数理解析研究所講究録 第2093巻 2018年 1-15

Remarks on strong instability of standing waves for nonlinear Schrödinger equations

Masahito Ohta

Department of Mathematics,
Tokyo University of Science

1 Introduction

In this note, based on our recent papers [13, 14, 11, 12], we give some remarks on the strong instability of standing wave solutions for nonlinear Scrhrödinger equations.

First, we consider nonlinear Schrödinger equation of the simplest form:

$$i\partial_t u = -\Delta u - |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{1.1}$$

where $1 < p < 2^* - 1$. Here and hereafter, $2^* = 2N/(N-2)$ if $N \geq 3$, and $2^* = \infty$ if $N = 1, 2$.

In this section, we give a simple proof for a classical result on the strong instability of standing waves for (1.1) by Berestycki and Cazenave [1].

It is well known that the Cauchy problem for (1.1) is locally well-posed in the energy space $H^1(\mathbb{R}^N)$ (see, e.g., [2, Chapter 4]).

Proposition 1.1. *Let $1 < p < 2^* - 1$. For any $u_0 \in H^1(\mathbb{R}^N)$ there exist $T_{\max} = T_{\max}(u_0) \in (0, \infty]$ and a unique maximal solution*

$$u \in C([0, T_{\max}), H^1(\mathbb{R}^N)) \cap C^1([0, T_{\max}), H^{-1}(\mathbb{R}^N))$$

of (1.1) with initial condition $u(0) = u_0$. The solution $u(t)$ is maximal in the sense that if $T_{\max} < \infty$, then $\lim_{t \to T_{\max}} \|u(t)\|_{H^1} = \infty$.

Moreover, the solution $u(t)$ satisfies the conservation laws

$$\|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2, \quad E(u(t)) = E(u_0) \tag{1.2}$$

for all $t \in [0, T_{\max})$, where the energy E is defined by

$$E(v) = \frac{1}{2}\|\nabla v\|_{L^2}^2 - \frac{1}{p+1}\|v\|_{L^{p+1}}^{p+1}.$$

Next, we consider the stationary problem

$$-\Delta\phi+\omega\phi-|\phi|^{p-1}\phi=0, \quad x\in\mathbb{R}^N, \tag{1.3}$$

where $\omega>0$ is a paramter. Note that if $\phi(x)$ solves (1.3), then $e^{i\omega t}\phi(x)$ is a solution of (1.1). Moreover, (1.3) is written as $S_\omega'(\phi)=0$, where

$$\begin{aligned} S_\omega(v)&=E(v)+\frac{\omega}{2}\|v\|_{L^2}^2 \\ &=\frac{1}{2}\|\nabla v\|_{L^2}^2+\frac{\omega}{2}\|v\|_{L^2}^2-\frac{1}{p+1}\|v\|_{L^{p+1}}^{p+1} \end{aligned}$$

is the action. The set of all ground states for (1.3) is defined by

$$\mathcal{G}_\omega=\{\phi\in\mathcal{A}_\omega : S_\omega(\phi)\le S_\omega(v) \text{ for all } v\in\mathcal{A}_\omega\}, \tag{1.4}$$

where

$$\mathcal{A}_\omega=\{v\in H^1(\mathbb{R}^N) : S_\omega'(v)=0,\ v\neq 0\}$$

is the set of all nontrivial solutions for (1.3).

The existence of ground states for (1.3) is well known.

Proposition 1.2. *Let $1<p<2^*-1$ and $\omega>0$. Then, the set $\mathcal{G}_\omega$ is not empty, and it is characterized by*

$$\mathcal{G}_\omega=\{v\in H^1(\mathbb{R}^N) : S_\omega(v)=d(\omega),\ K_\omega(v)=0,\ v\neq 0\}, \tag{1.5}$$

where

$$K_\omega(v)=\partial_\lambda S_\omega(\lambda v)|_{\lambda=1}=\|\nabla v\|_{L^2}^2+\omega\|v\|_{L^2}^2-\|v\|_{L^{p+1}}^{p+1}$$

is the Nehari functional, and

$$d(\omega)=\inf\{S_\omega(v) : v\in H^1(\mathbb{R}^N),\ K_\omega(v)=0,\ v\neq 0\}. \tag{1.6}$$

It is also known that there exists a unique positive radial soliution ϕ_ω of (1.3) (see [6] for the uniqueness), and the set $\mathcal{G}_\omega$ is given by

$$\mathcal{G}_\omega=\{e^{i\theta}\tau_y\phi_\omega : \theta\in\mathbb{R},\ y\in\mathbb{R}^N\},$$

where $\tau_y v(x)=v(x-y)$.

The following is the classical result by Berestycki and Cazenave [1] (see also [2, Theorem 8.2.2]).

Theorem 1.3 (Berestycki and Cazenave [1]). *Let $1+4/N<p<2^*-1$. Then, for any $\omega>0$, the standing wave solution $e^{i\omega t}\phi_\omega$ of (1.1) is strongly unstable in the following sense. For any $\varepsilon>0$ there exists $u_0\in H^1(\mathbb{R}^N)$ such that $\|u_0-\phi_\omega\|_{H^1}<\varepsilon$ and the solution $u(t)$ of (1.1) with $u(0)=u_0$ blows up in finite time.*

Note that when $1<p<1+4/N$, for all $\omega>0$, the standing wave solution $e^{i\omega t}\phi_\omega$ of (1.1) is orbitally stable in the following sense (see [3]). For any $\varepsilon>0$ there exists $\delta>0$ such that if $u_0\in H^1(\mathbb{R}^N)$ satisfies $\|u_0-\phi_\omega\|_{H^1}<\delta$, then the solution $u(t)$ of (1.1) with $u(0)=u_0$ exists globally and satisfies

$$\sup_{t\ge 0}\inf_{\theta\in\mathbb{R},y\in\mathbb{R}^N}\|u(t)-e^{i\theta}\tau_y\phi_\omega\|_{H^1}<\varepsilon.$$

The proof of finite time blowup for nonlinear Schrödinger equation (1.1) relies on the virial identity (1.7) below. We define

$$\Sigma=\{v\in H^1(\mathbb{R}^N):|x|v\in L^2(\mathbb{R}^N)\}.$$

Proposition 1.4. *Let $1<p<2^*-1$. If $u_0\in\Sigma$, then the solution $u(t)$ of (1.1) with $u(0)=u_0$ satisfies $u\in C([0,T_{\max}),\Sigma)$. Moreover, the function*

$$t\mapsto\|xu(t)\|_{L^2}^2=\int_{\mathbb{R}^N}|xu(t,x)|^2\,dx$$

is in $C^2[0,T_{\max})$, and satisfies

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2=8P(u(t)),\quad t\in[0,T_{\max}),\tag{1.7}$$

where

$$P(v)=\|\nabla v\|_{L^2}^2-\frac{\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1},\quad \alpha=\frac{N(p-1)}{2}.$$

For the proof of Proposition 1.4, see, e.g., [2, Proposition 6.5.1].

Note that by the scaling $v^\lambda(x)=\lambda^{N/2}v(\lambda x)$ for $\lambda>0$, we have

$$\|\nabla v^\lambda\|_{L^2}^2=\lambda^2\|\nabla v\|_{L^2}^2,\quad \|v^\lambda\|_{L^2}^2=\|v\|_{L^2}^2,\quad \|v^\lambda\|_{L^{p+1}}^{p+1}=\lambda^\alpha\|v\|_{L^{p+1}}^{p+1},$$

and

$$\begin{aligned}E(v^\lambda)&=\frac{\lambda^2}{2}\|\nabla v\|_{L^2}^2-\frac{\lambda^\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1},\\ P(v^\lambda)&=\lambda^2\|\nabla v\|_{L^2}^2-\frac{\alpha\lambda^\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1}=\lambda\partial_\lambda E(v^\lambda)=\lambda\partial_\lambda S_\omega(v^\lambda).\end{aligned}$$

Remark also that $\alpha > 2$ if $p > 1 + 4/N$,

The proof of Theorem 1.3 by Berestycki and Cazenave [1] is based on the fact that $d(\omega) = S_\omega(\phi_\omega)$ can be characterized as

$$d(\omega) = \inf\{S_\omega(v) : v \in H^1(\mathbb{R}^N),\ P(v) = 0,\ v \neq 0\} \tag{1.8}$$

for the case $1 + 4/N < p < 2^* - 1$. Using this fact, it is proved in [1] that if $u_0 \in \mathcal{B}_\omega \cap \Sigma$, then the solution $u(t)$ of (1.1) with $u(0) = u_0$ blows up in finite time (see Theorem 1.7 below), where

$$\mathcal{B}_\omega = \{v \in H^1(\mathbb{R}^N) : S_\omega(v) < d(\omega),\ P(v) < 0\}. \tag{1.9}$$

On the other hand, Zhang [15] and Le Coz [7] give alternative proofs of Theorem 1.3. Instead of considering the minimization problem (1.8), they proved that

$$d(\omega) \le \inf\{S_\omega(v) : v \in H^1(\mathbb{R}^N),\ P(v) = 0,\ K_\omega(v) < 0\} \tag{1.10}$$

holds for all $\omega > 0$ if $1 + 4/N < p < 2^* - 1$. Using this fact, it is proved in [15, 7] that if $u_0 \in \mathcal{B}_\omega^{ZL} \cap \Sigma$, then the solution $u(t)$ of (1.1) with $u(0) = u_0$ blows up in finite time, where

$$\mathcal{B}_\omega^{ZL} = \{v \in H^1(\mathbb{R}^N) : S_\omega(v) < d(\omega),\ P(v) < 0,\ K_\omega(v) < 0\}.$$

Remark that the method of [15, 7] does not need to solve the minimization problem (1.8).

The following lemma is a modification of the ideas of Zhang [15] and Le Coz [7], and it was introduced in [12] (see also [13, 14, 11]).

Lemma 1.5. *Let $1 + 4/N < p < 2^* - 1$ and $\omega > 0$. If $v \in H^1(\mathbb{R}^N)$ satisfies $P(v) \le 0$ and $v \neq 0$, then*

$$d(\omega) \le S_\omega(v) - \frac{1}{2}P(v).$$

Proof. Consider the function

$$(0, \infty) \ni \lambda \mapsto K_\omega(v^\lambda) = \lambda^2 \|\nabla v\|_{L^2}^2 + \omega \|v\|_{L^2}^2 - \lambda^\alpha \|v\|_{L^{p+1}}^{p+1}$$

for $\lambda > 0$. Then, $\lim_{\lambda \to +0} K_\omega(v^\lambda) = \omega \|v\|_{L^2}^2 > 0$.

Moreover, since $\alpha > 2$, we have $\lim_{\lambda \to +\infty} K_\omega(v^\lambda) = -\infty$. Thus, there exists $\lambda_0 \in (0, \infty)$ such that $K_\omega(v^{\lambda_0}) = 0$.

Then, by the definition (1.6) of $d(\omega)$, we have $d(\omega) \le S_\omega(v^{\lambda_0})$.
Moreover, since $\alpha > 2$, the function

$$(0, \infty) \ni \lambda \mapsto S_\omega(v^\lambda) - \frac{\lambda^2}{2} P(v) = \frac{\alpha\lambda^2 - 2\lambda^\alpha}{2(p+1)} \|v\|_{L^{p+1}}^{p+1} + \frac{\omega}{2}\|v\|_{L^2}^2$$

attains its maximum at $\lambda = 1$.

Thus, using $P(v) \le 0$ again, we have

$$d(\omega) \le S_\omega(v^{\lambda_0}) \le S_\omega(v^{\lambda_0}) - \frac{\lambda_0^2}{2} P(v) \le S_\omega(v) - \frac{1}{2} P(v).$$

This completes the proof. □

Once we have the key Lemma 1.5, the rest of the proof is the same as in the classical argument of Berestycki and Cazenave [1].

Lemma 1.6. *Let $1 + 4/N < p < 2^* - 1$ and $\omega > 0$. The set $\mathcal{B}_\omega$ defined by (1.9) is invariant under the flow of (1.1). That is, if $u_0 \in \mathcal{B}_\omega$, then the solution $u(t)$ of (1.1) with $u(0) = u_0$ satisfies $u(t) \in \mathcal{B}_\omega$ for all $t \in [0, T_{\max})$.*

Proof. Let $u_0 \in \mathcal{B}_\omega$ and let $u(t)$ be the solution of (1.1) with $u(0) = u_0$. Then, by the conservation laws (1.2), we have

$$S_\omega(u(t)) = E(u(t)) + \frac{\omega}{2}\|u(t)\|_{L^2}^2 = S_\omega(u_0) < d(\omega)$$

for all $t \in [0, T_{\max})$.

Next, we prove that $P(u(t)) < 0$ for all $t \in [0, T_{\max})$. Suppose that this were not true. Then, there exists $t_0 \in (0, T_{\max})$ such that $P(u(t_0)) = 0$. Moreover, since $u(t_0) \neq 0$, it follows from Lemma 1.5 that

$$d(\omega) \le S_\omega(u(t_0)) - \frac{1}{2} P(u(t_0)) = S_\omega(u(t_0)).$$

This contradicts the fact that $S_\omega(u(t)) < d(\omega)$ for all $t \in [0, T_{\max})$.

Therefore, $P(u(t)) < 0$ for all $t \in [0, T_{\max})$. □

Theorem 1.7. *Let $1 + 4/N < p < 2^* - 1$ and $\omega > 0$. If $u_0 \in \mathcal{B}_\omega \cap \Sigma$, then the solution $u(t)$ of (1.1) with $u(0) = u_0$ blows up in finite time.*

Proof. Let $u_0 \in \mathcal{B}_\omega \cap \Sigma$ and let $u(t)$ be the solution of (1.1) with $u(0) = u_0$. Then, it follows from Lemma 1.6 and Proposition 1.4 that $u(t) \in \mathcal{B}_\omega \cap \Sigma$ for all $t \in [0, T_{\max})$.

Moreover, by the virial identity (1.7), the conservation laws (1.2) and Lemma 1.5, we have

$$\begin{aligned}\frac{1}{16}\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 &= \frac{1}{2}P(u(t)) \\ &\le S_\omega(u(t)) - d(\omega) = S_\omega(u_0) - d(\omega) < 0\end{aligned}$$

for all $t \in [0, T_{\max})$, which implies $T_{\max} < \infty$. □

Finally, we give the proof of Theorem 1.3.

Proof of Theorem 1.3. First, by the elliptic regularity theory, $\phi_\omega \in \Sigma$ (see, e.g., [2, Theorem 8.1.1]).

Next, since $S'_\omega(\phi_\omega) = 0$ and $\alpha > 2$, the function

$$(0,\infty) \ni \lambda \mapsto S_\omega(\phi_\omega^\lambda) = \frac{\lambda^2}{2}\|\nabla\phi_\omega\|_{L^2}^2 + \frac{\omega}{2}\|\phi_\omega\|_{L^2}^2 - \frac{\lambda^\alpha}{p+1}\|\phi_\omega\|_{L^{p+1}}^{p+1}$$

attains its maximum at $\lambda = 1$, and we see that

$$S_\omega(\phi_\omega^\lambda) < S_\omega(\phi_\omega) = d(\omega), \quad P(\phi_\omega^\lambda) = \lambda\partial_\lambda S_\omega(\phi_\omega^\lambda) < 0$$

for all $\lambda > 1$.

Thus, for $\lambda > 1$, $\phi_\omega^\lambda \in \mathcal{B}_\omega \cap \Sigma$, and it follows from Theorem 1.7 that the solution $u(t)$ of (1.1) with $u(0) = \phi_\omega^\lambda$ blows up in finite time.

Finally, since $\lim_{\lambda\to 1}\|\phi_\omega^\lambda - \phi_\omega\|_{H^1} = 0$, the proof is completed. □

2 NLS with double power nonlinearities

In this section, we consider nonlinear Schrödinger equations with double power nonlinearities:

$$i\partial_t u = -\Delta u - a|u|^{p-1}u - b|u|^{q-1}u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^N, \tag{2.1}$$

where $1 < p < q < 2^* - 1$. For simplicity, we consider the case $a > 0$ and $b > 0$ only.

The energy for (2.1) is defined by

$$E(v) = \frac{1}{2}\|\nabla v\|_{L^2}^2 - \frac{a}{p+1}\|v\|_{L^{p+1}}^{p+1} - \frac{b}{q+1}\|v\|_{L^{q+1}}^{q+1}.$$

The Cauchy problem for (2.1) is locally well-posed in the energy space $H^1(\mathbb{R}^N)$, and the same statement as in Proposition 1.1 holds.

Next, we consider the stationary problem

$$-\Delta\phi + \omega\phi - a|\phi|^{p-1}\phi - b|\phi|^{q-1}\phi = 0, \quad x \in \mathbb{R}^N, \tag{2.2}$$

where $\omega > 0$. The action is defined by

$$S_\omega(v) = \frac{1}{2}\|\nabla v\|_{L^2}^2 + \frac{\omega}{2}\|v\|_{L^2}^2 - \frac{a}{p+1}\|v\|_{L^{p+1}}^{p+1} - \frac{b}{q+1}\|v\|_{L^{q+1}}^{q+1}.$$

The existence of ground states for (2.2) is also well known, and we have the same statement as in Proposition 1.2. However, the uniqueness of positive radial solutions for (2.2) is not known for the whole range of parameters p, q, a, b and ω.

2.1 The case $1 + 4/N \le p < q < 2^* - 1$

First, we give a simple proof of the following theorem, which is included in Berestycki and Cazenave [1], by the same argument as in Section 1.

Theorem 2.1 (Berestycki and Cazenave [1]). *Assume that $1 + 4/N \le p < q < 2^* - 1$, $a > 0$, $b > 0$. Let $\omega > 0$ and ϕ_ω be a ground state of (2.2). Then, for any $\omega > 0$, the standing wave solution $e^{i\omega t}\phi_\omega$ of (2.1) is strongly unstable.*

For (2.1) with initial data $u(0) = u_0 \in \Sigma$, we have the virial identity (1.7) with

$$P(v) = \|\nabla v\|_{L^2}^2 - \frac{a\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1} - \frac{b\beta}{q+1}\|v\|_{L^{q+1}}^{q+1},$$
$$\alpha = \frac{N(p-1)}{2}, \quad \beta = \frac{N(q-1)}{2}.$$

The following lemma is the key for the proof of Theorem 2.1.

Lemma 2.2. *Assume that $1 + 4/N \le p < q < 2^* - 1$, $a > 0$, $b > 0$. If $v \in H^1(\mathbb{R}^N)$ satisfies $P(v) \le 0$ and $v \ne 0$, then*

$$d(\omega) \le S_\omega(v) - \frac{1}{2}P(v).$$

Proof. Consider the function

$$\begin{aligned}(0,\infty) \ni \lambda &\mapsto K_\omega(v^\lambda) \\ &= \lambda^2\|\nabla v\|_{L^2}^2 + \omega\|v\|_{L^2}^2 - \lambda^\alpha a\|v\|_{L^{p+1}}^{p+1} - \lambda^\beta b\|v\|_{L^{q+1}}^{q+1}.\end{aligned}$$

Then, $\lim_{\lambda\to+0} K_\omega(v^\lambda) = \omega\|v\|_{L^2}^2 > 0$. Moreover, since $\beta > \alpha \geq 2$, we have $\lim_{\lambda\to+\infty} K_\omega(v^\lambda) = -\infty$. Thus, there exists $\lambda_0 \in (0,\infty)$ such that $K_\omega(v^{\lambda_0}) = 0$. Then, by the definition (1.6) of $d(\omega)$, we have $d(\omega) \leq S_\omega(v^{\lambda_0})$.

Moreover, since $\beta > \alpha \geq 2$, the function

$$\begin{aligned}(0,\infty) \ni \lambda &\mapsto S_\omega(v^\lambda) - \frac{\lambda^2}{2}P(v) \\ &= \frac{\alpha\lambda^2 - 2\lambda^\alpha}{2(p+1)}a\|v\|_{L^{p+1}}^{p+1} + \frac{\beta\lambda^2 - 2\lambda^\beta}{2(q+1)}b\|v\|_{L^{q+1}}^{q+1} + \frac{\omega}{2}\|v\|_{L^2}^2\end{aligned}$$

attains its maximum at $\lambda = 1$.

Thus, using $P(v) \leq 0$ again, we have

$$d(\omega) \leq S_\omega(v^{\lambda_0}) \leq S_\omega(v^{\lambda_0}) - \frac{\lambda_0^2}{2}P(v) \leq S_\omega(v) - \frac{1}{2}P(v).$$

This completes the proof. □

Once we have the key Lemma 2.2, Theorem 2.1 is proved in the same way as Theorem 1.3

2.2 The case $1 < p < 1 + 4/N < q < 2^* - 1$

Next, we consider the case $1 < p < 1 + 4/N < q < 2^* - 1$. For this case, it is known that the standing wave solution $e^{i\omega t}\phi_\omega$ of (2.1) is orbitally unstable for sufficiently large ω (see [10]), while $e^{i\omega t}\phi_\omega$ is orbitally stable for sufficiently small ω (see [4] and also [9, 8] for more results in one dimensional case).

The following theorem is proved by Ohta and Yamaguchi [13].

Theorem 2.3 (Ohta and Yamaguchi [13]). *Let $1 < p < 1+4/N < q < 2^*-1$, $a > 0$, $b > 0$. Let $\omega > 0$ and ϕ_ω be a ground state of (2.2). If $E(\phi_\omega) > 0$, then the standing wave solution $e^{i\omega t}\phi_\omega$ of (2.1) is strongly unstable.*

Corollary 2.4. *Let $1 < p < 1 + 4/N < q < 2^* - 1$, $a > 0$, $b > 0$. Let $\omega > 0$ and ϕ_ω be a ground state of (2.2). Then there exists $\omega_1 > 0$ such that the standing wave solution $e^{i\omega t}\phi_\omega$ of (2.1) is strongly unstable for all $\omega \in (\omega_1, \infty)$.*

Proof of Corollary 2.4. Since $P(\phi_\omega)=0$, we see that $E(\phi_\omega)>0$ if and only if

$$\frac{(2-\alpha)a}{p+1}\|\phi_\omega\|_{L^{p+1}}^{p+1} < \frac{(\beta-2)b}{q+1}\|\phi_\omega\|_{L^{q+1}}^{q+1}. \tag{2.3}$$

Note that $0<\alpha<2<\beta$. Moreover, as in the proof of Theorem 2 of [10], we can prove that

$$\lim_{\omega\to\infty}\frac{\|\phi_\omega\|_{L^{p+1}}^{p+1}}{\|\phi_\omega\|_{L^{q+1}}^{q+1}}=0.$$

Thus, there exists $\omega_1>0$ such that (2.3) holds for all $\omega\in(\omega_1,\infty)$. □

In the following, we give a proof of Theorem 2.3, which is slightly different from that in [13] (see Remark 2.8 below).

The key lemma for the proof of Theorem 2.3 is the following.

Lemma 2.5. *Let $1<p<1+4/N<q<2^*-1$, $a>0$, $b>0$. If $v\in H^1(\mathbb{R}^N)$ satisfies $E(v)\ge 0$, $P(v)\le 0$ and $v\ne 0$, then*

$$d(\omega)\le S_\omega(v)-\frac{1}{2}P(v).$$

Proof. Consider the function

$$\begin{aligned}&(0,\infty)\ni\lambda\mapsto K_\omega(v^\lambda)\\ &=\lambda^2\|\nabla v\|_{L^2}^2+\omega\|v\|_{L^2}^2-\lambda^\alpha a\|v\|_{L^{p+1}}^{p+1}-\lambda^\beta b\|v\|_{L^{q+1}}^{q+1}.\end{aligned}$$

Then, $\lim_{\lambda\to+0}K_\omega(v^\lambda)=\omega\|v\|_{L^2}^2>0$. Moreover, since $0<\alpha<2<\beta$, we have $\lim_{\lambda\to+\infty}K_\omega(v^\lambda)=-\infty$. Thus, there exists $\lambda_0\in(0,\infty)$ such that $K_\omega(v^{\lambda_0})=0$. Then, by the definition (1.6) of $d(\omega)$, we have $d(\omega)\le S_\omega(v^{\lambda_0})$.

Next, we consider the function

$$\begin{aligned}g(\lambda)&:=S_\omega(v^\lambda)-\frac{\lambda^2}{2}P(v)\\ &=\frac{\alpha\lambda^2-2\lambda^\alpha}{2(p+1)}a\|v\|_{L^{p+1}}^{p+1}+\frac{\beta\lambda^2-2\lambda^\beta}{2(q+1)}b\|v\|_{L^{q+1}}^{q+1}+\frac{\omega}{2}\|v\|_{L^2}^2\end{aligned}$$

for $\lambda>0$. Then, we have

$$g'(1)=\partial_\lambda S_\omega(v^\lambda)\big|_{\lambda=1}-P(v)=0.$$

Moreover, since $P(v)\le 0$ and $E(v)\ge 0$, we have

$$g(1)=S_\omega(v)-\frac{1}{2}P(v)\ge S_\omega(v)\ge\frac{\omega}{2}\|v\|_{L^2}^2=g(+0).$$

Further, since $0 < \alpha < 2 < \beta$, there exists $\lambda_1 \in (0,1)$ such that $g'(\lambda_1) = 0$, $g'(\lambda) < 0$ for $\lambda \in (0, \lambda_1) \cup (1, \infty)$, and $g'(\lambda) > 0$ for $\lambda \in (\lambda_1, 1)$. Thus, $g(\lambda)$ attains its maximum at $\lambda = 1$.

Therefore, using $P(v) \leq 0$ again, we have

$$d(\omega) \leq S_\omega(v^{\lambda_0}) \leq S_\omega(v^{\lambda_0}) - \frac{\lambda_0^2}{2} P(v) \leq S_\omega(v) - \frac{1}{2} P(v).$$

This completes the proof. □

By the key Lemma 2.5, we have the following Lemma 2.6 and Theorem 2.7 as in Section 1.

Lemma 2.6. *Let $1 < p < 1 + 4/N < q < 2^* - 1$, $a > 0$, $b > 0$. The set*

$$\mathcal{B}_\omega^1 = \{v \in H^1(\mathbb{R}^N) : S_\omega(v) < d(\omega),\ P(v) < 0,\ E(v) \geq 0\} \tag{2.4}$$

is invariant under the flow of (2.1).

Theorem 2.7. *Let $1 < p < 1 + 4/N < q < 2^* - 1$, $a > 0$, $b > 0$. Let $\omega > 0$ and ϕ_ω be a ground state of* (2.2). *If $u_0 \in \mathcal{B}_\omega^1 \cap \Sigma$, then the solution $u(t)$ of* (2.1) *with $u(0) = u_0$ blows up in finite time.*

Remark 2.8. In [13], instead of (2.4), the set

$$\begin{aligned}\mathcal{B}_\omega^2 = \{v \in H^1(\mathbb{R}^N) : \|v\|_{L^2}^2 = \|\phi_\omega\|_{L^2}^2,\ 0 < E(v) < E(\phi_\omega),& \\ P(v) < 0,\ K_\omega(v) < 0\}&\end{aligned}$$

is defined, and it is proved that if $u_0 \in \mathcal{B}_\omega^2 \cap \Sigma$, then the solution $u(t)$ of (2.1) with $u(0) = u_0$ blows up in finite time. Remark that $\mathcal{B}_\omega^2 \subset \mathcal{B}_\omega^1$.

Finally, we give the proof of Theorem 2.3.

Proof of Theorem 2.3. By the elliptic regularity theory, $\phi_\omega \in \Sigma$.

Next, we consider the function

$$\begin{aligned}&(0,\infty) \ni \lambda \mapsto S_\omega(\phi_\omega^\lambda) \\ &= \frac{\lambda^2}{2}\|\nabla\phi_\omega\|_{L^2}^2 + \frac{\omega}{2}\|\phi_\omega\|_{L^2}^2 - \frac{a\lambda^\alpha}{p+1}\|\phi_\omega\|_{L^{p+1}}^{p+1} - \frac{b\lambda^\beta}{q+1}\|\phi_\omega\|_{L^{q+1}}^{q+1}.\end{aligned}$$

Since $0 < \alpha < 2 < \beta$, $S_\omega'(\phi_\omega) = 0$ and $E(\phi_\omega) > 0$, there exists $\lambda_1 \in (1, \infty)$ such that

$$S_\omega(\phi_\omega^\lambda) < S_\omega(\phi_\omega) = d(\omega), \quad P(\phi_\omega^\lambda) = \lambda\partial_\lambda S_\omega(\phi_\omega^\lambda) < 0, \quad E(\phi_\omega^\lambda) > 0$$

for $\lambda \in (1, \lambda_1)$. Thus, for $\lambda \in (1, \lambda_1)$, $\phi_\omega^\lambda \in \mathcal{B}_\omega^1 \cap \Sigma$, and it follows from Theorem 2.7 that the solution $u(t)$ of (2.1) with $u(0) = \phi_\omega^\lambda$ blows up in finite time.

Finally, since $\lim_{\lambda \to 1} \|\phi_\omega^\lambda - \phi_\omega\|_{H^1} = 0$, the proof is completed. □

Remark 2.9. It is proved in [10] that if $\partial_\lambda^2 E(\phi_\omega^\lambda)\big|_{\lambda=1} < 0$, then the standing wave solution $e^{i\omega t}\phi_\omega$ of (2.1) is orbitally unstable. We remark that $E(\phi_\omega) > 0$ implies $\partial_\lambda^2 E(\phi_\omega^\lambda)\big|_{\lambda=1} < 0$ for the case where $1 < p < 1 + 4/N < q < 2^* - 1$, $a > 0$, $b > 0$. We conjecture that the standing wave solution $e^{i\omega t}\phi_\omega$ of (2.1) may be strongly unstable under the assumption $\partial_\lambda^2 E(\phi_\omega^\lambda)\big|_{\lambda=1} < 0$. See [11] for a related result on nonlinear Schrödinger equations with a harmonic potential.

3 NLS with delta potential

In this section, we consider nonlinear Schrödinger equations with a delta potential in one space dimension:

$$i\partial_t u = -\partial_x^2 u - \gamma\delta(x)u - |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \tag{3.1}$$

where $\gamma > 0$ is a constant, $\delta(x)$ is the delta measure at the origin, and $1 < p < \infty$. The energy for (3.1) is defined by

$$E(v) = \frac{1}{2}\|\partial_x v\|_{L^2}^2 - \frac{\gamma}{2}|v(0)|^2 - \frac{1}{p+1}\|v\|_{L^{p+1}}^{p+1}$$

for $v \in H^1(\mathbb{R})$, and the Cauchy problem for (3.1) is locally well-posed in the energy space $H^1(\mathbb{R})$.

We study the strong instability of standing wave solutions $e^{i\omega t}\phi_\omega(x)$ of (3.1), where $\omega > \gamma^2/4$, and

$$\phi_\omega(x) = \left\{ \frac{(p+1)\omega}{2} \operatorname{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{2}|x| + \tanh^{-1}\left(\frac{\gamma}{2\sqrt{\omega}}\right)\right)\right\}^{\frac{1}{p-1}}, \tag{3.2}$$

which is a unique positive solution of

$$-\partial_x^2 \phi - \gamma\delta(x)\phi + \omega\phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}. \tag{3.3}$$

The following theorem is proved by Fukuizumi, Ohta and Ozawa [5].

Theorem 3.1 (Fukuizumi, Ohta and Ozawa [5]). *Let $\gamma > 0$ and $\omega > \gamma^2/4$.*

(i) *When $1 < p \le 5$, the standing wave solution $e^{i\omega t}\phi_\omega$ of (3.1) is orbitally stable for any $\omega \in (\gamma^2/4, \infty)$.*

(ii) *When $p > 5$, there exists $\omega_1 = \omega_1(p, \gamma) \in (\gamma^2/4, \infty)$ such that the standing wave solution $e^{i\omega t}\phi_\omega$ of (3.1) is orbitally stable for $\omega \in (\gamma^2/4, \omega_1)$, and orbitally unstable for $\omega \in (\omega_1, \infty)$.*

The following theorem is proved by Ohta and Yamaguchi [14].

Theorem 3.2 (Ohta and Yamaguchi [14]). *Let $\gamma > 0$, $p > 5$, $\omega > \gamma^2/4$, and let ϕ_ω be the function defined by (3.2). If $E(\phi_\omega) > 0$, then the standing wave solution $e^{i\omega t}\phi_\omega$ of (3.1) is strongly unstable.*

We repeat the same argument as in Subsection 2.2 to give a proof of Theorem 3.2 slightly different from [14].

We define the functionals S_ω, K_ω and P by

$$
\begin{aligned}
S_\omega(v) &= \frac{1}{2}\|\partial_x v\|_{L^2}^2 - \frac{\gamma}{2}|v(0)|^2 + \frac{\omega}{2}\|v\|_{L^2}^2 - \frac{1}{p+1}\|v\|_{L^{p+1}}^{p+1}, \\
K_\omega(v) &= \|\partial_x v\|_{L^2}^2 - \gamma|v(0)|^2 + \omega\|v\|_{L^2}^2 - \|v\|_{L^{p+1}}^{p+1}, \\
P(v) &= \|\partial_x v\|_{L^2}^2 - \frac{\gamma}{2}|v(0)|^2 - \frac{\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1}, \quad \alpha = \frac{p-1}{2}.
\end{aligned}
$$

Note that by the scaling $v^\lambda(x) = \lambda^{1/2}v(\lambda x)$ for $\lambda > 0$, we have

$$
\begin{aligned}
S_\omega(v^\lambda) &= \frac{\lambda^2}{2}\|\partial_x v\|_{L^2}^2 - \frac{\gamma}{2}\lambda|v(0)|^2 + \frac{\omega}{2}\|v\|_{L^2}^2 - \frac{\lambda^\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1}, \\
P(v^\lambda) &= \lambda^2\|\partial_x v\|_{L^2}^2 - \frac{\gamma}{2}\lambda|v(0)|^2 - \frac{\alpha\lambda^\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1} = \lambda\partial_\lambda S_\omega(v^\lambda).
\end{aligned}
$$

Moreover, we define

$$
d(\omega) = \inf\{S_\omega(v) : v \in H^1(\mathbb{R}),\ K_\omega(v) = 0,\ v \ne 0\}. \tag{3.4}
$$

Then, we have $d(\omega) = S_\omega(\phi_\omega)$ for the case $p > 1$, $\gamma > 0$ and $\omega > \gamma^2/4$.

Lemma 3.3. *Let $\gamma > 0$, $p > 5$, $\omega > \gamma^2/4$. If $v \in H^1(\mathbb{R})$ satisfies $E(v) \ge 0$, $P(v) \le 0$ and $v \ne 0$, then*

$$
d(\omega) \le S_\omega(v) - \frac{1}{2}P(v).
$$

Proof. Consider the function

$$f(\lambda) := K_\omega(v^\lambda) = \lambda^2\|\nabla v\|_{L^2}^2 - \gamma\lambda|v(0)|^2 + \omega\|v\|_{L^2}^2 - \lambda^\alpha\|v\|_{L^{p+1}}^{p+1}$$

for $\lambda > 0$. Then, $\lim_{\lambda\to+0} f(\lambda) = \omega\|v\|_{L^2}^2 > 0$. Moreover, since $\alpha > 2$, we have $\lim_{\lambda\to+\infty} f(\lambda) = -\infty$. Thus, there exists $\lambda_0 \in (0,\infty)$ such that $K_\omega(v^{\lambda_0}) = 0$. Then, by (3.4), we have $d(\omega) \le S_\omega(v^{\lambda_0})$.

Next, we consider the function

$$\begin{aligned} g(\lambda) &:= S_\omega(v^\lambda) - \frac{\lambda^2}{2}P(v) \\ &= \frac{\lambda^2 - 2\lambda}{4}\gamma|v(0)|^2 + \frac{\alpha\lambda^2 - 2\lambda^\alpha}{2(p+1)}\|v\|_{L^{p+1}}^{p+1} + \frac{\omega}{2}\|v\|_{L^2}^2 \end{aligned}$$

for $\lambda > 0$. Then, we have

$$g'(1) = \partial_\lambda S_\omega(v^\lambda)\big|_{\lambda=1} - P(v) = 0.$$

Moreover, since $P(v) \le 0$ and $E(v) \ge 0$, we have

$$g(1) = S_\omega(v) - \frac{1}{2}P(v) \ge S_\omega(v) \ge \frac{\omega}{2}\|v\|_{L^2}^2 = g(+0).$$

Further, since $\alpha > 2$, there exists $\lambda_1 \in (0,1)$ such that $g'(\lambda_1) = 0$, $g'(\lambda) < 0$ for $\lambda \in (0,\lambda_1) \cup (1,\infty)$, and $g'(\lambda) > 0$ for $\lambda \in (\lambda_1, 1)$. Thus, $g(\lambda)$ attains its maximum at $\lambda = 1$.

Therefore, using $P(v) \le 0$ again, we have

$$d(\omega) \le S_\omega(v^{\lambda_0}) \le S_\omega(v^{\lambda_0}) - \frac{\lambda_0^2}{2}P(v) \le S_\omega(v) - \frac{1}{2}P(v).$$

This completes the proof. □

By the key Lemma 3.3, we have the following theorem, which improves Theorem 1.6 of [14] (see Remark 2.8 above).

Theorem 3.4. *Let $\gamma > 0$, $p > 5$, $\omega > \gamma^2/4$, and define*

$$\mathcal{B}_\omega^1 = \{v \in H^1(\mathbb{R}) : S_\omega(v) < d(\omega),\ P(v) < 0,\ E(v) \ge 0\}.$$

If $u_0 \in \mathcal{B}_\omega^1 \cap \Sigma$, then the solution $u(t)$ of (3.1) with $u(0) = u_0$ blows up in finite time.

The proof of Theorem 3.2 is exactly the same as in that of Theorem 2.3.

Remark 3.5. We conjecture that the standing wave solution $e^{i\omega t}\phi_\omega$ of (3.1) may be strongly unstable under the assumption $\partial_\lambda^2 E(\phi_\omega^\lambda)\big|_{\lambda=1} < 0$ (see also Remark 2.9).

References

[1] H. Berestycki and T. Cazenave, *Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaires*, C. R. Acad. Sci. Paris Sér. I Math. **293** (1981), 489–492.

[2] T. Cazenave, "Semilinear Schrödinger equations," Courant Lect. Notes in Math., 10, New York University, Courant Institute of Mathematical Sciences, New York; Amer. Math. Soc., Providence, RI, 2003.

[3] T. Cazenave and P.-L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys. **85** (1982), 549–561.

[4] R. Fukuizumi, *Remarks on the stable standing waves for nonlinear Schrödinger equations with double power nonlinearity*, Adv. Math. Sci. Appl. **13** (2003), 549–564.

[5] R. Fukuizumi, M. Ohta and T. Ozawa, *Nonlinear Schrödinger equation with a point defect*, Ann. Inst. H. Poincaré, Anal. Non Linéaire **25** (2008), 837–845.

[6] M. K. Kwong, *Uniqueness of positive solutions of* $\Delta u - u + u^p = 0$ *in* $\mathbf{R}^n$, Arch. Rational Mech. Anal. **105** (1989), 243–266.

[7] S. Le Coz, *A note on Berestycki-Cazenave's classical instability result for nonlinear Schrödinger equations*, Adv. Nonlinear Stud. **8** (2008), 455–463.

[8] M. Maeda, *Stability and instability of standing waves for 1-dimensional nonlinear Schrödinger equation with multiple-power nonlinearity*, Kodai Math. J. **31** (2008), 263–271.

[9] M. Ohta, *Stability and instability of standing waves for one-dimensional nonlinear Schrödinger equations with double power nonlinearity*, Kodai Math. J. **18** (1995), 68–74.

[10] M. Ohta, *Instability of standing waves for the generalized Davey-Stewartson system*, Ann. Inst. H. Poincaré, Phys. Théor. **62** (1995), 69–80.

[11] M. Ohta, *Strong instability of standing waves for nonlinear Schrödinger equations with harmonic potential*, Preprint, arXiv:1604.06957.

[12] M. Ohta, *Strong instability of standing waves for nonlinear Schrödinger equations with a partial confinement*, Preprint, arXiv:1706.02100.

[13] M. Ohta and T. Yamaguchi, *Strong instability of standing waves for nonlinear Schrödinger equations with double power nonlinearity*, SUT J. Math. **51** (2015), 49–58.

[14] M. Ohta and T. Yamaguchi, *Strong instability of standing waves for nonlinear Schrödinger equations with a delta potential*, in "Harmonic analysis and nonlinear partial differential equations", RIMS Kôkyûroku Bessatsu **B56** (2016), 79–92.

[15] J. Zhang, *Cross-constrained variational problem and nonlinear Schrödinger equation*, in "Foundations of computational mathematics", World Sci. Publ., River Edge, NJ, (2002), 457–469.

Department of Mathematics
Tokyo University of Science
1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601
JAPAN
E-mail address: mohta@rs.tus.ac.jp

東京理科大学・理学部数学科
太田 雅人