704 & \text{if } \chi(-1) = -1. \end{cases}$$

His proof is based on integral representations of the Dirichlet $L$-function.

Let $\chi$ be a primitive Dirichlet character of conductor $q > 1$. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be such that $f(t) = F(t)/t$ in $C^2(\mathbb{R})$ (even at 0), vanishes at $t = \mp\infty$ and its first and second derivatives belong to $L^1(\mathbb{R})$. We make the following assumptions; $F$ is even if $\chi$ is odd and that $F$ is odd if $\chi$ is even. Then for any $\delta > 0$ and under the above assumptions, Ramaré [25] gave a new approximate formulas for $L(1,\chi)$ depending on Fourier transforms:

$$L(1,\chi) = \sum_{n\geq 1} \frac{(1-F(\delta n))\,\chi(n)}{n} + \frac{\chi(-1)\tau(\chi)}{q} \sum_{m\geq 1} \overline{\chi}(m) \int_{-\infty}^{+\infty} \frac{F(t)}{t}\, e\left(mt/(\delta q)\right) dt. \tag{2}$$

With a proper choice of the function $F$ in the above formula

$$F_1(t) = \frac{\sin(\pi t)}{\pi}\left(\log 4 + \sum_{n\geq 1}(-1)^n\left(\frac{2n}{t^2-n^2} + \frac{2}{n}\right)\right),$$

$$F_2(t) = 1 - \frac{\sin(\pi t)}{\pi t},$$

$$F_3(t) = \left(\frac{\sin(\pi t)}{\pi}\right)^2\left(\frac{2}{t} + \sum_{m\in\mathbb{Z}}\frac{\operatorname{sgn}(m)}{(t-m)^2}\right),$$

$$F_4(t) = 1 - \left(\frac{\sin(\pi t)}{\pi t}\right)^2.$$

He proved that

$$L(1,\chi)=\begin{cases}\displaystyle\sum_{n\geq 1}\frac{(1-F_1(\delta n))\,\chi(n)}{n}-\frac{2\tau(\chi)}{q}\sum_{1\leq m\leq \delta q/2}\overline{\chi}(m)\log\left|\sin\frac{\pi m}{\delta q}\right| & \text{if } \chi(-1)=1,\\ \displaystyle\sum_{n\geq 1}\frac{(1-F_2(\delta n))\,\chi(n)}{n}-\frac{i\pi\tau(\chi)}{q}\sum_{1\leq m\leq \delta q/2}\overline{\chi}(m)\left(1-\frac{2m}{\delta q}\right) & \text{if } \chi(-1)=-1,\end{cases}$$

and that

$$L(1,\chi)=\begin{cases}\displaystyle\sum_{n\geq 1}\frac{(1-F_3(\delta n))\,\chi(n)}{n}-\frac{\tau(\chi)}{q}\sum_{1\leq m\leq \delta q}\overline{\chi}(m)j\left(\frac{m}{\delta q}\right) & \text{if } \chi(-1)=1,\\ \displaystyle\sum_{n\geq 1}\frac{(1-F_4(\delta n))\,\chi(n)}{n}+\frac{i\pi\tau(\chi)}{q}\sum_{1\leq m\leq \delta q}\overline{\chi}(m)\left(1-\frac{m}{\delta q}\right)^2 & \text{if } \chi(-1)=-1,\end{cases}$$

where

$$j(t)=2\int_{|t|}^{1}(\pi(1-u)\cot(\pi u)+1)\,du.$$

Taking $\delta$ to be around $1/\sqrt{q}$ in the first formula of $L(1,\chi)$ above, Ramaré obtained the following explicit upper bounds

$$|L(1,\chi)|\leq\begin{cases}\frac{1}{2}\log q+0.006 & \text{if } \chi(-1)=+1,\\ \frac{1}{2}\log q+0.9 & \text{if } \chi(-1)=-1.\end{cases}$$

By using the second one, he gave the best upper bound for $|L(1,\chi)|$.

$$|L(1,\chi)|\leq\begin{cases}\frac{1}{2}\log q & \text{if } \chi(-1)=+1,\\ \frac{1}{2}\log q+0.7082 & \text{if } \chi(-1)=-1.\end{cases}$$

To understand the difference between these two results, one needs to compare the function $F_1$ to $F_3$ and $F_2$ to $F_4$. For a nice comparison see [27].

More a general form of Eq. (2) is given by Ramaré [26] in 2004. Let $\chi$ be a primitive Dirichlet character modulo $q$, and let $h$ be an integer prime to $q$. Under the same assumptions, on the function $F$, given above. Ramaré proved that

$$\prod_{p|h}\left(1-\frac{\chi(p)}{p}\right)L(1,\chi)=\sum_{\substack{n\geq 1\\(n,h)=1}}\frac{(1-F(\delta n))\,\chi(n)}{n}$$
$$+\frac{\chi(-h)\tau(\chi)}{qh}\sum_{m\geq 1}c_h(m)\overline{\chi}(m)\int_{-\infty}^{+\infty}\frac{F(t)}{t}\,e\left(mt/(\delta qh)\right)\,dt.$$

Here $c_h(m)$ is the Ramanujan sums defined by

$$c_h(m) = \sum_{a \mod {}^*h} e(am/q).$$

Of course $e(x) = e^{2i\pi x}$, and $a \mod {}^*h$ denotes summation over all invertible residue classes modulo $h$. In the case that $q$ is odd, he deduced that

$$|(1 - \chi(2)/2)\, L(1, \chi)| \leq \frac{1}{4} (\log q + \kappa(\chi))\,,$$

where $\kappa(\chi) = 4 \log 2$ if $\chi$ is even, and $\kappa(\chi) = 5 - 2\log(3/2)$ otherwise.

For a particular case when $\chi(2) = 0$ and $\chi(3) = -1$, Le [14] gave the following upper bound:

$$|L(1, \chi)| \leq \frac{1}{8} \log q + \frac{3 \log 6 + 8}{8}.$$

This result has been later improved by Louboutin [18].

Let $S$ be a given finite set of pairwise distinct rational primes. Then, for any primitive Dirichlet character $\chi$ of conductor $q > 1$, Louboutin [19] collected his previous results in the following formula

$$\left| \left\{ \prod_{p \in S} \left(1 - \frac{\chi(p)}{p}\right) \right\} L(1, \chi) \right| \leq \frac{1}{2} \left| \left\{ \prod_{p \in S} \left(1 - \frac{1}{p}\right) \right\} \right| \times \left( \log q + \kappa_\chi + \omega \log 4 + 2 \sum_{p \in S} \frac{\log p}{p - 1} \right) + o(1),$$

where

$$\kappa_\chi = \begin{cases} \kappa_{\text{even}} = 2 + \gamma - \log(4\pi) = 0.046191 \cdots & \text{if } \chi(-1) = +1, \\ \kappa_{\text{odd}} = 2 + \gamma - \log(\pi) = 1.432485 \cdots & \text{if } \chi(-1) = -1. \end{cases}$$

Here $\omega \geq 0$ is the number of primes $p \in S$ which does not divide $q$, and where $o(1)$ is an explicit error term which tends rapidly to zero when $q$ goes to infinity. Moreover, if $S = \phi$ or if $S = 2$, then this error term $o(1)$ is always less than or equal to zero, and if none of the primes in $S$ divides $q$ then this error term $o(1)$ is less than or equal to zero for $q$ large enough.

In 2013, the author considered the most difficult case when $\chi(2) = 1$ and showed that the constant $C$ in Eq. (1) can be negative, see [28]. For $\chi$ an even primitive Dirichlet character of conductor $q > 1$, we proved that

$$|L(1, \chi)| \leq \frac{1}{2} \log q - 0.02012.$$

This result is the best upper bound of $|L(1,\chi)|$ up to date. Which gives us an improvement of the Ramaré result. As an example of application, we deduced an explicit upper bound for the class number for any real quadratic field $\mathbb{Q}\left(\sqrt{q}\right)$, improving on a result by Le [14]. For every real quadratic field of discriminant $q > 1$ and $\chi(2) = 1$, we showed that

$$h\left(\mathbb{Q}\left(\sqrt{q}\right)\right) \leq \frac{\sqrt{q}}{2}\left(1 - \frac{1}{25\log q}\right),$$

where $h\left(\mathbb{Q}(\sqrt{q})\right)$ is the class number of $\mathbb{Q}\left(\sqrt{q}\right)$. Since Oriat [21] has computed the class number of this field when $1 < q < 24572$. We proved the above result for $q \geq 24572$.

Using the previous Ramaré formula of $L(1,\chi)$, Platt and the author [24] gave a sharper upper bound of $|L(1,\chi)|$ when 3 divides the conductor

$$|L(1,\chi)| \leq \frac{1}{3}\log q + \begin{cases} 0.368296 & \text{when } \chi(-1) = 1, \\ 0.838374 & \text{when } \chi(-1) = -1. \end{cases}$$

We proved this result for $q > 2\cdot 10^6$. To check that it is valid for $1 < q \leq 2\cdot 10^6$, Platt has checked by using his algorithm from his thesis [23], ( which is rigorous and efficient for computing $L(1,\chi)$ for all primitive $\chi$ of conductor $2 \leq q \leq 2\cdot 10^6$ ). These bounds are improvement of the following result, due to Louboutin [19],

$$|L(1,\chi)| \leq \frac{1}{3}\log q + \begin{cases} 0.3816 & \text{when } \chi(-1) = 1, \\ 0.8436 & \text{when } \chi(-1) = -1. \end{cases}$$

# 4 The mean values of the Dirichlet $L$-function at $s = 1$

The asymptotic properties for the $2k$-th power mean value of $L$-functions at $s = 1$ have been studied by many authors. We again consider the case $q = p$ is a prime number. The classical result of the second power mean value of the Dirichlet $L$-function at $s = 1$ is due to Paley and Selberg, see [1]. They proved that

$$\sum_{\substack{\chi \bmod p \\ \chi \neq \chi_0}} |L(1,\chi)|^2 = \zeta(2)p + O\left((\log p)^2\right),$$

where $\chi$ runs over all Dirichlet characters modulo $p$ except for the principal character $\chi_0$. This result has been improved by several authors. In this section, we mention some of them. In 1985, Slavutskii [31] and [32] showed that

$$\sum_{\substack{\chi \bmod p \\ \chi \neq \chi_0}} |L(1,\chi)|^2 = \zeta(2)p - (\log p)^2 + O\left(\log p\right).$$

Later, the above error term was improved to $O(\log\log p)$ by Zhang [34]. In 1994, Katsurada and Matsumoto [13] obtained a sharper asymptotic expansion for the second power mean value of $|L(1,\chi)|$. For any integer $N \geq 1$, they proved that

$$\sum_{\substack{\chi \bmod p \\ \chi \neq \chi_0}} |L(1,\chi)|^2 = \zeta(2)p - (\log p)^2 + \left(\gamma_0^2 - 2\gamma_1 - 3\zeta(2)\right) - \left(\gamma_0^2 - 2\gamma_1 - 2\zeta(2)\right) p^{-1}$$
$$+ 2\left(1 - p^{-1}\right)\left[\sum_{n=1}^{N-1} (-1)^n \zeta(1-n)\zeta(1+n)p^{-n} + O\left(p^{-N}\right)\right].$$

Here the $O$-constant depends only on $N$, and the constants $\gamma_0$ and $\gamma_1$ are the Laurent expansion coefficients of the zeta function at 1. As for general $k$, Zhang and Wang [37] gave the following interesting result, for any $q \geq 3$,

$$\sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} |L(1,\chi)|^{2k} = \varphi(q) \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^2} + O\left(\exp\left(\frac{2k\log q}{\log\log q}\right)\right),$$

where $d_k(n) = \sum_{r_1 \cdots r_k = n} 1$ is the $k$th divisor function. In that paper, for $k = 2$ they also deduced that

$$\sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} |L(1,\chi)|^4 = \frac{5}{72}\pi^4 \varphi(q) \prod_{p|q} \frac{(p^2-1)^3}{p^4(p+1)} + O\left(\exp\left(\frac{4\log q}{\log\log q}\right)\right).$$

## 5 The mean values of the logarithmic derivatives of the Dirichlet $L$-function at $1 + it_0$

In this section, we are interested by the values of the logarithmic derivatives of the Dirichlet $L$-function at $1 + it_0$ . We shall only give an announcement of our recent results in this direction of research. For more details see [20].

In 1992, Zhang [36] studied the fourth power mean value of $(L'/L)(s,\chi)$ at $s = 1$. For the real number $Q > 3$, he gave the following asymptotic formula

$$\sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} \left|\frac{L'(1,\chi)}{L(1,\chi)}\right|^4 = Q \sum_p \frac{(p^2+1)\log^4 p}{p(p+1)(p^2-1)^2} + 4Q\left(\sum_p \frac{\log^2 p}{p^2-1}\right)\left(\sum_p \frac{\log^2 p}{p(p+1)}\right)$$
$$- 4Q \sum_p \frac{(p^2-p+1)\log^4 p}{p^2(p^2-1)^2} + 4Q\left(\sum_p \frac{\log^2 p}{p(p^2-1)}\right)^2 + O\left(\log^5 Q\right),$$

where $\sum_p$ denotes the summation over all primes. He proved his result by using the estimates of the character sums and the Bombieri-Vinogradov theorem.

Ihara and Matsumoto [9], [10] gave a result related to the value-distributions of $\{(L'/L)(s,\chi)\}_\chi$ and of $\{(\zeta'/\zeta)(s+i\tau)\}_\tau$, where $\chi$ runs over Dirichlet characters with prime conductors and $\tau$ runs over $\mathbb{R}$.

Let $p$ be a prime and $X_p$ denote the set of all non-principal multiplicative characters $\chi$ such that

$$\chi : (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times.$$

Recently, motivated by connections of the values of $(L'/L)(1,\chi)$ with the Euler-Kronecker invariants of global fields (especially the cyclotomic fields), Ihara, Murty and Shimura [11] studied the maximal absolute value of the logarithmic derivatives $(L'/L)(1,\chi)$ and showed that the following result

$$\max_{\chi\in X_p}\left|\frac{L'(1,\chi)}{L(1,\chi)}\right| \le (2+o(1))\log\log p,$$

holds under **GRH**. For any $\varepsilon>0$, they Unconditionally proved that

$$\frac{1}{|X_p|}\sum_{\chi\in X_p}\left|\frac{L'(1,\chi)}{L(1,\chi)}\right|^{2k} = \sum_{m\ge 1}\frac{\left(\sum\limits_{m=m_1\cdot m_2\cdots m_k}\Lambda(m_1)\cdots\Lambda(m_k)\right)^2}{m^2} + O\left(p^{\varepsilon-1}\right), \qquad (3)$$

where $\Lambda(.)$ denotes the von Mangoldt function. The proof of this result is based on the study of distribution of zeros of $L$-functions. More recently, Matsumoto and the author [20] gave an asymptotic formula for the $2k$-th power mean value of $|(L'/L)(1+it_0,\chi)|$ when $\chi$ runs over all Dirichlet characters modulo $q$ and any fixed real number $t_0$. We proved that,

$$\frac{1}{\varphi(q)}\sum_{\chi \bmod q}\left|\frac{L'(1+it_0,\chi)}{L(1+it_0,\chi)}\right|^{2k} = \sum_{\substack{m\ge 1\\ (m,q)=1}}\frac{\left(\sum\limits_{m=m_1\cdot m_2\cdots m_k}\Lambda(m_1)\cdots\Lambda(m_k)\right)^2}{m^2}$$
$$+O\left(\frac{1}{q}(\log q)^{4k+4+\varepsilon}+\frac{1}{\varphi(q)}\left(\frac{1}{|t_0|^{2k-1}}+(\log(q(|t_0|+2)))^{2k}\right)\right), \qquad (4)$$

for any fixed real number $t_0\neq 0$ and an arbitrary positive integer $k$. Here $\varphi$ is the Euler totient function. In the case when $t_0=0$, we deduced that

$$\frac{1}{\varphi(q)}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}\left|\frac{L'(1,\chi)}{L(1,\chi)}\right|^{2k} = \sum_{\substack{m\ge 1\\ (m,q)=1}}\frac{\left(\sum\limits_{m=m_1\cdot m_2\cdots m_k}\Lambda(m_1)\cdots\Lambda(m_k)\right)^2}{m^2} + O\left(\frac{(\log q)^{8k+\varepsilon}}{q}\right).$$

This result provides an improvement (and a generalization to the case of general modulus $q$) on Eq. (3). In fact, when $q=p$ is a prime, it is shown in [11] that the factor $p^\varepsilon$ in the error term in Eq. (3) can be replaced by a certain log-power under the assumption of the **GRH**. Our result gives a same type of improvement unconditionally.

As a consequence of those results, we showed that the values $|(L'/L)(1+it_0,\chi)|^2$ behave according to a distribution law. Our main result is proved by profiting from the known zero-free regions of the functions $L(s,\chi)$.

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Sumaia Saad Eddin
Graduate School of Mathematics, Nagoya University Nagoya 464-8602, Japan.
Institute of Financial Mathematics and Applied Number Theory, Johannes Kepler University, 4040 Linz, Austria (New address).
e-mail: saad.eddin@math.nagoya-u.ac.jp
e-mail: sumaia.saad_eddin@jku.at