数理解析研究所講究録 第2093巻 2018年 38-46

LIFESPAN OF PERIODIC SOLUTIONS TO NONLINEAR SCHRÖDINGER EQUATIONS

KAZUMASA FUJIWARA(藤原 和将)
TOHRU OZAWA(小澤 徹)

1. Introduction

We study the Cauchy problem for the nongauge invariant nonlinear Schrödinger equations of the form

$$i\partial_t u + \Delta u = \lambda |u|^p, \quad (t,x) \in [0,T) \times \mathbb{T}^n \tag{NLS}$$

and the derivative nonlinear Schrödinger equations of the form

$$i\partial_t u + \partial^2 u = \lambda \partial(|u|^{p-1}u), \quad (t,x) \in [0,T) \times \mathbb{T}, \tag{DNLS}$$

where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $\lambda \in \mathbb{C}\backslash\{0\}$, $\partial = \partial/\partial x$, $p > 1$, and $T > 0$. The purpose of this note is to present some explicit upperbounds for the lifespan of periodic solutions to (NLS) and (DNLS) in terms of the Cauchy data and to examine their optimality by exact solutions. Part of the contents of this note is devoted to a detailed description of the argument of our recent papers [4, 5].

2. Nongauge Invariant NLS

In this section, we study (NLS) in $[0,T) \times \mathbb{T}^n$ with $T > 0$ and $n \geq 1$. The Cauchy problem for (NLS) is proved to be locally well-posed in the Sobolev space $H^s(\mathbb{T}^n)$ with $s > n/2$ and $p \in 2\mathbb{N} \cup (s, \infty)$, where

$$H^s(\mathbb{T}^n) = \{u \in L^2(\mathbb{T}^n)\,;\ \sum_{k \in \mathbb{Z}^n} (1 + |k|^2)^s |\hat{u}(k)|^2 < \infty\}$$

and $\hat{u}(k)$ is the Fourier coefficient

$$\hat{u}(k) = (2\pi)^{-n} \int_{\mathbb{T}^n} e^{-ik\cdot x} u(x) dx, \quad k \in \mathbb{Z}^n$$

in the Fourier series expansion

$$u(x) = \sum_{k \in \mathbb{Z}^n} \hat{u}(k) e^{ix\cdot k}, \quad x \in \mathbb{T}^n.$$

The blowup problem for (NLS) has been studied in [13] (see also [14]), where the Cauchy data $u_0 = u(0)$ is supposed to satisfy

$$(\mathrm{Re}\lambda)\, \mathrm{Im} \int_{\mathbb{T}^n} u_0 < 0 \qquad \text{or} \qquad (\mathrm{Im}\lambda)\, \mathrm{Re} \int_{\mathbb{T}^n} u_0 > 0. \tag{A0}$$

The argument in [13] depends on a test function method [1, 20, 21], originally introduced for nonlinear heat and damped wave equations. In the argument based on a test function method, however, the condition (A0) arises in a rather implicit setting, so that it is unlikely that (A0) provides a simple and direct description of the blowup mechanism. In this section, we introduce a twisted total signed density of wavefunctions over $\mathbb{T}^n$ and prove its finite time blowup by differential inequalities. Our argument clarifies how the blowup phenomena occur by ODE mechanism on the basis of monotonicity. Moreover, a clear picture is given on how necessary conditions on the Cauchy data come into play in the proof of blowup in a rather general framework. Furthermore, an explicit and optimal upperbound of the lifespan of solution is naturally introduced in our argument.

The main result is this section is the following:

Theorem 1. *Let $u \in C([0,T_m)\,;\,(H^2 \cap L^p)(\mathbb{T}^n)) \cap C^1([0,T_m)\,;\,L^2(\mathbb{T}^n))$ be the maximal solution of (NLS) with $u_0 = u(0) \in H^2(\mathbb{T}^n)\backslash\{0\}$. Assume that :*

$$\operatorname{Im}\Big(\overline{\lambda}\int_{\mathbb{T}^n} u_0\Big) \le 0 \qquad \text{or} \qquad \operatorname{Re}\Big(\overline{\lambda}\int_{\mathbb{T}^n} u_0\Big) \ne 0. \tag{A1}$$

Then, $T_m < +\infty$. Moreover:

(1) If $\operatorname{Im}(\overline{\lambda}\int_{\mathbb{T}^n} u_0) < 0$, then T_m is estimated as

$$T_m \le \frac{(2\pi)^{n(p-1)}}{(p-1)|\lambda|}\left|\operatorname{Im}\Big(\frac{\overline{\lambda}}{|\lambda|}\int_{\mathbb{T}^n} u_0\Big)\right|^{1-p}. \tag{2.1}$$

(2) If $\operatorname{Im}(\overline{\lambda}\int_{\mathbb{T}^n} u_0) = 0$, then there exists $t_0 \in (0, T_m)$ such that $\operatorname{Im}(\overline{\lambda}\int_{\mathbb{T}^n} u(t_0)) < 0$ and T_m is estimated as

$$T_m \le t_0 + \frac{(2\pi)^{n(p-1)}}{(p-1)|\lambda|}\left|\operatorname{Im}\Big(\frac{\overline{\lambda}}{|\lambda|}\int_{\mathbb{T}^n} u(t_0)\Big)\right|^{1-p}. \tag{2.2}$$

(3) If $\operatorname{Im}(\overline{\lambda}\int_{\mathbb{T}^n} u_0) > 0$ and $\operatorname{Re}(\overline{\lambda}\int_{\mathbb{T}^n} u_0) \ne 0$, then T_m is estimated as

$$T_m \le \frac{(2\pi)^{n(p-1)}}{(p-1)|\lambda|}\left(1 + 4\Big(\frac{\operatorname{Im}(\overline{\lambda}\int_{\mathbb{T}^n} u_0)}{\operatorname{Re}(\overline{\lambda}\int_{\mathbb{T}^n} u_0)}\Big)^2\right)^{p/2}\left|\operatorname{Im}\Big(\frac{\overline{\lambda}}{|\lambda|}\int_{\mathbb{T}^n} u_0\Big)\right|^{1-p}. \tag{2.3}$$

Proof of Theorem 1. First, we prove that $T_m < +\infty$, provided that there exists $\alpha \in \mathbb{C}\backslash\{0\}$ such that

$$\operatorname{Re}(\alpha\lambda) > 0 \ge \operatorname{Im}\Big(\alpha\int_{\mathbb{T}^n} u_0\Big). \tag{2.4}$$

For that purpose, we assume $T_m = +\infty$ and derive a contradiction. We define

$$M(t) = -\operatorname{Im}\Big(\alpha\int_{\mathbb{T}^n} u(t)\Big), \quad t \ge 0.$$

Then

$$M(0) = -\operatorname{Im}\Big(\alpha\int_{\mathbb{T}^n} u_0\Big) \ge 0. \tag{2.5}$$

Differentiating M in t and using (NLS), we have

$$\begin{aligned} M'(t) &= -\operatorname{Im}\Big(\alpha\int_{\mathbb{T}^n} \partial_t u\Big) \\ &= \operatorname{Re}\Big(\alpha\int_{\mathbb{T}^n} i\partial_t u\Big) \\ &= \operatorname{Re}\Big(\alpha\int_{\mathbb{T}^n} (-\Delta u + \lambda|u|^p)\Big) = \operatorname{Re}(\alpha\lambda)\|u(t)\|_p^p, \end{aligned} \tag{2.6}$$

where we have used

$$\int_{\mathbb{T}^n} \Delta u(t) = -\sum_{k\in\mathbb{Z}^n} |k|^2 \hat{u}(t,k) \int_{\mathbb{T}^n} e^{ik\cdot x} = -\sum_{k\in\mathbb{Z}^n} |k|^2 \hat{u}(t,k)\delta_{0k} = 0$$

with Kronecker's delta δ_{jk} in $\mathbb{Z}^n$. By (2.5) and (2.6), $M(t)$ is nonnegative for all $t \geq 0$. By the Hölder inequality, $M(t)$ is bounded by

$$0 \leq M(t) \leq |\alpha| \int_{\mathbb{T}^n} |u(t)| \leq |\alpha|(2\pi)^{n(p-1)/p} \|u(t)\|_p. \tag{2.7}$$

By (2.6) and (2.7), we have

$$M'(t) \geq \mathrm{Re}(\alpha\lambda)|\alpha|^{-p}(2\pi)^{n(1-p)} M(t)^p. \tag{2.8}$$

We now distinguish two cases: (i) $M(0) > 0$. (ii) $M(0) = 0$.

(i) If $M(0) > 0$, then by (2.6), $M(t)$ is strictly positive for all $t \geq 0$ and (2.8) implies

$$\begin{aligned} \frac{d}{dt}(M(t)^{1-p}) &= -(p-1)M(t)^{-p}M'(t) \\ &\leq -(p-1)\mathrm{Re}(\alpha\lambda)|\alpha|^{-p}(2\pi)^{n(1-p)}, \end{aligned} \tag{2.9}$$

which in turn implies

$$M(t) \geq (M(0)^{1-p} - (p-1)\mathrm{Re}(\alpha\lambda)|\alpha|^{-p}(2\pi)^{n(1-p)}t)^{-1/(p-1)} \tag{2.10}$$

for all $t \geq 0$. This is a contradiction to $T_m = +\infty$ since $M(t)$ tends to infinity in a finite time.

(ii) Let $M(0) = 0$. We prove that there exists $t_0 > 0$ such that $M(t_0) > 0$. Otherwise, $M(t)$ vanishes identically and so does $M'(t)$. By (2.6), this shows that $\|u(t)\|_p = 0$ for all $t \geq 0$. In particular, $u_0 = 0$, which is a contradiction.

Again by (2.6), $M(t)$ is strictly positive for all $t \geq t_0$ and (2.9) holds on $[t_0, \infty)$. Integrating (2.9) on $[t_0, t]$, we obtain

$$M(t) \geq (M(t_0)^{1-p} - (p-1)\mathrm{Re}(\alpha\lambda)|\alpha|^{-p}(2\pi)^{n(1-p)}(t-t_0))^{-1/(p-1)} \tag{2.11}$$

for all $t \geq t_0$. This is a contradiction to $T_m = +\infty$ as above.

(1) If $\mathrm{Im}(\overline{\lambda} \int_{\mathbb{T}^n} u_0) < 0$, we set $\alpha = \overline{\lambda}$. Then (2.4) holds and $M(0) > 0$. Moreover, (2.1) follows from (2.10).

(2) If $\mathrm{Im}(\overline{\lambda} \int_{\mathbb{T}^n} u_0) = 0$, we set $\alpha = \overline{\lambda}$. Then (2.4) holds and $M(0) = 0$. Moreover, (2.2) follows from (2.11).

(3) If $\mathrm{Im}(\overline{\lambda} \int_{\mathbb{T}^n} u_0) > 0$ and $\mathrm{Re}(\overline{\lambda} \int_{\mathbb{T}^n} u_0) \neq 0$, we set $\alpha = \overline{\lambda}(1 - ia)$ with

$$a = 2\frac{\mathrm{Im}(\overline{\lambda} \int_{\mathbb{T}^n} u_0)}{\mathrm{Re}(\overline{\lambda} \int_{\mathbb{T}^n} u_0)}.$$

Then, $\mathrm{Re}(\alpha\lambda) = |\lambda|^2 \mathrm{Re}(1 - ia) = |\lambda|^2 > 0$ and

$$\begin{aligned} M(0) &= -\mathrm{Im}\left(\overline{\lambda}(1-ia)\int_{\mathbb{T}^n} u_0\right) \\ &= -\mathrm{Im}\left(\overline{\lambda}\int_{\mathbb{T}^n} u_0\right) + a\mathrm{Re}\left(\overline{\lambda}\int_{\mathbb{T}^n} u_0\right) = \mathrm{Im}\left(\overline{\lambda}\int_{\mathbb{T}^n} u_0\right). \end{aligned}$$

Therefore, (2.4) holds and $M(0) > 0$. Moreover, (2.3) follows from (2.10) since $|\alpha|^2 = |\lambda|^2(1 + a^2)$.

□

Remark 1. *The condition (A1) is optimal in the sense that there exist global solutions if (A1) fails. For instance, let* $c \in \mathbb{C}\backslash\{0\}$ *satisfy* $c = i\frac{\lambda}{|\lambda|}|c|$. *Then*

$$u(t,x) = c(1 + (p-1)|\lambda||c|^{p-1}t)^{-1/(p-1)}$$

is a global solution with $u_0(x) = c$. *In this case,*

$$\mathrm{Im}\left(\overline{\lambda}\int_{\mathbb{T}^n} u_0\right) = (2\pi)^n \mathrm{Im}(\overline{\lambda}c) = (2\pi)^n|\lambda||c| > 0,$$

$$\mathrm{Re}\left(\overline{\lambda}\int_{\mathbb{T}^n} u_0\right) = (2\pi)^n \mathrm{Re}(\overline{\lambda}c) = (2\pi)^n \mathrm{Re}(i|\lambda||c|) = 0.$$

Remark 2. *The lifespan estimate* (2.1) *is optimal. Let* $c \in \mathbb{C}\backslash\{0\}$ *satisfy* $c = -i\frac{\lambda}{|\lambda|}|c|$. *Then*

$$u(t,x) = c(1 - (p-1)|\lambda||c|^{p-1}t)^{-1/(p-1)}$$

is a blowup solution with $u_0(x) = c$. *The blowup time is given by*

$$T = \frac{1}{(p-1)|\lambda||c|^{p-1}} = \frac{1}{(p-1)|\lambda|}\left|\mathrm{Im}\left(\frac{\overline{\lambda}}{|\lambda|}c\right)\right|^{1-p}$$
$$= \frac{(2\pi)^{n(p-1)}}{(p-1)|\lambda|}\left|\mathrm{Im}\left(\frac{\overline{\lambda}}{|\lambda|}\int_{\mathbb{T}^n} u_0\right)\right|^{1-p},$$

which is exactly the same as the right hand of (2.1). *In this case,*

$$\mathrm{Im}\left(\overline{\lambda}\int_{\mathbb{T}^n} u_0\right) = -(2\pi)^n|\lambda||c| < 0,$$

$$\mathrm{Re}\left(\overline{\lambda}\int_{\mathbb{T}^n} u_0\right) = 0.$$

A characterization of (A1) is shown to be given by (2.4). In fact, we have the following proposition.

Proposition 1. *Let* $\lambda \in \mathbb{C}\backslash\{0\}$ *and let* $u_0 \in L^1(\mathbb{T}^n)$. *Then the following statements are equivalent.*

(A1) $\mathrm{Im}\left(\overline{\lambda}\int_{\mathbb{T}^n} u_0\right) \leq 0$ *or* $\mathrm{Re}\left(\overline{\lambda}\int_{\mathbb{T}^n} u_0\right) \neq 0$.

(A2) There exists $a \in \mathbb{R}$ *such that* $a\mathrm{Re}\left(\overline{\lambda}\int_{\mathbb{T}^n} u_0\right) \leq -\mathrm{Im}\left(\overline{\lambda}\int_{\mathbb{T}^n} u_0\right)$.

(A3) There exists $\alpha \in \mathbb{C}$ *such that* $\mathrm{Re}(\alpha\lambda) > 0 \geq \mathrm{Im}\left(\alpha\int_{\mathbb{T}^n} u_0\right)$.

Proposition 1 is reduced to the following elementary proposition.

Proposition 2. *Let* $\lambda \in \mathbb{C}\backslash\{0\}$ *and let* $\mu \in \mathbb{C}$. *Then the following statements are equivalent.*

(i) $\mathrm{Im}(\overline{\lambda}\mu) \leq 0$ *or* $\mathrm{Re}(\overline{\lambda}\mu) \neq 0$.

(ii) There exists $a \in \mathbb{R}$ *such that* $a\mathrm{Re}(\overline{\lambda}\mu) \leq -\mathrm{Im}(\overline{\lambda}\mu)$.

(iii) There exists $\alpha \in \mathbb{C}$ *such that* $\mathrm{Re}(\alpha\lambda) > 0 \geq \mathrm{Im}(\alpha\mu)$.

Remark 3. *(A0) is regarded as a special case of (A3). Indeed, if* $\alpha = \mathrm{Re}\lambda \neq 0$, *then (A3) becomes*

$$(\mathrm{Re}\lambda)^2 > 0 \geq (\mathrm{Re}\lambda)\,\mathrm{Im}\int_{\mathbb{T}^n} u_0$$

and if $\alpha = -i\mathrm{Im}\lambda \neq 0$, *then (A3) becomes*

$$(\mathrm{Im}\lambda)^2 > 0 \geq -(\mathrm{Im}\lambda)\,\mathrm{Re}\int_{\mathbb{T}^n} u_0.$$

(A1) is recovered by a specific choice of α *in (A3). Indeed, if* $\alpha = \overline{\lambda}$, *then (A3) becomes*

$$|\lambda|^2 > 0 \geq \mathrm{Im}\Big(\overline{\lambda}\int_{\mathbb{T}^n} u_0\Big)$$

and if $\alpha = \pm\overline{\int_{\mathbb{T}^n} u_0}$, *then* (A3) *becomes*

$$\pm\mathrm{Re}\Big(\overline{\lambda}\int_{\mathbb{T}^n} u_0\Big) > 0 = \mathrm{Im}\Big(\pm\overline{\int_{\mathbb{T}^n} u_0}\int_{\mathbb{T}^n} u_0\Big).$$

Proof of Proposition 2. (i) $\Rightarrow$ (ii): If $\mathrm{Im}(\overline{\lambda}\mu) \geq 0$, then we take $a = 0$. If $\mathrm{Im}(\overline{\lambda}\mu) < 0$, then (i) implies $\mathrm{Re}(\overline{\lambda}\mu) \neq 0$ and we take

$$a = -(\mathrm{Im}(\overline{\lambda}\mu) + 1)\mathrm{Re}(\overline{\lambda}\mu)/|\mathrm{Re}(\overline{\lambda}\mu)|^2,$$

which yields

$$a\mathrm{Re}(\overline{\lambda}\mu) = -(\mathrm{Im}(\overline{\lambda}\mu) + 1) < -\mathrm{Im}(\overline{\lambda}\mu).$$

(ii) $\Rightarrow$ (iii): Let $\alpha = (1 + ia)\overline{\lambda}$. Then

$$\mathrm{Re}(\alpha\lambda) = |\lambda|^2 > 0$$

and

$$\mathrm{Im}(\alpha\mu) = \mathrm{Im}(\overline{\lambda}\mu) + a\mathrm{Re}(\overline{\lambda}\mu) \leq 0.$$

(iii) $\Rightarrow$ (i): Assume that $\mathrm{Im}(\overline{\lambda}\mu) > 0$ and $\mathrm{Re}(\overline{\lambda}\mu) = 0$. Then for any $\alpha \in \mathbb{C}$,

$$\mathrm{Im}(\alpha\mu) = \mathrm{Im}\Big(\frac{\alpha\lambda}{|\lambda|^2}\cdot\overline{\lambda}\mu\Big) = \frac{1}{|\lambda|^2}\mathrm{Re}(\alpha\lambda)\cdot\mathrm{Im}(\overline{\lambda}\mu).$$

If $\mathrm{Im}(\alpha\mu) = 0$, then $\mathrm{Re}(\alpha\lambda) = 0$, which contradicts (iii). If $\mathrm{Im}(\alpha\mu) \neq 0$, then $\mathrm{Im}(\alpha\mu)$ and $\mathrm{Re}(\alpha\lambda)$ have the same sign, which is also a contradiction to (iii). □

3. Derivative NLS

In this section, we study (DNLS) in $[0,T) \times \mathbb{T}$ with $T > 0$ and $p > 1$. The original derivative nonlinear Schrödinger equation takes the form

$$i\partial_t u + \partial^2 u = \pm i\partial(|u|^2 u),$$

namely, (DNLS) with $\mathrm{Re}\lambda = 0$ and $p = 3$. There is a large literature on the Cauchy problem for (DNLS). We refer the reader to [3, 6, 7, 8, 9, 10, 11, 12, 15, 18, 19] for instance. The blowup problem for DNLS is still open, however (see [2, 16, 17] for related results).

In this section, we consider the maximal solution

$$u \in C([0,T_m); H^2(\mathbb{T})) \cap C^1([0,T_m); L^2(\mathbb{T}))$$

with Cauchy data $u_0 = u(0) \in H^2(\mathbb{T})\backslash\{0\}$. The periodic boundary condition is explicitly given by

$$u(t,0) = u(t,2\pi), \qquad \partial_t u(t,0) = \partial_t u(t,2\pi) \tag{3.1}$$

for all $t \in [0, T_m)$. We note that constants are global solutions to (DNLS). To study blowup problem for (DNLS), we impose the following renormalization excluding constant solutions:

$$\int_0^{2\pi} u_0 = 0. \tag{3.2}$$

The renormalization condition is shown to be preserved in time. Indeed, by (3.1) and (3.2), we have

$$\begin{aligned} \int_0^{2\pi} u(t) &= \int_0^{2\pi} \left(u_0 + \int_0^t \partial_s u(s) ds \right) \\ &= -i \int_0^t \left(\int_0^{2\pi} i\partial_s u(s) \right) ds \\ &= -i \int_0^t \left(\int_0^{2\pi} \partial(-\partial u + \lambda |u|^{p-1} u)(s) \right) ds = 0 \end{aligned} \tag{3.3}$$

for all $t \in [0, T_m)$.

We introduce the positive and negative momentum of integrated wavefunctions by

$$\begin{aligned} M_\pm(t) &= \pm \mathrm{Im} \int_0^{2\pi} u(t) \left(\int_0^{\cdot} \overline{u(t)} \right) \\ &= \pm \mathrm{Im} \int_0^{2\pi} u(t,x) \left(\int_0^x \overline{u(t,y)} dy \right) dx. \end{aligned}$$

The main result in this section is the following:

Theorem 2. *Let $u \in C([0,T_m); H^2(\mathbb{T})) \cap C^1([0,T_m); L^2(\mathbb{T}))$ be the maximal solution of (DNLS) with $u_0 = u(0) \in H^2(\mathbb{T}) \backslash \{0\}$. Assume:*

- *Sign condition I:* $\mathrm{Re}\lambda \neq 0.$
- *Sign condition II:* $(\mathrm{Re}\lambda)\mathrm{Im} \int_0^{2\pi} u_0(x) \left(\int_0^x \overline{u_0(y)} dy \right) dx \geq 0.$
- *Renormalization condition:* $\int_0^{2\pi} u_0(x) dx = 0.$

Then, $T_m < +\infty$. Moreover:
(1) If $M_\pm(0) > 0$, then T_m is estimated as

$$T_m \leq \frac{2^{p-1}\pi^p}{(p-1)|\mathrm{Re}\lambda|} M_\pm(0)^{\frac{1-p}{2}}. \tag{3.4}$$

(2) If $M_\pm(0) = 0$, then there exists $t_0 \in (0, T_m)$ such that $M_\pm(t_0) > 0$ and T_m is estimated as

$$T_m \leq t_0 + \frac{2^{p-1}\pi^p}{(p-1)|\mathrm{Re}\lambda|} M_\pm(t_0)^{\frac{1-p}{2}}. \tag{3.5}$$

Remark 4. *The original derivative NLS does not satisfy sign condition I.*

Remark 5. *Sign condition II is understood to assume that*

$$M_+(0) \geq 0 \quad \text{if} \quad \mathrm{Re}\lambda > 0$$

and that

$$M_-(0) \leq 0 \quad \text{if} \quad \mathrm{Re}\lambda < 0.$$

Proof of Theorem 2. Let λ satisfy $\mathrm{Re}\lambda > 0$ [respectively, $\mathrm{Re}\lambda < 0$] and let u_0 satisfy $M_+(0) \geq 0$ [respectively, $M_-(0) \leq 0$]. We denote both cases by $\pm\mathrm{Re}\lambda > 0$ and $\pm M_\pm(0) \geq 0$. We assume $T_m = +\infty$ and derive a contradiction. Differentiating $M_\pm$ in t and using (DNLS), (3.1), (3.3), and integration by parts, we have

$$
\begin{aligned}
M'_\pm(t) &= \pm\mathrm{Im}\Big[\int_0^{2\pi} \partial_t u\Big(\int_0^{\cdot} \overline{u}\Big) + \int_0^{2\pi} u\Big(\int_0^{\cdot} \overline{\partial_t u}\Big)\Big] \\
&= \pm \mathrm{Re}\Big[-\int_0^{2\pi} i\partial_t u\Big(\int_0^{\cdot} \overline{u}\Big) + \int_0^{2\pi} u\Big(\int_0^{\cdot} \overline{i\partial_t u}\Big)\Big] \\
&= \pm \mathrm{Re}\Big[-\int_0^{2\pi} (-\partial(\partial u - \lambda|u|^{p-1}u))\Big(\int_0^{\cdot} \overline{u}\Big) + \int_0^{2\pi} u\Big(\int_0^{\cdot} \partial(-\overline{\partial u} + \lambda|u|^{p-1}\overline{u})\Big)\Big] \\
&= \pm \mathrm{Re}\Big[-\int_0^{2\pi} (\partial u - \lambda|u|^{p-1}u)\overline{u} + \int_0^{2\pi} u(-\overline{\partial u} + \lambda|u|^{p-1}\overline{u})\Big] \\
&= \pm 2\mathrm{Re}\lambda \int_0^{2\pi} |u|^{p+1} \geq 0.
\end{aligned} \tag{3.6}
$$

By (3.6) and the sign condition II, $M_\pm(t)$ are nonnegative for all $t \geq 0$. By the Hölder inequality, $M_\pm(t)$ are bounded by

$$
\begin{aligned}
0 \leq M_\pm(t) &\leq \int_0^{2\pi} |u(t,x)|\Big(\int_0^x |u(t,y)|dy\Big)dx \\
&= \frac{1}{2}\int_0^{2\pi} \frac{d}{dx}\Big(\int_0^x |u(t,y)|dy\Big)^2 dx \\
&= \frac{1}{2}\Big(\int_0^{2\pi} |u(t,y)|dy\Big)^2 \\
&\leq \frac{1}{2}((2\pi)^{\frac{p}{p+1}}\|u(t)\|_{p+1})^2.
\end{aligned} \tag{3.7}
$$

By (3.6) and (3.7), we obtain

$$
\begin{aligned}
0 \leq M_\pm(t)^{\frac{p+1}{2}} &\leq 2^{\frac{p-1}{2}}\pi^p\|u(t)\|_{p+1}^{p+1} \\
&= 2^{\frac{p-1}{2}}\pi^p \cdot \frac{1}{2|\mathrm{Re}\lambda|}M'_\pm(t).
\end{aligned} \tag{3.8}
$$

We now distinguish two cases: (1) $M_\pm(0) > 0$. (2) $M_\pm(0) = 0$.
(1) If $M_\pm(0) > 0$, then by (3.6), $M_\pm(t)$ are strictly positive for all $t \geq 0$ and (3.8) implies

$$
\begin{aligned}
\frac{d}{dt}(M_\pm(t)^{-\frac{p-1}{2}}) &= -\frac{p-1}{2}M_\pm(t)^{-\frac{p+1}{2}}M'_\pm(t) \\
&\leq -\frac{p-1}{2}\frac{2|\mathrm{Re}\lambda|}{2^{\frac{p-1}{2}}\pi^p} = -\frac{1}{c_0},
\end{aligned} \tag{3.9}
$$

where

$$
c_0 = \frac{2^{\frac{p-1}{2}}\pi^p}{(p-1)|\mathrm{Re}\lambda|}.
$$

Integrating both hand sides of (3.9), we have

$$
M_\pm(t)^{-\frac{p-1}{2}} - M_\pm(0)^{-\frac{p-1}{2}} \leq -\frac{1}{c_0}t,
$$

which are equivalent to

$$M_{\pm}(t) \geq \left(M_{\pm}(0)^{-\frac{p-1}{2}} - \frac{1}{c_0}t\right)^{-\frac{2}{p-1}} \tag{3.10}$$

for all $t \geq 0$. This is a contradiction to $T_m = +\infty$ since $M_{\pm}(t)$ tend to infinity in a finite time. Moreover, (3.4) follows from (3.10) which holds for all $t \in [0, T_m)$.

(2) Let $M_{\pm}(0) = 0$. For definiteness, we consider M_+ only. The other case may be treated similarly. We prove that there exists $t_0 > 0$ such that $M_+(t_0) > 0$. Otherwise, $M_+(t)$ vanishes identically and so does $M_+'(t)$. By (3.6), this shows that $\|u(t)\|_{p+1} = 0$ for all $t \geq 0$. In particular, $u_0 = 0$, which is a contradiction.

Again by (3.6), $M_+(t)$ is strictly positive for all $t \geq t_0$. Integrating (3.9) on $[t_0, t]$, we obtain

$$M_+(t) \geq \left(M_+(t_0)^{-\frac{p-1}{2}} - \frac{1}{c_0}(t - t_0)\right)^{-\frac{2}{p-1}} \tag{3.11}$$

for all $t \geq t_0$. This is a contradiction to $T_m = +\infty$ as above. Moreover, (3.5) follows from (3.11) which holds for all $t \in [0, T_m)$. □

Remark 6. *The sign condition II is optimal in the sense that there exist global solutions if it fails. For any $\lambda \in \mathbb{R}$ with $\mp\lambda > 0$ and any $c \in \mathbb{C}\backslash\{0\}$, functions $u^{\pm} : [0, \infty) \times \mathbb{T} \to \mathbb{C}$ given by*

$$u^{\pm}(t, x) = ce^{-it \pm ix}(1 \mp \lambda(p-1)|c|^{p-1}t)^{-1/(p-1)}$$

are global solutions to (DNLS) with $u^{\pm}(0, x) = ce^{\pm ix}$. In this case,

$$\int_0^{2\pi} u^{\pm}(0, x)dx = 0$$

and $M_{\pm}(0) = 2\pi|c|^2$, so that $\pm\lambda M_{\pm}(0) < 0$, violating sign condition II.

Remark 7. *Let $\pm\lambda > 0$ and let $c \in \mathbb{C}\backslash\{0\}$. Functions $u_{\pm} : [0, T_m) \times \mathbb{T} \to \mathbb{C}$ given by*

$$u_{\pm}(t, x) = ce^{-it \pm ix}\left(1 - \frac{t}{T_m}\right)^{-1/(p-1)}$$

are solutions to (DNLS) if and only if

$$T_m = \frac{1}{(p-1)|\lambda||c|^{p-1}}.$$

The upperbound given by the right hand side of (3.4) is greater than T_m since

$$\frac{2^{\frac{p-1}{2}}\pi^p}{(p-1)|\mathrm{Re}\lambda|}M_{\pm}(0)^{\frac{1-p}{2}} = \frac{\pi^{\frac{p+1}{2}}}{(p-1)|\mathrm{Re}\lambda||c|^{p-1}}.$$

References

[1] P. Baras and M. Pierre, "Critre d'existence de solutions positives pour des quations semi-linaires non monotones", Annales de l'I.H.P. Analyse non linaire, **2** (1985), 185–212.

[2] H. A. Biagioni and F. Linares, "Ill-posedness for the derivative Schrödinger and generalized Benjamin-Ono equations", Trans. Amer. Math. Soc., **353**(2001), 3649–3659.

[3] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, "A refined global well-posedness result for Schrödinger equations with derivative", SIAM J. Math. Anal., **34**(2002), 64–86.

[4] K. Fujiwara and T. Ozawa, "Lifespan of strong solutions to the periodic nonlinear Schrödinger equation without gauge invariance", J. Evol. Equ., (2016), 1–8.

[5] K. Fujiwara and T. Ozawa, "Lifespan of strong solutions to the periodic derivative nonlinear Schrödinger equation", to appear in Evol. Equ. Control Theory.
[6] M. Hayashi and T. Ozawa, "Well-posedness for a generalized derivative nonlinear Schrödinger equation", J. Differential Equations, **261**(2016), 5424–5445.
[7] N. Hayashi, "The initial value problem for the derivative nonlinear Schrödinger equation in the energy space", Nonlinear Anal., **20**(1993), 823–833.
[8] N. Hayashi and T. Ozawa, "On the derivative nonlinear Schrödinger equation", Phys. D, **55**(1992), 14–36.
[9] N. Hayashi and T. Ozawa, "Finite energy solutions of nonlinear Schrödinger equations of derivative type", SIAM J. Math. Anal., **25**(1994), 1488–1503.
[10] S. Herr, "On the Cauchy problem for the derivative nonlinear Schrödinger equation with periodic boundary condition", Int. Math. Res. Not., (2006), Art. ID 96763, 33.
[11] K. Mio, K. Minami, T. Ogino, and S. Takeda, "Modified nonlinear Schrödinger equation for Alfvén waves propagating along the magnetic field in cold plasmas", J. Phys. Soc. Japan, **41**(1976), 265–271.
[12] R. Mosincat, "Global well-posedness of the derivative nonlinear Schrödinger equation with periodic boundary condition in $H^{\frac{1}{2}}$", J. Differential Equations, **263**(2017), 4658–4722.
[13] T. Oh, "A blowup result for the periodic NLS without gauge invariance", C. R. Math. Acad. Sci. Paris, **350**(2012), 389–392.
[14] T. Ozawa and Y. Yamazaki, "Life-span of smooth solutions to the complex Ginzburg-Landau type equation on a torus", Nonlinearity, **16**(2003), 2029–2034.
[15] C. Sulem and P. Sulem, The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse, Springer New York, 1999.
[16] H. Sunagawa, "The lifespan of solutions to nonlinear Schrödinger and Klein-Gordon equations", Hokkaido Math. J., **37**(2008), 825–838.
[17] S. B. Tan, "Blow-up solutions for mixed nonlinear Schrödinger equations", Acta Math. Sin. (Engl. Ser.), **20**(2004), 115–124.
[18] M. Tsutsumi and I. Fukuda, "On solutions of the derivative nonlinear Schrödinger equation. Existence and uniqueness theorem", Funkcial. Ekvac., **23**(1980), 259–277.
[19] Y. Y. S. Win, "Global well-posedness of the derivative nonlinear Schrödinger equations on T", Funkcial. Ekvac., **53**(2010), 51–88.
[20] Q. S. Zhang, "Blow-up results for nonlinear parabolic equations on manifolds", Duke Math. J., **97**(1999), 515–539.
[21] Q. S. Zhang, "A blow-up result for a nonlinear wave equation with damping: the critical case", C. R. Acad. Sci. Paris Sér. I Math., **333**(2001), 109–114.

CENTRO DI RICERCA MATEMATICA ENNIO DE GIORGI, SCUOLA NORMALE SUPERIORE, PISA, ITALY.
E-mail address: kazumasa.fujiwara@sns.it

DEPARTMENT OF APPLIED PHYSICS, WASEDA UNIVERSITY, TOKYO, JAPAN
E-mail address: txozawa@waseda.jp