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On a rigidity result for the Camassa-Holm equation

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Abstract

The Camassa-Holm equation possesses peaked solitary waves called peakons. In this note we present a rigidity result, proven in [30], for uniformly almost localized (up to translations) H^1-global solutions of the Camassa-Holm equation with a momentum density that is a non negative finite measure.

1 Introduction

The Camassa-Holm equation (C-H),

$$u_t - u_{txx} = -3uu_x + 2u_x u_{xx} + uu_{xxx}, \quad (t,x) \in \mathbb{R}^2, \tag{1.1}$$

was introduced by Camassa and Holm [7] as a model for the propagation of unidirectional shalow water waves over a flat bottom. A rigorous derivation of the Camassa-Holm equation from the full water waves problem is obtained in [1] and [13].

(C-H) is completely integrable (see [7],[8], [10] and [12]) and enjoys also a geometrical derivation (cf. [24], [25]). It possesses among others the following invariants

$$M(v) = \int_{\mathbb{R}} (v - v_{xx})\, dx, \;\; E(v) = \int_{\mathbb{R}} v^2(x) + v_x^2(x)\, dx \text{ and } F(v) = \int_{\mathbb{R}} v^3(x) + v(x) v_x^2(x)\, dx\,. \tag{1.2}$$

It is also worth noticing that (1.4) can be rewritted as

$$y_t + u y_x + 2u_x y = 0 \tag{1.3}$$

which is a transport equation for the momentum density $y = u - u_{xx}$.

Camassa and Holm [7] exhibited peaked solitary waves solutions to (C-H) that are given by

$$u(t,x) = \varphi_c(x - ct) = c\varphi(x - ct) = ce^{-|x-ct|}, \; c \in \mathbb{R}^*.$$

They are called peakon whenever $c > 0$ and antipeakon whenever $c < 0$. Their orbital stability has been proven by Constantin and Strauss [16]. Note that the initial value problem associated with (C-H) has to be rewriten as

$$\begin{cases} u_t + uu_x + (1 - \partial_x^2)^{-1} \partial_x (u^2 + u_x^2/2) = 0 \\ u(0) = u_0, \end{cases} \tag{1.4}$$

to give a meaning to these solutions.

In a series of papers (see for instance [27], [28]) Martel and Merle developped an approach, based on a Liouville property for uniformly almost localized global solutions close to the solitary waves, to prove the asymptotic stability for a wide class of dispersive equations. The Liouville property is based on the study of a dual equation related to the linearized equation around the solitary waves.

In this note we present a Liouville result for uniformly almost localized (up to translations) global solutions to the CH equation. We emphasize that our result is global and not local around the peakon profile. The main ingredient to prove our Liouville result is the finite speed propagation of the momentum density of the solution.

Before stating our results let us introduce the function space where our initial data will take place. Following [15], we introduce the following space of functions

$$Y = \{u \in H^1(\mathbb{R}) \text{ such that } u - u_{xx} \in \mathcal{M}(\mathbb{R})\}\ . \tag{1.5}$$

We denote by Y_+ the closed subset of Y defined by $Y_+ = \{u \in Y \,/\, u - u_{xx} \in \mathcal{M}_+\}$. Let $I \subset \mathbb{R}$ be an interval. Throughout this paper, $y \in C_w(I;\mathcal{M})$ will signify that for any $\phi \in C(\mathbb{R})$, $t \mapsto \Big\langle y(t), \phi \Big\rangle$ is continuous on I and $y_n \rightharpoonup\!* \; y$ in $C_w(I;\mathcal{M})$ will signify that for any $\phi \in C(\mathbb{R})$, $\Big\langle y_n(\cdot), \phi\Big\rangle \to \Big\langle y(\cdot), \phi\Big\rangle$ in $C(I)$.

Definition 1.1. We say that a solution $u \in C(\mathbb{R}; H^1(\mathbb{R}))$ with $u - u_{xx} \in C_w(\mathbb{R};\mathcal{M}_+)$ of (1.4) is Y-almost localized if there exist $c > 0$ and a C^1-function $x(\cdot)$, with $x_t \geq c > 0$, for which for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that for all $t \in \mathbb{R}$ and all $\Phi \in C(\mathbb{R})$ with $0 \leq \Phi \leq 1$ and $\operatorname{supp}\Phi \subset [-R_\varepsilon, R_\varepsilon]^c$.

$$\int_{\mathbb{R}} (u^2(t) + u_x^2(t))\Phi(\cdot - x(t))\, dx + \Big\langle \Phi(\cdot - x(t)), u(t) - u_{xx}(t)\Big\rangle \leq \varepsilon\ . \tag{1.6}$$

Theorem 1.1. *Let $u \in C(\mathbb{R}; H^1(\mathbb{R}))$, with $u - u_{xx} \in C_w(\mathbb{R};\mathcal{M}_+)$, be a Y-almost localized solution of* (1.4) *that is not identically vanishing. Then there exists $c^* > 0$ and $x_0 \in \mathbb{R}$ such that*

$$u(t) = c^*\,\varphi(\cdot - x_0 - c^* t), \quad \forall t \in \mathbb{R}\,.$$

Remark 1.1. This theorem implies, in particular, that a Y-almost localized solution with non negative momentum density cannot be smooth for any time. More precisely, if $u \in C(\mathbb{R}; H^1)$, with $u - u_{xx} \in C_w(\mathbb{R};\mathcal{M}_+)$, is a Y-almost localized solution of the Camassa-Holm equation that belongs to $H^{\frac{3}{2}}(\mathbb{R})$ for some $t \in \mathbb{R}$ then u must be the trivial null solution.

Remark 1.2. It turns out that the above rigidity result also holds for other equations with peakons as the Degasperis-Procesi equation.

2 Sketch of the proof of Theorem 1.1

The main ingredients are the following :

1. A global well-posedness result with continuity with respect to initial data in strong H^1- topology.

2. An almost monotonicity result that ensures that an Y almost localized solution of (1.4) actually enjoys a uniform exponential decay.

3. The "finite speed" propagation of the momentum density y that will ensures that y has a compact support at the right.

4. An exact formula for the evolution of the jump of u_x at $x(t) + x_+(t)$ defined by

$$x_+(t) = \inf\{x \in \mathbb{R},\ \operatorname{supp} y(t) \subset]-\infty, x(t) + x]\}\ .$$

2.1 Global well-posedness results

We first recall some obvious estimates that will be useful in the sequel of this paper. Noticing that $p(x) = \frac{1}{2}e^{-|x|}$ satisfies $p * y = (1-\partial_x^2)^{-1}y$ for any $y \in H^{-1}(\mathbb{R})$ we easily get

$$\|u\|_{W^{1,1}} = \|p*(u-u_{xx})\|_{W^{1,1}} \lesssim \|u-u_{xx}\|_{\mathcal{M}}$$

and

$$\|u_{xx}\|_{\mathcal{M}} \leq \|u\|_{L^1} + \|u-u_{xx}\|_{\mathcal{M}}$$

which ensures that

$$Y \hookrightarrow \{u \in W^{1,1}(\mathbb{R}) \text{ with } u_x \in \mathcal{B}V(\mathbb{R})\} . \tag{2.1}$$

It is also worth noticing that since for $v \in C_0^\infty(\mathbb{R})$,

$$v(x) = \frac{1}{2}\int_{-\infty}^{x} e^{x'-x}(v-v_{xx})(x')dx' + \frac{1}{2}\int_{x}^{+\infty} e^{x-x'}(v-v_{xx})(x')dx'$$

and

$$v_x(x) = -\frac{1}{2}\int_{-\infty}^{x} e^{x'-x}(v-v_{xx})(x')dx' + \frac{1}{2}\int_{x}^{+\infty} e^{x-x'}(v-v_{xx})(x')dx' ,$$

we get $v_x^2 \leq v^2$ as soon as $v - v_{xx} \geq 0$ on $\mathbb{R}$. By the density of $C_0^\infty(\mathbb{R})$ in Y, we deduce that

$$|v_x| \leq v \text{ for any } v \in Y_+ . \tag{2.2}$$

Finally, throughout this paper, we will denote $\{\rho_n\}_{n\geq 1}$ the mollifiers defined by

$$\rho_n = \Bigl(\int_{\mathbb{R}} \rho(\xi)\, d\xi\Bigr)^{-1} n\rho(n\cdot) \text{ with } \rho(x) = \begin{cases} e^{1/(x^2-1)} & \text{for } \ |x|<1 \\ 0 & \text{for } \ |x| \geq 1 \end{cases} \tag{2.3}$$

The following global well-posedness result is mainly proven in [15].

Proposition 2.1. *(Global weak solution [15])*
Let $u_0 \in Y_+$ be given.

1. Uniqueness and global existence : (1.4) *has a unique solution* $u \in C^1(\mathbb{R}; L^2(\mathbb{R})) \cap C(\mathbb{R}; H^1(\mathbb{R}))$ *such that* $y = (1-\partial_x^2)u \in C_w(\mathbb{R}; \mathcal{M}+)$. *Moreover,* $E(u)$ $F(u)$ *and* $M(u) = \Bigl\langle y, 1\Bigr\rangle$ *are conservation laws .*

2. Continuity with respect to initial data in $H^1(\mathbb{R})$: *For any sequence* $\{u_{0,n}\}$ *bounded in* Y_+ *such that* $u_{0,n} \to u_0$ *in* $H^1(\mathbb{R})$, *the emanating sequence of solution* $\{u_n\} \subset C^1(\mathbb{R}_+; L^2(\mathbb{R})) \cap C(\mathbb{R}_+; H^1(\mathbb{R}))$ *satisfies for any* $T > 0$

$$u_n \to u \textit{ in } C([-T,T]; H^1(\mathbb{R})) \tag{2.4}$$

and

$$(1-\partial_x^2)u_n \rightharpoonup * \; y \textit{ in } C_w([-T,T], \mathcal{M}) . \tag{2.5}$$

2.2 Uniform exponential decay of Y-almost localized solution

Proposition 2.2. *Let* $u \in C(\mathbb{R}; H^1)$ *with* $y = (1-\partial_x^2)u \in C_w(\mathbb{R}; \mathcal{M}+)$ *be a* Y*-almost localized solution of* (1.4) *with* $\inf_{\mathbb{R}} \dot{x} \geq c_0 > 0$. *Then there exists* $C > 0$ *such that for all* $t \in \mathbb{R}$, *all* $R > 0$ *and all* $\Phi \in C(\mathbb{R})$ *with* $0 \leq \Phi \leq 1$ *and* $\operatorname{supp}\Phi \subset [-R,R]^c$.

$$\int_{\mathbb{R}} (u^2(t) + u_x^2(t))\Phi(\cdot - x(t))\, dx + c_0\Bigl\langle \Phi(\cdot - x(t)), y(t)\Bigr\rangle \leq C \exp(-R/6) . \tag{2.6}$$

To prove this proposition, the main tool is an almost monotonicity result for $E(u)+c_0M(u)$ at the right of an almost localized solution. Actually, the almost monotonicity is more general and says somehow that if $z(t)$ moves to the right with a positive speed strictly less that $\dot{x}(t)$ then the part of $E(u)+c_0M(u)$ at the right of $z(t)$ is almost decreasing as soon as $|z(t)-x(t)|$ stays large enough.

2.3 Compact support at the right of the momentum density

Proposition 2.3. *Let $u\in C(\mathbb{R};Y_+)$ be a Y-almost localized solution of (1.4) with $x_t\geq c_0>0$. There exists $r_0>0$ such that for all $t\in\mathbb{R}$, it holds*

$$\operatorname{supp} y(t,\cdot+x(t))\subset]-\infty,r_0]\,. \tag{2.7}$$

Proof. Clearly, it suffices to prove the result for $t=0$. Let $u\in C(\mathbb{R};H^1)$,with $u-u_{xx}\in C_w(\mathbb{R};\mathcal{M}_+)$, be a Y-almost localized solution to (1.4) and let $\phi\in C^\infty(\mathbb{R})$ with $\phi\equiv 0$ on $\mathbb{R}_-$, $\phi'\geq 0$ and $\phi\equiv 1$ on $[1,+\infty]$. We claim that there exists $r_0>0$ such that

$$\Big\langle y(0),\phi(\cdot-(x(0)+r_0))\Big\rangle=0 \tag{2.8}$$

which proves the result since $y\in\mathcal{M}_+$.
We approximate $u_0=u(0)$ by the sequence of smooth functions $u_{0,n}=\rho_n*u_0$ that belongs to $H^\infty(\mathbb{R})\cap Y_+$ so that (2.4)-(2.5) hold for any $T>0$. We denote by u_n the solution to (1.4) emanating from $u_{0,n}$ and by $y_n=u_n-u_{n,xx}$ its momentum density. Let us recall that classical LWP results ensure that $u_n\in C(\mathbb{R};H^\infty(\mathbb{R}))$ and $y_n\in C_w((\mathbb{R};L^1(\mathbb{R}))$. We fix $T>0$ and we take $n_0\in\mathbb{N}$ large enough so that for all $n\geq n_0$,

$$\|u_n-u\|_{L^\infty(]-T,T[;H^1)}<\frac{1}{10}\min(c_0,\|u(0)\|_{H^1}) \tag{2.9}$$

and

$$\|y_{0,n}-y_0\|_{\mathcal{M}}<\frac{\varepsilon_0}{2}\quad . \tag{2.10}$$

where $\varepsilon_0>0$ will be specified later. Thanks to the Y-almost localization of u , there exists $r_0>0$ such that

$$\|u(t)\|_{H^1(\mathbb{R}/]x(t)-r_0,x(t)+r_0[)}\leq\frac{1}{10}\min(c_0,\|u(0)\|_{H^1}),\forall t\in\mathbb{R}\,. \tag{2.11}$$

Note that by Sobolev injections, it also holds

$$u(t,x(t)+x)\leq\frac{1}{10}\min(c_0,\|u(0)\|_{H^1}),\forall(|x|,t)\in[r_0,+\infty[\times\mathbb{R}\,. \tag{2.12}$$

Combining these two estimates with (2.9) we infer that for $n\geq n_0$,

$$\|u_n(t)\|_{H^1(\mathbb{R}/]x(t)-r_0,x(t)+r_0[)}\leq\frac{1}{5}\min(c_0,\|u(0)\|_{H^1}),\forall t\in[-T,T] \tag{2.13}$$

and

$$u_n(t,x(t)+x)\leq\frac{1}{5}\min(c_0,\|u(0)\|_{H^1}),\forall(|x|,t)\in[r_0,+\infty[\times[-T,T]\,. \tag{2.14}$$

Now, we introduce the flow q_n associated with u_n defined by

$$\left\{\begin{array}{rcll} q_{n,t}(t,x) &=& u_n(t,q_n(t,x)) & ,\ (t,x)\in\mathbb{R}^2\\ q_n(0,x) &=& x & ,\ x\in\mathbb{R}\end{array}\right. \quad . \tag{2.15}$$

Following [9], we know that for any $t \in \mathbb{R}$,

$$y_n(0,x) = y_n(t, q_n(t,x)) q_{n,x}(t,x)^2 \tag{2.16}$$

We claim that for all $n \geq n_0$ and $t \in [-T, 0]$,

$$q_n(t, x(0)+r_0) - x(t) \geq r_0 + \frac{c_0}{2}|t| \, . \tag{2.17}$$

Indeed, fixing $n \geq n_0$, in view of (2.14) and the continuity of u_n there exists $t_0 \in [-T, 0[$ such that for all $t \in [t_0, 0]$,

$$u_n(t, q_n(t, x(0)+r_0)) \leq \frac{c_0}{4}$$

and thus according to (2.15), for all $t \in [t_0, 0]$,

$$\frac{d}{dt} q_n(t, x(0)+r_0) \leq \frac{c_0}{4}$$

which leads to

$$q_n(t, x(0)+r_0) - x(t) \geq r_0 + \frac{c_0}{2}|t|, \quad t \in [t_0, 0] \, .$$

This proves (2.17) by a continuity argument. We thus deduce from Proposition 2.2 that for all $t \in [-T, 0]$ and all $x \geq 0$,

$$u(t, q_n(t, x(0)+r_0+x) \leq C \exp\Big(-\frac{1}{6}(r_0 + c_0|t|/2)\Big) \tag{2.18}$$

Therefore, in view of (2.4) and (2.2), there exists $n_1 \geq n_0$ such that for all $t \in [-T, 0]$ and all $x \geq 0$,

$$u_n(t, q_n(t, x(0)+r_0+x) + |u_{n,x}(t, q_n(t, x(0)+r_0+x)| \leq 4C \exp\Big(-\frac{1}{6}(r_0 + c_0|t|/2)\Big) \tag{2.19}$$

The formula

$$q_{n,x}(t,x) = \exp\Big(-\int_t^0 u_{n,x}(s, q_n(s,x))\, ds\Big) \tag{2.20}$$

thus ensures that $\forall t \in [-T, 0]$, $\forall x \geq 0$ and $\forall n \geq n_0$,

$$\exp\Big(-4C \int_{-T}^0 e^{-\frac{1}{6}(r_0+c_0|s|/2)}\, ds\Big) \leq q_{n,x}(t, x(0)+r_0+x) \leq \exp\Big(4C \int_{-T}^0 e^{-\frac{1}{6}(r_0+c_0|s|/2)}\, ds\Big)$$

Setting $C_0 := e^{\frac{48Ce^{-r_0/6}}{c_0}}$ this leads to

$$\frac{1}{C_0} \leq q_{n,x}(t, x(0)+r_0+x) \leq C_0 \, , \; \forall t \in [-T, 0] \, . \tag{2.21}$$

Now, we claim that any $n \geq n_1$ it holds

$$\int_{x(0)+r_0}^{+\infty} y_n(0,x)\, dx \leq C_0 \int_{x(t)+r_0+c_0|t|/2}^{+\infty} y_n(t,z)\, dz \, , \; \forall t \leq [-T, 0] \, . \tag{2.22}$$

Letting $n \to +\infty$ using (2.5) and then letting $T \to \infty$, this ensures that

$$\Big\langle y(0), \phi(\cdot - x(t) - r_0)\Big\rangle \leq C_0 \Big\langle y(t), \phi(\cdot - x(t) - r_0 - c_0|t|/2 + 1)\Big\rangle \, , \; \forall t \leq 0$$

which proves (2.8) since the Y-uniform localization of u forces the right-hand side member to goes to 0 as $t \to -\infty$. Therefore, to complete the proof of (2.7), it remains to prove (2.22). First, it follows from (2.16) that for any $t \leq 0$ and any $r_0' > r_0$,

$$\int_{x(0)+r_0}^{x(0)+r_0'} y_n(0,x)\,dx = \int_{x(0)+r_0}^{x(0)+r_0'} y_n(t,q_n(t,x))q_n(t,x)^b\,dx$$

and (2.21) leads to

$$\int_{x(0)+r_0}^{x(0)+r_0'} y_n(0,x)\,dx \leq C_0 \int_{x(0)+r_0}^{x(0)+r_0'} y_n(t,q_n(t,x))q_{n,x}(t,x)\,dx$$

The change of variables $z = q_n(t,x)$ then yields

$$\int_{x(0)+r_0}^{x(0)+r_0'} y_n(0,x)\,dx \leq C_0 \int_{q_n(t,x(0)+r_0)}^{q_n(t,x(0)+r_0')} y_n(t,z)\,dz$$

and (2.22) then follows from (2.17) by letting r_0' tend to $+\infty$.

$\square$

2.4 An exact formula for the discontinuity of u_x at the right border of the compact support of y

We define

$$x_+(t) = \inf\{x \in \mathbb{R},\ \operatorname{supp} y(t) \subset]-\infty, x(t)+x]\}$$

According to Proposition 2.3, $t \mapsto x_+(t)$ is well defined with values in $]-\infty, r_0]$ and

$$u(t,x(t)+x_+(t)) = -u_x(t,x(t)+x_+(t)) \geq \alpha_0\,. \tag{2.23}$$

Clearly, if u would belong to $C(\mathbb{R};H^3(\mathbb{R}))$ then $t \mapsto x(t)+x_+(t)$ would be an integral line of u (this is because $y \equiv 0$ at the right of $t \mapsto x(t)+x_+(t)$ and y is transport by the flow of u). Actually, this fact remains for our class of solutions as stated in the following lemma :

Lemma 2.4. *For all $t \in \mathbb{R}$, it holds*

$$x(t)+x_+(t) = q(t,x(0)+x_+(0))\,. \tag{2.24}$$

where $q(\cdot,\cdot)$ is defined by

$$\left\{\begin{array}{rcll} q_t(t,x) &=& u(t,q(t,x)) &,\ (t,x)\in\mathbb{R}^2 \\ q(0,x) &=& x &,\ x\in\mathbb{R} \end{array}\right. . \tag{2.25}$$

In the sequel we define $q^* : \mathbb{R} \to \mathbb{R}$ by

$$q^*(t) = q(t,x(0)+x_+(0)) = x(t)+x_+(t), \quad \forall t \in \mathbb{R}\,. \tag{2.26}$$

The following key proposition gives an exact formula for the evolution of the jump of $u_x(t)$ at $q^*(t)$.

Proposition 2.5. *Let $a : \mathbb{R} \to \mathbb{R}$ be the function defined by*

$$a(t) = u_x(t,q^*(t)-) - u_x(t,q^*(t)+), \quad \forall t \in \mathbb{R}\,. \tag{2.27}$$

Then $a(\cdot)$ is a bounded non decreasing derivable function on $\mathbb{R}$ with values in $[\frac{\alpha_0}{8}, 2\sqrt{E(u)}]$ such that

$$a'(t) = \frac{1}{2}(u^2 - u_x^2)(t, q^*(t)-), \ \forall t \in \mathbb{R}. \tag{2.28}$$

where

$$\alpha_0 := \frac{e^{-2r_0}}{4\sqrt{r_0}}\sqrt{E(u)}$$

Combining (2.28) and (2.2) we obtain that $t \mapsto a(t)$ is a not decreasing function and thus enjoys a limit at $\mp\infty$. Moreover, it is not too hard to prove that $t \mapsto a'(t)$ is Lipschitz on $\mathbb{R}$ which ensures that $a'(t) \to 0$ as $t \to \mp\infty$. Therefore, the identity

$$\begin{aligned} 0 \le a'(t) = \frac{1}{2}(u^2 - u_x^2)(t, x(t) + x_+(t)-) &= \frac{a(t)}{2}(u - u_x)(t, x(t) + x_+(t)-) \\ &= \frac{a(t)}{2}\Big(2u(t, x(t) + x_+(t)) - a(t)\Big) \end{aligned} \tag{2.29}$$

ensures that

$$\lim_{t\to+\infty} u(t, x(t) + x_+(t)) = \lim_{t\to+\infty} a(t)/2 = a_+/2 \, , \tag{2.30}$$

$$\lim_{t\to-\infty} u(t, x(t) + x_+(t)) = \lim_{t\to-\infty} a(t)/2 = a_-/2 \, , \tag{2.31}$$

2.5 End of the proof ot Theorem 1.1.

We conclude by proving that the jump of $u_x(0,\cdot)$ at $x(0)+x_+(0)$ is equal to $-2u(0, x(0)+x_+(0))$. This saturates for all $v \in Y_+$, the relation between the jump of v_x and the value of v at a point $\xi \in \mathbb{R}$ and forces $u(0,\cdot)$ to be equal to $u(0, x(0)+x_+(0))\varphi(\cdot - x(0) + x_+(0))$.

We use the invariance of the (CH) equation under the transformation $(t,x) \mapsto (-t,-x)$. This invariance ensures that $v(t,x) = u(-t,-x)$ is also a solution of the (C-H) equation that belongs to $C(\mathbb{R}; H^1(\mathbb{R}))$, with $u - u_{xx} \in C_w(\mathbb{R}; \mathcal{M}_+)$ and shares the property of Y-almost localization with $x(\cdot)$ replaced by $-x(-\cdot)$ and the same fonction $\varepsilon \mapsto R_\varepsilon$ (See Definition 1.1). Therefore, by applying Propositions 2.3, 2.5 and Lemma 2.4 for v we infer that there exists a C^1-function $x_- : \mathbb{R} \mapsto]-\infty, r_0]$ and a derivable non decreasing function $\tilde{a} : \mathbb{R} \to [\alpha_0/8, 2\|u_0\|_{H^1}]$ with $\lim_{t\to\mp\infty} \tilde{a}(t) = \tilde{a}_\mp$ such that

$$\tilde{a}(t) = v_x(t, (-x(-t) + x_+(t))+) - v_x(t, (-x(-t) + x_+(t))-), \quad \forall t \in \mathbb{R}. \tag{2.32}$$

Moreover,

$$\lim_{t\to\mp\infty} v(t, -x(-t) + x_+(t)) = \lim_{t\to\mp\infty} \tilde{a}(t)/2 = \tilde{a}_\mp/2 \, .$$

Coming back to u this ensures that

$$\lim_{t\to+\infty} u(t, x(t) - x_-(-t)) = \lim_{t\to-\infty} \tilde{a}(t)/2 = \tilde{a}_-/2 \, , \tag{2.33}$$

$$\lim_{t\to-\infty} u(t, x(t) - x_-(-t)) = \lim_{t\to+\infty} \tilde{a}(t)/2 = \tilde{a}_+/2 \, , \tag{2.34}$$

At this stage let us underline that since

$$x_-(-t) = \sup\{x \in \mathbb{R}, \operatorname{supp} y(-t) \in [x(t) - x(-t), +\infty[\}$$

and $u \not\equiv 0$ we must have $x(-t) + x(t) \ge 0$ for all $t \in \mathbb{R}$. We claim that this forces

$$\tilde{a}_- = \tilde{a}_+ = a_- = a_+ \, . \tag{2.35}$$

Note first that since $\tilde{a}_- \leq \tilde{a}_+$ and $a_- \leq a_+$, it suffices to prove that $\tilde{a}_- \geq a_+$ and $\tilde{a}_+ \leq a_-$. This follows easily by a contradiction argument. Indeed, assume for instance that $\tilde{a}_- < a_+$.Then, there exists $t_0 \in \mathbb{R}$ and $\varepsilon > 0$ such that $u(t, x(t) - x_-(-t)) < u(t, x(t) + x_+(t)) - \varepsilon$ for all $t \geq t_0$. Since $x(t) - x_-(-t) = q(t - t_0, x(t_0) - x_-(-t_0))$ and $x(t) + x_+(t) = q(t - t_0, x(t_0) + x_+(t_0))$, it follows from (2.25) that

$$x_+(t) + x_-(-t)) \geq \varepsilon(t - t_0) \underset{t\to+\infty}{\longrightarrow} +\infty$$

which contradicts that $(x_+(t), x_-(t)) \in]-\infty, r_0]^2$. Exactly the same argument but with $t \to -\infty$ ensures that $\tilde{a}_+ \leq a_-$ and completes the proof of the claim (2.35).

We deduce from (2.35) that $a(t) = a_+$ for all $t \in \mathbb{R}$ and thus (2.28), (2.24) and (2.27) force

$$u(t, x(0) + x_+(0) + \frac{a_+}{2}t) = \frac{a_+}{2}, \quad \forall t \in \mathbb{R}$$

and

$$u_x\Big(t, (x(0) + x_+(0) + \frac{a_+}{2}t)-\Big) - u_x\Big(t, (x(0) + x_+(0) + \frac{a_+}{2}t)+\Big) = a_+, \quad \forall t \in \mathbb{R} .$$

In particular, in view of the definition of $a(\cdot)$ in (2.27),

$$u(0, x(0) + x_+(0)) = \frac{a_+}{2} \text{ and } y(0) = a_+\delta_{x(0)+x_+(0)} + \mu$$

with $\mu \in \mathcal{M}_+(\mathbb{R})$. But this forces $\mu = 0$ since

$$(1 - \partial_x^2)^{-1}(a_+ \, \delta_{x(0)+x_+(0)}) = \frac{a_+}{2} \exp\Big(-|\cdot - (x(0) + x_+(0))|\Big)$$

and for any $\mu \in \mathcal{M}_+(\mathbb{R})$, with $\mu \neq 0$, it holds

$$(1 - \partial_x^2)^{-1}\nu = \frac{1}{2}e^{-|x|} * \nu > 0 \text{ on } \mathbb{R} .$$

We thus conclude that $y(0) = a_+\delta_{x(0)+x_+(0)}$ which leads to

$$u(t, x) = \frac{a_+}{2} \exp\Big(-\Big|x - x(0) - x_+(0) - \frac{a_+}{2}t\Big|\Big)$$

References

[1] B. ALVAREZ-SAMANIEGO AND D. LANNES, Large time existence for 3D water-waves and asymptotics, *Invent. Math.* **171** *(2009), 165–186.*

[2] R. BEALS, D.H. SATTINGER AND J. SZMIGIELSKI, Multi-peakons and the classical moment problem, *Adv. Math.* **154** *(2000), no. 2, 229–257.*

[3] T. B. BENJAMIN, *The stability of solitary waves. Proc. Roy. Soc. London Ser. A* **328** *(*1972*),* 153 − −183.

[4] A. BRESSAN, G. CHEN, AND Q. ZHANG, Uniqueness of conservative solutions to the Camassa-Holm equation via characteristics, *Discr. Cont. Dyn. Syst.* **35** *(2015), 25?42.*

[5] A. BRESSAN AND A. CONSTANTIN, Global conservative solutions of the Camassa-Holm equation, *Arch. Rational Mech. Anal.* **187** *(2007), 215–239.*

[6] A. Bressan and A. Constantin, Global dissipative solutions of the Camassa-Holm equation, *Analysis and Applications* **5** *(2007), 1–27.*

[7] R. Camassa and D. Holm, An integrable shallow water equation with peaked solitons, *Phys. rev. Lett.* **71** *(1993), 1661–1664.*

[8] R. Camassa, D. Holm and J. Hyman, An new integrable shallow water equation, *Adv. Appl. Mech.* **31** *(1994)*

[9] A. Constantin , Existence of permanent and breaking waves for a shallow water equations: a geometric approach, *Ann. Inst. Fourier* **50** *(2000), 321-362.*

[10] A. Constantin , On the scattering problem for the Camassa-Holm equation, *Proc. Roy. Soc. London Ser. A .* **457** *(2001), 953-970.*

[11] A. Constantin and J. Escher, Global existence and blow-up for a shallow water equation, *Annali Sc. Norm. Sup. Pisa* **26** *(1998), 303–328.*

[12] A. Constantin, V. Gerdjikov and R. Ivanov, Inverse scattering transform for the Camassa-Holm equation, *Inverse problems* **22** *(2006), 2197-2207.*

[13] A. Constantin and D. Lannes, The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi Equations *Arch. Rat. Mech. Anal.,* **192** *(2009), 165–186.*

[14] A. Constantin and B. Kolev, Geodesic flow on the diffeomorphism group of the circle, *Comment. Math. Helv.* **78** *(2003), 787-804.*

[15] A. Constantin and L. Molinet, Global weak solutions for a shallow water equation, *Comm. Math. Phys.* **211** *(2000), 45–61.*

[16] A. Constantin and W. Strauss, Stability of peakons, *Commun. Pure Appl. Math.* **53** *(2000), 603-610.*

[17] J. Eckhardt, and G.Teschl On the isospectral problem of the dispersionless Camassa-Holm equation, *Adv. Math.* **235** *(2013), 469–495.*

[18] K. El Dika and Y. Martel, Stability of N solitary waves for the generalized BBM equations, *Dyn. Partial Differ. Equ.* **1** *(2004), 401-437.*

[19] K. El Dika and L. Molinet, Stability of multipeakons, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26** *(2009), no. 4, 1517-1532.*

[20] K. El Dika and L. Molinet, Stability of train of anti-peakons -peakons , *Discrete Contin. Dyn. Syst. Ser. B* **12** *(2009), no. 3, 561–577.*

[21] M. Grillakis, J. Shatah and W. Strauss, Stability theory of solitary waves in the presence of symmetry, *J. Funct. Anal.* **74** *(1987), 160-197.*

[22] D. Iftimie, Large time behavior in perfect incompressible flows, *Partial differential equations and applications,119–179, Sémin. Congr.,* **15**, *Soc. Math. France, Paris, 2007.*

[23] R.S. Johnson, Camassa-Holm, Korteweg-de Vries and related models for water waves, *J. Fluid Mech.* **455** *(2002), 63–82.*

[24] B. KOLEV, Lie groups and mechanics: an introduction, *J. Nonlinear Math. Phys.* **11** *(2004), 480–498.*

[25] B. KOLEV, Poisson brackets in hydrodynamics, *Discrete Contin. Dyn. Syst.* **19** *(2007), 555–574.*

[26] Y. MARTEL, F. MERLE AND T-P. TSAI Stability and asymptotic stability in the energy space of the sum of N solitons for subcritical gKdV equations. *Comm. Math. Phys.* **231** *(2002), 347–373.*

[27] Y. MARTEL AND F. MERLE Asymptotic stability of solitons for subcritical generalized KdV equations. *Arch. Ration. Mech. Anal.* **157** *(2001), no. 3, 219-254.*

[28] Y. MARTEL AND F. MERLE Asymptotic stability of solitons of the gKdV equations with general nonlinearity. *Math. Ann.* **341** *(2008), no. 2, 391-427.*

[29] L. MOLINET On well-posedness results for Camassa-Holm equation on the line: a survey. *J. Nonlinear Math. Phys.* **11** *(2004), 521–533.*

[30] L. MOLINET A Liouville property with application to asymptotic stability for the Camassa-Holm equation (submitted).