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OBSERVATION INEQUALITIES FOR KLEIN-GORDON EQUATIONS

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1. Introduction

In this short communication we present an estimate for Klein-Gordon equations, both linear and nonlinear, which permit controlling the energy norm of the solution point-wise in time (say at time $t = 0$) by a space-time average of the solution over an interval around that time. These estimates arose naturally in the author's work with Nicolas Burq and Genevieve Raugel, [BurRauSch1, BurRauSch1], where they play an essential role. In fact, the contents of this note are part of a larger and more systematic discussion of problem of this type, see [BurRauSch3]. The approach chosen here in order to pass from the linear to the nonlinear equation is concentration-compactness [BahGer]. An alternative method is presented in [BurRauSch3].

2. Observation inequalities

2.1. **Basic linear estimates.** We begin with the free equation. The following lemma explains what kind of estimate this entire note is concerned with.

Lemma 1. *Let u solve $\partial_{tt}u - \Delta u + u = 0$ with data in $\mathcal{H}$. Then for any $0 < b < 1$*

$$\|\vec{u}(0)\|_{\mathcal{H}} \leqslant C(b)\|\partial_t u\|_{L^2([0,b],L^2(\mathbb{R}^d))} \tag{1}$$

with $C(b) = C_0\, b^{-\frac{3}{2}}$ and C_0 absolute. One also has

$$\|\vec{u}(0)\|_{\mathcal{H}_{-1}} \leqslant C(b)\|u\|_{L^2([0,b],L^2(\mathbb{R}^d))} \tag{2}$$

where $\mathcal{H}_{-1} = L^2 \times H^{-1}(\mathbb{R}^d)$.

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Proof. We write $\vec{u}(0) = (f,g)$. Then with $y = 2b\langle\xi\rangle \geqslant 2b$,

$$\begin{aligned}\int_0^b \|\partial_t u(t)\|_2^2\,dt &= \frac{b}{2}\int_{\mathbb{R}^d}\Big\{\big(1-\frac{\sin y}{y}\big)\langle\xi\rangle^2|\hat{f}(\xi)|^2 \\ &\quad - 2\frac{1-\cos y}{y}\Re\langle\xi\rangle\hat{f}(\xi)\overline{\hat{g}(\xi)} + \big(1+\frac{\sin y}{y}\big)|\hat{g}(\xi)|^2\Big\}\,d\xi\end{aligned} \tag{3}$$

The expression on the right-hand side is a quadratic form with matrix

$$\begin{bmatrix} 1-\frac{\sin y}{y} & \frac{1-\cos y}{y} \\ \frac{1-\cos y}{y} & 1+\frac{\sin y}{y}\end{bmatrix} \tag{4}$$

has eigenvalues $\mu_\pm = 1 \pm \sqrt{\phi(y)}$, where $\phi(x) = 2\frac{1-\cos x}{x^2}$. One checks that $\phi(x) < 1$ for $x > 0$ and $1 - \sqrt{\phi(x)} \gtrsim x^2$ for $0 < x \lesssim 1$. Thus, (3) yields

$$\int_0^b \|\partial_t u(t)\|_2^2\,dt \geqslant cb^3\|(f,g)\|_{\mathcal{H}}^2$$

as desired. The second estimate (2) is established by an analogous calculation. The matrix (4) only changes by interchanging the elements on the diagonal which does not affect the eigenvalues of the associated quadratic form. □

The lemma extends trivially to $b > 1$ with an absolute constant C replacing $C(b)$ (or in fact, it decays like $b^{-\frac{1}{2}}$).

Next, we allow a potential V in the linear Klein-Gordon equation. Because of eigenvalues that might be present, we can no longer bound $u(0)$ in terms of $\partial_t u$ in that case. Agmon [Agmon75] showed that $H = -\Delta + V$ where V is real-valued and continuous[1] in $\mathbb{R}^d$ with

$$|V(x)| \leqslant C\langle x\rangle^{-\gamma}, \qquad \gamma > 1 \tag{5}$$

admits a *distorted Fourier transform* in the sense that there exists a unitary map $\mathcal{F} : L^2_{\mathrm{ac}}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ (where $L^2_{\mathrm{ac}}(\mathbb{R}^d)$ is the absolutely continuous subspace of L^2 relative to H) given by

$$\begin{aligned}\mathcal{F}f(x) = \hat{f}(\xi) &= \lim_{R\to\infty}\int_{\mathbb{R}^d}\chi_{[|x|\leqslant R]}\,f(x)\,\overline{\phi(x,\xi)}\,dx, \\ f(x) &= \lim_{R\to\infty}\int_{\mathbb{R}^d}\chi_{[|\xi|\leqslant R]}\,\hat{f}(\xi)\,\phi(x,\xi)\,d\xi\end{aligned} \tag{6}$$

Here $\phi(x,\xi)$ are the generalized eigenfunctions or plane waves associated with H. By Kato's theorem there are no embedded eigenvalues in the continuous spectrum of H, which moreover satisfies the asymptotic completeness property (no singular

[1] This can be weakened considerably but is enough for our purposes.

continuous spectrum and wave operators are isometries onto $L^2_{\mathrm{ac}}(\mathbb{R}^d)$). $\mathcal{F}$ extends to a map on all of $L^2(\mathbb{R}^d)$ with kernel given by the pure point subspace of L^2 (the closure of the span of all eigenfunctions of H).

Under the previous condition on V, H may have infinitely many negative eigenvalues. But it is known (Birman-Schwinger, Cwickel-Lieb-Rozenbluym, Newton) that for $|V(x)| \leqslant K\langle x\rangle^{-\sigma}$ in $\mathbb{R}^d$ with $\sigma > 2$ the number of negative (or zero) eigenvalues counted with multiplicity is bounded by $C(d, K, \sigma)$.

Lemma 2. *Let $|V(x)| \lesssim \langle x\rangle^{-\sigma}$ in $\mathbb{R}^d$ with $\sigma > 2$. Let u solve $\partial_{tt}u - \Delta u + Vu + u = 0$ with data in $\mathcal{H}$. Then*

$$\|\partial_t u\|_{L^\infty(I,L^2)} \leqslant C(V, |I|)\|\partial_t u\|_{L^2(I,L^2(\mathbb{R}^d))} \tag{7}$$

for any finite interval I. Moreover, one has

$$\|\vec{u}\|_{L^\infty(I,\mathcal{H})} \leqslant C(V, |I|)(\|\partial_t u\|_{L^2(I,L^2(\mathbb{R}^d))} + \inf_{t\in I}\|\Pi_0\, u(t)\|_{L^2}) \tag{8}$$

with Π_0 being the projection onto the zero eigenspace of H. Finally, one has

$$\|\vec{u}\|_{L^\infty(I,\mathcal{H}_{-1})} \leqslant C(V, |I|)\|u\|_{L^2(I,L^2(\mathbb{R}^d))} \tag{9}$$

Proof. By time translation, we may assume that $0 \in I$. The operator

$$H = -\Delta + V + 1$$

has eigenvalues $\lambda_k \leqslant 1$, $1 \leqslant k \leqslant K < \infty$, with orthonormal (relative to L^2) eigenfunctions ψ_k. If $\lambda_k < 1$, then ψ_k decays exponentially, whereas for $\lambda_k = 1$ the decay is at least r^{-2}. The solution $u(t)$ is of the form

$$u(t) = \sum_{k=1}^{K} c_k(t)\psi_k + w(t)$$

where $w(t) \perp \psi_k$ for all k and all t. The c_k are given by $\ddot{c}_k + \lambda_k c_k(t) = 0$ and

$$c_k(0) = \langle u(0), \psi_k\rangle_{L^2}, \quad \dot{c}_k(0) = \langle \partial_t u(0), \psi_k\rangle_{L^2}.$$

Thus,

$$\begin{aligned}
c_k(t) &= \cos(t\sqrt{\lambda_k})c_k(0) + \frac{\sin(t\sqrt{\lambda_k})}{\sqrt{\lambda_k}}\dot{c}_k(0) \quad, \lambda_k > 0 \\
c_k(t) &= \cosh(t\sqrt{-\lambda_k})c_k(0) + \frac{\sinh(t\sqrt{-\lambda_k})}{\sqrt{-\lambda_k}}\dot{c}_k(0) \quad, \lambda_k < 0 \\
c_k(t) &= c_k(0) + t\dot{c}_k(0) \quad, \lambda_k = 0
\end{aligned}$$

Moreover, the distorted Fourier transform for w from above yields

$$w(t, x) = \int_{\mathbb{R}^d} \hat{w}(t, \xi)\phi(x, \xi)\, d\xi$$

with the Plancherel theorem

$$\|w(t)\|_2^2 = \int_{\mathbb{R}^d} |\hat{w}(t,\xi)|^2\, d\xi$$

More generally, we claim that for any $0 \leqslant s \leqslant 2$,

$$\int_{\mathbb{R}^d} \langle\xi\rangle^{2s} |\hat{w}(t,\xi)|^2\, d\xi \simeq \|w(t)\|_{H^s}^2 \tag{10}$$

By duality, these bounds extend to all $|s| \leqslant 2$. Indeed, this follows from the fact that for $M > 0$ large enough we have $(H+M)^{-1} : L^2 \to H^2$ as an isomorphism, whence

$$\int_{\mathbb{R}^d} \langle\xi\rangle^{4} |\hat{w}(t,\xi)|^2\, d\xi \simeq \|w(t)\|_{H^2}^2$$

The general case (10) follows by complex interpolation with $s = 0$.

One has

$$\|\partial_t u(0)\|_{L^2}^2 = \sum_k \dot{c}_k^2(0) + \|\partial_t w(0)\|_{L^2}^2$$

$$\|\partial_t u\|_{L^2([0,b],L^2(\mathbb{R}^d))}^2 = \sum_k \int_0^b \dot{c}_k^2(t)\, dt + \|\partial_t w\|_{L^2([0,b],L^2(\mathbb{R}^d))}^2$$

For the piece w coming from the continuous spectrum the proof of Lemma 1 applies. It therefore suffices to analyze the contribution of the discrete spectrum. If $\lambda_k \neq 0$, then it is elementary to check that

$$c_k^2(0) + \dot{c}_k^2(0) \leqslant C(b,\lambda_k) \int_0^b \dot{c}_k^2(t)\, dt \tag{11}$$

where the constant blows up as $\lambda_k \to 0$ (due to the existence of stationary solutions). The constant also decreases as $\lambda_k \to -\infty$. However, no eigenvalue lies to the left of $-\|V_-\|_\infty$ where $V_- = \max(0,-V)$. Estimate (11) shows that (8) holds if there is no zero eigenvalue.

On the other hand, if $\lambda_k = 0$, then by inspection

$$\dot{c}_k^2(0) \leqslant b^{-1} \int_0^b \dot{c}_k^2(t)\, dt$$

which proves (7). To obtain (8), we use that for any linear function $h(t)$

$$\sup_{t\in I} |h(t)| \leqslant \inf_{t\in I} |h(t)| + \int_I |h'(t)|\, dt \tag{12}$$

Applying Cauchy-Schwarz to the right-hand side and adding over the range of Π_0 finishes the proof of (8).

Finally, (9) follows by an analogous argument. The difference here lies with the 0 eigenvalue for which one uses

$$\sup_{t\in I} |h(t)|^2 \simeq \int_I |h(t)|^2\, dt$$

for any linear function h. The constants only depend on I. □

2.2. **The nonlinear equation.** For the nonlinear Klein-Gordon equation one has the following corollary to Lemma 1. For the sake of simplicity, we present the model equation

$$\partial_{tt}u - \Delta u + u - u^3 = 0 \tag{13}$$

in $\mathbb{R}^3$. The argument below extends to all subcritical nonlinearities.

Corollary 3. *There exists an absolute constant $1 \gg \delta_0 > 0$ with the following property: consider data $(u_0, u_1) \in \mathcal{H}$ so that the linear Klein-Gordon evolution $\vec{w}_0(t) = \vec{S}_0(t)(u_0, u_1)$ satisfies $\|w_0\|_{L^3([0,1],L^6(\mathbb{R}^3))} < \delta_0$. Then the solution u to (13) exists for all times $0 \leqslant t \leqslant 1$ and*

$$\|\vec{u}\|_{L^\infty([0,1],\mathcal{H}(\mathbb{R}^d))} \leqslant C\|\partial_t u\|_{L^2([0,1],L^2(\mathbb{R}^d))} \tag{14}$$

with an absolute constant C.

Proof. With $\omega = \langle\nabla\rangle$,

$$\begin{aligned} u(t) &= \cos(t\omega)u_0 + \omega^{-1}\sin(t\omega)u_1 + \int_0^t \omega^{-1}\sin((t-s)\omega)u^3(s)\, ds \\ &= w_0(t) + N(t) \end{aligned} \tag{15}$$

Thus $\|u\|_{\mathcal{S}} \lesssim \|w_0\|_{\mathcal{S}}$, where $\mathcal{S} := L^3([0,1], L^6(\mathbb{R}^3))$ as well as

$$\|u - w_0\|_{\mathcal{S}} + \|\vec{u} - \vec{w}_0\|_{L^\infty([0,1],\mathcal{H})} \lesssim \|N\|_{L^1([0,1],L^2)} \lesssim \|w_0\|_{\mathcal{S}}^3$$

Therefore, by Lemma 1,

$$\begin{aligned} \|\vec{u}\|_{L^\infty([0,1],\mathcal{H})} &\leqslant \|\vec{w}_0\|_{L^\infty([0,1],\mathcal{H})} + C\|w_0\|_{\mathcal{S}}^3 \\ &\leqslant C\|\partial_t u\|_{L^2([0,1],L^2)} + C\|w_0\|_{\mathcal{S}}^3 \end{aligned} \tag{16}$$

If $\|(u_0,u_1)\|_{\mathcal{H}} \geqslant 2C\delta_0^3$, then (16) implies that

$$\|\vec{u}\|_{L^\infty([0,1],\mathcal{H})} \lesssim \|\partial_t u\|_{L^2([0,1],L^2)} \tag{17}$$

as desired. On the other hand, assume that $\rho := \|(u_0,u_1)\|_{\mathcal{H}} \leqslant 2C\delta_0^3 \ll 1$. By standard Strichartz estimates $\|w_0\|_{\mathcal{S}} + \|u\|_{\mathcal{S}} \lesssim \rho$ and we may repeat the previous analysis to conclude that (17) holds. Indeed, (16) yields the desired bound since

$$\rho = \|(u_0,u_1)\|_{\mathcal{H}} \gg \rho^3 \gtrsim \|w_0\|_{\mathcal{S}}^3$$

The size of δ_0 is determined so as to ensure smallness of $C\delta_0^3$. $\square$

Next, we turn to the nonlinear damped Klein-Gordon equation $\mathbb{R}^3$

$$\partial_{tt}u - \Delta u + 2\alpha(t)\partial_t u + u - u^3 = 0 \tag{18}$$

with α satisfying our assumptions. We assume that a solution $u(t)$ exists on the interval $I = [t_0, t_1]$ with $\|\vec{u}(t)\|_{\mathcal{H}} \leqslant M$ for $t \in I$. Also, for simplicity, we restrict to the radial case. We shall drop this assumption in the following subsection.

Proposition 4. *There exists $C = C(M, |I|)$ so that*

$$\|\partial_t u\|_{L^\infty(I,L^2)} \leqslant C(M, |I|)\|\partial_t u\|_{L^2(I,L^2(\mathbb{R}^d))} \tag{19}$$

for radial solutions of (18).

Proof. Without loss of generality take $|I| \leqslant 1$. By local well-posedness one has with $\mathcal{S} = L^3(I, L^6(\mathbb{R}^3))$

$$\|u\|_{\mathcal{S}} \leqslant C(M)$$

If the proposition fails, then there exists a sequence u_n of solutions to (18) on I with

$$\sup_n \|\vec{u}_n(t)\|_{\mathcal{H}} \leqslant M \qquad \forall\, t \in I$$

and thus also

$$\sup_n \|u_n\|_{\mathcal{S}} \leqslant C(M)$$

such that

$$\|\partial_t u_n\|_{L^\infty(I,L^2)} \geqslant n\|\partial_t u_n\|_{L^2(I,L^2(\mathbb{R}^d))} \tag{20}$$

The left-hand side here is $\leqslant M$ so that

$$\|\partial_t u_n\|_{L^2(I,L^2(\mathbb{R}^d))} \to 0 \;\text{ as }\; n \to \infty \tag{21}$$

We can assume that $\|\partial_t u_n\|_{L^2(I,L^2(\mathbb{R}^d))} > 0$ for all n. We claim that there exists a sequence v_n with the same properties which solves (18) without damping. To see this, let $\tilde{v}_n$ solve the undamped equation

$$\partial_{tt}v - \Delta v + v - v^3 = 0 \tag{22}$$

on I with the same initial condition as $\vec{u}_n$ at time t_0. By Lemma 2.19 in [NakSch] we have

$$\|\vec{u}_n - \vec{v}_n\|_{L^\infty(I,\mathcal{H})} \lesssim \sup_{t\in I}|\alpha(t)|\|\partial_t u_n\|_{L^1(I,L^2(\mathbb{R}^d))} \lesssim |I|^{\frac{1}{2}}\|\partial_t u_n\|_{L^2(I,L^2(\mathbb{R}^d))}$$

Taking $|I|$ to be smaller than some absolute constant without loss of generality, one notes that (20) now holds for v_n as claimed. For the remainder of this proof we will use u_n for the sequence of solutions without damping.

Pick a time $t_2 \in I$ with $\|\partial_t u_n(t_2)\|_2 \to 0$. We apply the radial concentration-compactness decomposition to the sequence $\{\vec{u}_n(t_2)\}_{n=1}^{\infty}$, see Proposition 2.17 in [NakSch]. Thus there exist free Klein-Gordon solutions V^j, γ_n^J (up to passing to subsequence) and times $t_n^j \in \mathbb{R}$ so that

$$\vec{u}_n(t_2) = \sum_{1 \leqslant j < J} \vec{V}^j(t_n^j, \cdot) + \vec{\gamma}_n^J(0) \tag{23}$$

where

$$\begin{aligned} |t_n^j - t_n^k| &\to \infty, & n \to \infty,\ j \neq k \\ \limsup_{n \to \infty} \|\gamma_n^J\|_{L_t^\infty L_x^p \cap \mathcal{S}} &\to 0, & J \to \infty,\ 2 < p < 6 \\ \vec{\gamma}_n^J(-t_n^j) &\rightharpoonup 0, & n \to \infty,\ j < J \\ \|\vec{u}_n\|_{\mathcal{H}}^2 &= \sum_{1 \leqslant j < J} \|\vec{V}^j\|_{\mathcal{H}}^2 + \|\vec{\gamma}_n^J\|_{\mathcal{H}}^2 + o(1) & n \to \infty \end{aligned} \tag{24}$$

In the second line $\mathcal{S}$ is the $L^3 L^6$ Strichartz norm. The profiles V^j are obtained by considering all possibly weak limits $S_0(-t)\vec{u}_n(t_2)$ in $\mathcal{H}$. Thus, if such a weak limit exists and does not vanish, then it must appear in (23) for large enough J. Note here that due to the final orthogonality property of the $\mathcal{H}$-norm in (24) one knows that

$$\sum_{1 \leqslant j < \infty} \|\vec{V}^j\|_{\mathcal{H}}^2 \leqslant \sup_n \|\vec{u}_n\|_{\mathcal{H}}^2 < \infty$$

This point of view is important in the construction of the profiles $\vec{W}^j$ below.

We first assume that $|t_n^j| \to \infty$ as $n \to \infty$ for all j. By (23) we see that

$$\|S_0(\cdot)\vec{u}_n(t_2)\|_{\mathcal{S}} \to 0 \quad \text{as} \quad n \to \infty$$

where S_0 is the free Klein-Gordon evolution. Corollary 3 thus gives a contradiction to (20).

Now assume that one sequence of times, say $\{t_n^1\}_{n=1}^{\infty}$ remains bounded in n. We set $t_n^1 = 0$ for each n whence $\vec{\gamma}_n^J(0) \rightharpoonup 0$ in $\mathcal{H}$. Since $\|\partial_t u_n(t_2)\|_2 \to 0$, one has $\vec{V}^1 = (\phi, 0)$. For $j > 1$ necessarily $|t_n^j| \to \infty$ as $n \to \infty$. By construction,

$$\vec{S}_0(-t_n^j)\vec{u}_n(t_2) \rightharpoonup \vec{V}^j \quad \text{as} \quad n \to \infty$$

Since $\|\partial_t u_n(t_2)\|_2 \to 0$,

$$\vec{S}_0(t_n^j)\vec{u}_n(t_2) \rightharpoonup \vec{W}^j \quad \text{as} \quad n \to \infty$$

where $\vec{V}^j = (\phi_j, \psi_j)$ and $\vec{W}^j = (\phi_j, -\psi_j)$. Hence we know that $\vec{W}^j$ appears in (23), and that the times t_n^j must come with its mirror image $-t_n^j$. Hence also $|t_n^j \pm t_n^k| \to \infty$ as $n \to \infty$ if $j \neq k$, cf. (24).

In other words, abusing notation we can write (23) in the form (with $\pi_1(w_1, w_2) = (w_1, 0)$ the projection onto the first component in $\mathcal{H}$),

$$\begin{aligned}\vec{u}_n(t_2) &= (\phi, 0) + \sum_{1<j<J} \left[\vec{V}^j(t_n^j, \cdot) + \vec{W}^j(-t_n^j, \cdot)\right] + \vec{\gamma}_n^J(0) \\ &= (\phi, 0) + \sum_{1<j<J} \pi_1 \vec{S}_0(t_n^j)(2\phi_j, 2\psi_j) + \vec{\gamma}_n^J(0)\end{aligned} \tag{25}$$

where now $\vec{\gamma}_n^J(\pm t_n^j) \to 0$ in $\mathcal{H}$ as $n \to \infty$. By inspection, $\|\partial_t \gamma_n^J(0)\|_2 \to 0$ as $n \to \infty$ for each J.

Passing to the nonlinear evolution we claim the following representation

$$\begin{aligned}u_n(t) = U_1(t) &+ \sum_{1<j<J} \left[V^j(t - t_2 + t_n^j, \cdot) + W^j(t - t_2 - t_n^j, \cdot)\right] \\ &+ \gamma_n^J(t - t_2) + \eta_n(t) \quad \text{for all } \ t \in I\end{aligned} \tag{26}$$

where U_1 is the solution of (22) with data $(\phi, 0)$ at time t_2. The error satisfies $\eta_n(t) = o(1)$ in $\mathcal{H}$ as $n \to \infty$, uniformly in $t \in I$. This decomposition follows from Lemma 2.19 in [NakSch] and the preceding properties of the linear decomposition. This lemma in particular gives that U_1 exists on I with bounds on the $\mathcal{S}$ norm.

To be specific, we let $v = u_n$ in Lemma 2.19 and $u = U_1$ on the time interval I, as well as $\vec{w}_0$ being the linear solution with data at time $t = t_2$ equal to

$$\vec{w}_0(t_2) = \sum_{1<j<J} \left[\vec{V}^j(t_n^j, \cdot) + \vec{W}^j(-t_n^j, \cdot)\right] + \gamma_n^J(0)$$

In the notation of the lemma, we have $\mathrm{eq}(u) = \mathrm{eq}(v) = 0$ and $\|v\|_S \leqslant B = C(M)$ uniformly in n. Furthermore, given any $\varepsilon > 0$ taking first J large and then n large, we can ensure that

$$\|w_0\|_S \leqslant \varepsilon$$

Then the lemma implies that

$$\|\vec{\eta}_n\|_{L^\infty(I,\mathcal{H})} = \|\vec{u} + \vec{w}_0 - \vec{v}\|_{L^\infty(I,\mathcal{H})} \leqslant C(M)\varepsilon$$

which proves our claim.

Next, we claim the orthogonality property,

$$\begin{aligned}\|\partial_t u_n\|^2_{L^2(I,L^2(\mathbb{R}^d))} = \|\partial_t U_1\|^2_{L^2(I,L^2(\mathbb{R}^d))} &+ \sum_{1<j<J} \|\partial_t U_j^n\|^2_{L^2(I,L^2(\mathbb{R}^d))} \\ &+ \|\partial_t \gamma_n^J(\cdot - t_2)\|^2_{L^2(I,L^2(\mathbb{R}^d))} + o(1)\end{aligned} \tag{27}$$

where

$$V^j(t - t_2 + t_n^j, \cdot) + W^j(t - t_2 - t_n^j, \cdot) = U_j^n(t), \qquad t \in I \tag{28}$$

This follows (i) by the dispersive properties of linear and nonlinear flows as the times t_n^j diverge from each other arbitrarily far (ii) by the weak convergence $\vec{\gamma}_n^J(\pm t_n^j) \rightharpoonup 0$ that we already used above. To be specific, taking a time derivative in (26) and computing L^2-norms the claim (27) reduces to the asymptotic vanishing

$$\begin{aligned}\langle \partial_t U_1, \partial_t U_j^n\rangle &= \langle \partial_t U_k^n, \partial_t U_j^n\rangle \\ &= \langle \partial_t U_1, \partial_t \gamma_n^J(\cdot - t_2)\rangle = \langle \partial_t U_j^n, \partial_t \gamma_n^J(\cdot - t_2)\rangle = o(1)\end{aligned} \tag{29}$$

as $n \to \infty$. The pairings are in $L^2(I, L^2(\mathbb{R}^d))$, and $J > j, k > 1$, $j \neq k$. As a first step, we reduce those pairings involving γ_n^J to pairings in $L^2(I, \mathcal{H}(\mathbb{R}^d))$ by means of the following device:

$$\begin{aligned}\langle \partial_t U_j^n, \partial_t \gamma_n^J(\cdot - t_2)\rangle &= \int_I \frac{1}{2}\langle \vec{V}^j(t - t_2 + t_n^j, \cdot) - \vec{V}^j(-t + t_2 - t_n^j, \cdot), \vec{\gamma}_n^J(t - t_2)\rangle_{\mathcal{H}}\, dt \\ &+ \int_I \frac{1}{2}\langle \vec{W}^j(t - t_2 + t_n^j, \cdot) - \vec{W}^j(-t + t_2 - t_n^j, \cdot), \vec{\gamma}_n^J(t - t_2)\rangle_{\mathcal{H}}\, dt\end{aligned}$$

Moving the free evolutions $\vec{S}_0(\pm t_n^j)$ over to the second slot, and using the aforementioned vanishing $\vec{\gamma}_n^J(\pm t_n^j) \rightharpoonup 0$ in the weak sense, as well as the uniform strong $\mathcal{H}$-continuity of the free evolution as a function of $t \in I$ now shows that the expressions above are $o(1)$ as $n \to \infty$. The same argument applies to the third term in (29). On the other hand, for the first and second terms one uses the dispersive decay of the free evolution. First, approximating the free waves U_j^n be ones with smooth compactly supported data (which only produces an error of small $\mathcal{H}$ norm), we may assume that $\|\partial_t U_j^n(t)\|_\infty \to 0$ uniformly for $t \in I$ as $n \to \infty$. Using the finite propagation speed on the compact time interval I, we see that $U_j^n(t)$ is supported in some fixed compact set K for all $t \in I$. It follows that

$$|\langle \partial_t U_1(t), \partial_t U_j^n(t)\rangle| \leqslant \int_K |\partial_t U_1(t,x)|\, dx \|\partial_t U_j^n(t)\|_\infty = o(1)$$

uniformly in $t \in I$. For the terms $\langle \partial_t U_k^n, \partial_t U_j^n\rangle$ we need to consider all possible combinations of V^j, W^j, V^k, W^k as in (28). Writing each of these free waves in terms of $\cos(t\omega)$ and $\sin(t\omega)$ as in (15) leads us to consider expressions of the form

$$\langle \varphi, \cos((t_n^j \pm t_n^k)\omega\tilde{\varphi}\rangle, \qquad \langle \varphi, \sin((t_n^j \pm t_n^k)\omega\tilde{\varphi}\rangle$$

where $\varphi, \tilde{\varphi}$ are Schwartz functions. Since $|t_n^j \pm t_n^k| \to \infty$ for $k \neq j$, we conclude that all of these expressions vanish in the limit $n \to \infty$. Thus our second claim (27) holds.

Passing to the limit $n \to \infty$ in (27) gives $\partial_t U_1 = 0$, so $U_1 = \phi$ is a stationary radial solution with $-\Delta\phi + \phi = \phi^3$. Furthermore, applying Lemma 1 to the free waves U_j^n and γ_n^J whose time derivatives are $o(1)$ in $L^2(I, L^2)$ we infer that they in fact are

$o(1)$ in the energy norm $L^\infty(I,\mathcal{H})$. Thus we can write

$$u_n(t) = \phi + \eta_n(t), \qquad \|\vec{\eta}_n\|_{L^\infty(I,\mathcal{H})} = o(1) \tag{30}$$

where η_n now collects the second, third, and fourth terms in (26). By (20),

$$\|\partial_t \eta_n\|_{L^\infty(I,L^2)} \geqslant n \|\partial_t \eta_n\|_{L^2(I,L^2(\mathbb{R}^d))} \tag{31}$$

The error $\eta_n \neq 0$ solves

$$\partial_{tt}\eta_n - \Delta \eta_n + \eta_n - 3\phi^2 \eta_n = 3\phi\eta_n^2 + \eta_n^3 \tag{32}$$

Lemma 2 now leads to a contradiction. Indeed, if the right-hand side vanishes in (32), then that lemma contradicts (31) directly. To include the nonlinearity on the right-hand side of (32) requires Strichartz bounds such as $L^3(I, L^6(\mathbb{R}^3))$ to hold for the linear flow with potential $V = -3\phi^2$. This, however, follows perturbatively from the Strichartz estimates without potential since we are working locally in time on the interval I, and ϕ is bounded and decays rapidly. Coming back to (32) we thus have

$$\|\vec{\eta}_n\|_{\mathcal{S}} = o(1) \qquad n \to \infty \tag{33}$$

where $\mathcal{S}$ is any admissible Strichartz norm.

In Corollary 3 we passed from observation inequalities for the linear flow without potential to analogous bounds for the nonlinear flow. This same argument goes through for (32) provided $H = -\Delta + 1 - 3\phi^2$ does not have any eigenfunctions with 0 eigenvalue. If, however, H does have a nontrivial kernel then the presence of Π_0 in (8) is a serious obstruction. To avoid it, we differentiate (32) in time and use (9) instead.

The functions $\psi_n = \partial_t \eta_n$ solve

$$\partial_{tt}\psi_n - \Delta \psi_n + \psi_n - 3\phi^2 \psi_n = 6\phi\eta_n\psi_n + 3\eta_n^2\psi_n =: G_n \tag{34}$$

with data in $\mathcal{H}_{-1}$. The solution is

$$\psi_n(t) = S'(t)\psi_n(0) + S(t)\partial_t\psi_n(0) + \int_0^t S(t-s)G_n(s)\, ds$$

where $S(t)$ is the fundamental solution of the linear problem with potential. Thus, by (9), as well as the equivalence of Sobolev norms (10) we infer that

$$\|\psi_n\|_{L^\infty(I,L^2)} \lesssim \|\vec{\psi}_n\|_{L^\infty(I,\mathcal{H}_{-1})} \lesssim \|\psi_n\|_{L^2(I,L^2(\mathbb{R}^3))} + \|G_n\|_{L^1(I,H^{-1}(\mathbb{R}^3))} \tag{35}$$

with constants uniform in n. By the embedding $L^{\frac{6}{5}} \hookrightarrow H^{-1}$,

$$\begin{aligned} \|\phi\eta_n\psi_n\|_{L^1(I,H^{-1})} &\lesssim \|\phi\|_6 \|\eta_n\|_{L^1(I,L^6)} \|\psi_n\|_{L^\infty(I,L^2)} \\ \|\eta_n^2\psi_n\|_{L^1(I,H^{-1})} &\lesssim \|\eta_n\|_{L^2(I,L^6)}^2 \|\psi_n\|_{L^\infty(I,L^2)} \end{aligned} \tag{36}$$

In view of (33) we may therefore absorb the G_n term in (35) into the left-hand side to conclude that

$$\|\psi_n\|_{L^\infty(I,L^2)} \lesssim \|\psi_n\|_{L^2(I,L^2(\mathbb{R}^3))}$$

with uniform constants in n. This contradicts (31) and we are done. □

We now briefly indicate how to modify the argument for other nonlinearities, say for the equations with $p < 5$

$$\partial_{tt}u - \Delta u + u - |u|^{p-1}u = 0 \tag{37}$$

in $\mathbb{R}^3$, still radial (for nonradial solutions the concentration-compactness argument is different due to spatial translations). Corollary 3 is a simple application of Strichartz estimates and carries over immediately to other powers. The concentration compactness decomposition as well as the perturbative Lemma 2.19 from [NakSch] apply to all subcritical NLKG equations, see [IbrMasNak] for details.

The previous proof therefore applies more or less unchanged in the case where there is no stationary profile. Only the final step involving $u_n(t) = \phi + \eta_n(t)$, see (30), requires some modifications. For example, for $p = 4$ one has

$$\begin{aligned} \partial_{tt}\eta_n - \Delta\eta_n + \eta_n - 4|\phi|^3\eta_n &= R_n \\ |R_n| &\lesssim \phi^2\eta_n^2 + \eta_n^4 \end{aligned} \tag{38}$$

and the time-differentiated version is

$$\begin{aligned} \partial_{tt}\psi_n - \Delta\psi_n + \psi_n - 4|\phi|^3\psi_n &= \tilde{R}_n\psi_n \\ |\tilde{R}_n| &\lesssim \phi^2|\eta_n| + |\eta_n|^3 \end{aligned} \tag{39}$$

The difference now lies in the bounds (36). Here one has

$$\begin{aligned} \|\phi^2\eta_n\psi_n\|_{L^1(I,H^{-1})} &\lesssim \|\phi\|_{12}^2\|\eta_n\|_{L^1(I,L^6)}\|\psi_n\|_{L^\infty(I,L^2)} \\ \|\eta_n^3\psi_n\|_{L^1(I,H^{-1})} &\lesssim \|\eta_n\|_{L^3(I,L^9)}^2\|\psi_n\|_{L^\infty(I,L^2)} \end{aligned} \tag{40}$$

Since $L^3(I, L^9(\mathbb{R}^3))$ is an admissible Strichartz estimate for Klein-Gordon (see (2.130) and (2.121) in [NakSch]) we can conclude as before. Note that on the level of the η_n equation (38) we place η_n^4 in $L^1(I, L^2)$ which leads to $L^4(I, L^8(\mathbb{R}^3))$ which is again an admissible norm (locally in time).

To verify that this process extends to the full subcritical range, we can check it for the endpoint $p = 5$. Then

$$\begin{aligned} \partial_{tt}\eta_n - \Delta\eta_n + \eta_n - 5\phi^4\eta_n &= R_n \\ |R_n| &\lesssim |\phi^3|\eta_n^2 + |\eta_n|^5 \end{aligned} \tag{41}$$

and the time-differentiated version is

$$\partial_{tt}\psi_n - \Delta\psi_n + \psi_n - 5\phi^4\psi_n = \tilde{R}_n\psi_n$$
$$|\tilde{R}_n| \lesssim |\phi^3||\eta_n| + |\eta_n|^4 \tag{42}$$

The difference now lies in the bounds (36). Here one has

$$\|\phi^3\eta_n\psi_n\|_{L^1(I,H^{-1})} \lesssim \|\phi\|_{18}^3\|\eta_n\|_{L^1(I,L^6)}\|\psi_n\|_{L^\infty(I,L^2)}$$
$$\|\eta_n^4\psi_n\|_{L^1(I,H^{-1})} \lesssim \|\eta_n\|_{L^4(I,L^{12})}^4\|\psi_n\|_{L^\infty(I,L^2)} \tag{43}$$

Once again, $L^4(I, L^{12}(\mathbb{R}^3)$ is an admissible Strichartz norm, whereas for (38) we place η_n^5 in $L^1(I, L^2)$ which leads to $L^5(I, L^{10}(\mathbb{R}^3))$ which is again an admissible norm (locally in time). Note, however, that the critical case $p = 5$ is not included for the full observation inequality argument, since the concentration compactness decomposition takes a different form then (one needs to include the dilation symmetry, and there is no mass term).

2.3. **The nonlinear equation, nonradial data.** In this section we indicate how to modify the preceding arguments so as to encompass nonradial data. The main difference lies with the profile decomposition in which the translation symmetry needs to be taken into account. In particular, the analysis will require a version of Lemma 2 for potentials consisting of several "bumps".

Lemma 5. *Fix some nonempty finite interval $I \subset \mathbb{R}$. Suppose $V = \sum_{\ell=1}^{L} V_\ell(\cdot - x_\ell)$ where*

$$\max_\ell |V_\ell(x)| \lesssim \langle x\rangle^{-\sigma} \qquad \forall\, x \in \mathbb{R}^d$$

with $\sigma > 2$. There exists $S > 0$ depending the V_ℓ and the interval I so that if $\min_{\ell\neq m}|x_\ell - x_m| > S$, then the following holds: let u solve

$$\partial_{tt}u - \Delta u + Vu + u = 0$$

with data in $\mathcal{H}$. Then

$$\|\vec{u}\|_{L^\infty(I,\mathcal{H}_{-1})} \leqslant C\|u\|_{L^2(I,L^2(\mathbb{R}^d))} \tag{44}$$

The constant depends on I, L, and the V_ℓ, but not on the translations x_ℓ.

Proof. Fix a bump function $\chi \geqslant 0$, $\chi(x) = 1$ for $|x| \leqslant 1$, and $\chi(x) = 0$ if $|x| \geqslant 2$. We use the partition of unity

$$1 = \chi_\infty(x) + \sum_{\ell=1}^{L}\chi((x - x_\ell)/R) \tag{45}$$

where $R > 1$ will be fixed later. The "infinite channel" χ_∞ is defined by the previous equation. Given a solution $u(t)$ as above we write accordingly

$$\begin{aligned} u(t,x) &= u(t,x)\chi_\infty(x) + \sum_{\ell=1}^{L} u(t,x)\chi((x-x_\ell)/R) \\ &= u_\infty(t,x) + \sum_{\ell=1}^{L} u_\ell(t,x) \end{aligned} \tag{46}$$

Set $\chi_\ell(x) = \chi((x-x_\ell)/R)$, which is supported on $|x - x_\ell| \leqslant 2R$. We therefore take $S \geqslant 5R$. The constituents solve the equations

$$\begin{aligned} \partial_{tt}u_\infty - \Delta u_\infty + u_\infty &= -(\Delta\chi_\infty)u - 2\nabla\chi_\infty\nabla u - V(x)\chi_\infty u \\ \partial_{tt}u_\ell - \Delta u_\ell + V_\ell(x-x_\ell)u_\ell + u_\ell &= -(\Delta\chi_\ell)u - 2\nabla\chi_\ell\nabla u - \sum_{k\neq\ell}^{L} V_k(\cdot - x_k)\chi_\ell u \end{aligned} \tag{47}$$

To prove (44) we perturb around (9) of Lemma 2. For u_∞ we use Lemma 1. To be precise, by (2) and the Duhamel formula applied to the first line of (47) yield,

$$\begin{aligned} \|\vec{u}_\infty\|_{L^\infty(I,\mathcal{H}_{-1})} &\lesssim \|u_\infty\|_{L^2(I,L^2(\mathbb{R}^d))} + (\|\Delta\chi_\infty\|_\infty + \|\nabla\chi_\infty\|_\infty \\ &\qquad + \|V(x)\chi_\infty\|_\infty)\|u\|_{L^2(I,L^2(\mathbb{R}^d))} \\ &\lesssim \|u_\infty\|_{L^2(I,L^2(\mathbb{R}^d))} + R^{-1}\|u\|_{L^2(I,L^2(\mathbb{R}^d))} \end{aligned} \tag{48}$$

where the constants depend on L. As in (35) one gains a derivative in the Duhamel integral, which allows us to bound ∇u in terms of u. In fact, one has

$$\begin{aligned} \|\nabla\chi_\infty\nabla u\|_{H^{-1}} &\lesssim \sum_{\ell=1}^{L} \|\nabla\chi_\ell\nabla u\|_{H^{-1}} \lesssim L\|\langle\xi\rangle^{-1}(R^{d-1}|\hat{\chi}(R\cdot)| * \langle\eta\rangle\hat{u}(\eta))\|_2 \\ &\lesssim LR^{-1}\|u\|_2 \end{aligned} \tag{49}$$

The final bound follows by Schur's test applied to the kernel

$$K(\xi,\eta) = \langle\xi\rangle^{-1}R^{d-1}|\hat{\chi}(R(\xi-\eta))|\langle\eta\rangle$$

Indeed, from

$$\sup_\xi \int_{\mathbb{R}^d} |K(\xi,\eta)|\,d\eta + \sup_\eta \int_{\mathbb{R}^d} |K(\xi,\eta)|\,d\xi \lesssim R^{-1}$$

we deduce that

$$\Big\| \int_{\mathbb{R}^d} K(\xi,\eta)\, f(\eta)\, d\eta \Big\|_2 \lesssim R^{-1}\|f\|_2$$

The terms involving $\Delta\chi_\infty$ and $V(x)\chi_\infty$ in (48) are bounded directly in the stronger L^2 norm and give R^{-2} in L^∞ by our assumptions. For the potential we use $S \geqslant 5R$ and that $\chi_\infty(x) = 0$ if $|x - x_\ell| \leqslant R$ for some ℓ. This implies that $|V(x)\chi_\infty(x)| \lesssim R^{-2}$.

Analogous estimates applied to the second line of (47) lead to similar bounds for u_ℓ. We use Lemma 2 with the potential $V_\ell(x)$. The translation by x_ℓ on the left-hand side of (47) can be removed by translation invariance of that lemma. Furthermore, the equivalence of norms (10) allows us to pass from Sobolev estimates relative to $-\Delta$ with a potential to those without a potential to which the preceding argument applies (this is for the continuous spectrum, the finitely many eigenfunctions simply absorb the derivate). Finally, the potential term in (47) satisfies

$$\Big\| \sum_{k\neq\ell}^{L} V_k(\cdot - x_k)\chi_\ell \Big\|_\infty \leqslant R^{-2}$$

Hence, we obtain

$$\|\vec{u}_\ell\|_{L^\infty(I,\mathcal{H}_{-1})} \lesssim \|u_\ell\|_{L^2(I,L^2(\mathbb{R}^d))} + R^{-1}\|u\|_{L^2(I,L^2(\mathbb{R}^d))} \tag{50}$$

In combination with (48) we conclude that

$$\begin{aligned}
\|\vec{u}\|_{L^\infty(I,\mathcal{H}_{-1})} &\lesssim \|\vec{u}_\infty\|_{L^\infty(I,\mathcal{H}_{-1})} + \sum_{\ell=1}^{L} \|\vec{u}_\ell\|_{L^\infty(I,\mathcal{H}_{-1})} \\
&\lesssim \|u_\infty\|_{L^2(I,L^2(\mathbb{R}^d))} + \sum_{\ell=1}^{L} \|u_\ell\|_{L^2(I,L^2(\mathbb{R}^d))} + R^{-1}\|u\|_{L^2(I,L^2(\mathbb{R}^d))} \\
&\lesssim \|u\|_{L^2(I,L^2(\mathbb{R}^d))} + R^{-1}\|u\|_{L^\infty(I,L^2(\mathbb{R}^d))}
\end{aligned}$$

The final term satisfies $R^{-1}\|u\|_{L^\infty(I,L^2(\mathbb{R}^d))} \lesssim R^{-1}\|\vec{u}\|_{L^\infty(I,\mathcal{H}_{-1}))}$ and so can be absorbed in the left-hand side if R is large enough. Setting $S = 5R$ proves the lemma. □

We can now establish the nonradial version of Proposition 4.

Proposition 6. *There exists $C = C(M, |I|)$ so that*

$$\|\partial_t u\|_{L^\infty(I,L^2)} \leqslant C(M, |I|)\|\partial_t u\|_{L^2(I,L^2(\mathbb{R}^d))} \tag{51}$$

for all energy solutions of (18).

Proof. We will only sketch the argument and indicate the modifications of the radial proof. Once again, we assume (51) fails, then remove the damping. We pick a time $t_2 \in I$ with $\|\partial_t u_n(t_2)\|_2 \to 0$. We apply the nonracial concentration-compactness decomposition to the sequence $\{\vec{u}_n(t_2)\}_{n=1}^\infty$, see Proposition 2.24 in [NakSch]. Thus there exist free Klein-Gordon solutions V^j, γ_n^J (up to passing to subsequence) and times $t_n^j \in \mathbb{R}$ and translations $x_n^j \in \mathbb{R}^3$ so that

$$\vec{u}_n(t_2) = \sum_{1\leqslant j<J} \vec{V}^j(t_n^j, \cdot + x_n^j) + \vec{\gamma}_n^J(0) \tag{52}$$

where

$$
\begin{aligned}
&|t_n^j - t_n^k| + |x_n^j - x_n^k| \to \infty, && n \to \infty,\ j \neq k \\
&\limsup_{n\to\infty} \|\gamma_n^J\|_{L_t^\infty L_x^p \cap \mathcal{S}} \to 0, && J \to \infty,\ 2 < p < 6 \\
&\vec{\gamma}_n^J(-t_n^j, -x_n^j) \rightharpoonup 0, && n \to \infty,\ j < J \\
&\|\vec{u}_n\|_{\mathcal{H}}^2 = \sum_{1\leqslant j<J} \|\vec{V}^j\|_{\mathcal{H}}^2 + \|\vec{\gamma}_n^J\|_{\mathcal{H}}^2 + o(1) && n \to \infty
\end{aligned}
$$

In the second line $\mathcal{S}$ is the L^3L^6 Strichartz norm (the same holds for any admissible Strichartz norm other than the energy).

We first assume that $|t_n^j| \to \infty$ as $n \to \infty$ for all j. By (52) we see that

$$\|S_0(\cdot)\vec{u}_n(t_2)\|_{\mathcal{S}} \to 0 \quad \text{as} \quad n \to \infty$$

where S_0 is the free Klein-Gordon evolution. Corollary 3 thus gives a contradiction to (20) as in the radial case.

Now assume that one sequence of times, say $\{t_n^1\}_{n=1}^\infty$ remains bounded in n. We then set $t_n^1 = 0$ for each n. By translation invariance, we may also set $x_n^1 = 0$ for all n. If one has $|t_n^j| \to \infty$ for all $j \neq 1$, then the argument from the radial case carries over mutatis mutandis. We sketch the details: by construction, one has for each j that

$$\vec{S}_0(-t_n^j, -x_n^j)\vec{u}_n(t_2) \rightharpoonup \vec{V}^j \quad \text{as} \quad n \to \infty$$

Since $\|\partial_t u_n(t_2)\|_2 \to 0$, also

$$\vec{S}_0(t_n^j, -x_n^j)\vec{u}_n(t_2) \rightharpoonup \vec{W}^j \quad \text{as} \quad n \to \infty$$

where $\vec{V}^j = (\phi_j, \psi_j)$ and $\vec{W}^j = (\phi_j, -\psi_j)$. We can therefore write (52) in the form (with $\pi_1(w_1, w_2) = (w_1, 0)$ the projection onto the first component in $\mathcal{H}$),

$$
\begin{aligned}
\vec{u}_n(t_2) &= (\phi, 0) + \sum_{1<j<J} \left[\vec{V}^j(t_n^j, \cdot + x_n^j) + \vec{W}^j(-t_n^j, \cdot + x_n^j)\right] + \vec{\gamma}_n^J(0) \\
&= (\phi, 0) + \sum_{1<j<J} \pi_1 \vec{S}_0(t_n^j)(2\phi_j, 2\psi_j)(\cdot + x_n^j) + \vec{\gamma}_n^J(0)
\end{aligned}
\tag{53}
$$

where now $\vec{\gamma}_n^J(\pm t_n^j, -x_n^j) \rightharpoonup 0$ in $\mathcal{H}$ as $n \to \infty$. By inspection, $\|\partial_t \gamma_n^J(0)\|_2 \to 0$ as $n \to \infty$ for each J.

Passing to the nonlinear evolution one has the following representation

$$
\begin{aligned}
u_n(t) = U_1(t) + \sum_{1<j<J} &\left[V^j(t - t_2 + t_n^j, \cdot + x_n^j) + W^j(t - t_2 - t_n^j, \cdot + x_n^j)\right] \\
&+ \gamma_n^J(t - t_2) + \eta_n(t) \quad \text{for all} \quad t \in I
\end{aligned}
\tag{54}
$$

where U_1 is the solution of (22) with data $(\phi, 0)$ at time t_2. The error satisfies $\eta_n(t) = o(1)$ in $\mathcal{H}$ as $n \to \infty$, uniformly in $t \in I$. This decomposition follows from Lemma 2.19 in [NakSch] and the preceding properties of the linear decomposition in the exact same fashion as in the radial setting. As before, we establish the orthogonality property (27), viz.

$$\begin{aligned}\|\partial_t u_n\|^2_{L^2(I,L^2(\mathbb{R}^d))} = \|\partial_t U_1\|^2_{L^2(I,L^2(\mathbb{R}^d))} + \sum_{1<j<J} \|\partial_t U_j^n\|^2_{L^2(I,L^2(\mathbb{R}^d))} \\ + \|\partial_t \gamma_n^J(\cdot - t_2)\|^2_{L^2(I,L^2(\mathbb{R}^d))} + o(1)\end{aligned}$$

where

$$V^j(t - t_2 + t_n^j, \cdot + x_n^j) + W^j(t - t_2 - t_n^j, \cdot + x_n^j) = U_j^n(t), \qquad t \in I$$

This follows (i) by the dispersive properties of linear and nonlinear flows as the times t_n^j or the spatial translations x_n^j diverge from each other arbitrarily far (ii) by the weak convergence $\vec{\gamma}_n^J(\pm t_n^j, -x_n^j) \rightharpoonup 0$ that we already used above. Other than inserting spatial translations into that radial argument, and using that a sequence of translations of fixed functions that diverge from each other infinitely far are asymptotically perpendicular, the proof of (27) goes through as before,

After this point the argument proceeds exactly as in the radial case, obtaining a contradiction to the representation (30) via the linear observation inequality with potential of Lemma 2.

Recall that we assumed that exactly one sequence of times remains bounded. The main difference to the radial case occurs if this does not hold. It is exactly for this scenario that we need Lemma 5. Thus, suppose that $t_n^j = 0$ for all $1 \leqslant j \leqslant J_0 \leqslant J$ in the nonradial profile decomposition. We shall deduce later that J_0 is uniformly bounded, but for now we take this integer as a parameter. In analogy to (53)

$$\begin{aligned}\vec{u}_n(t_2) &= \sum_{1\leqslant j\leqslant J_0} (\phi(\cdot + x_n^j), 0) + \sum_{J_0<j<J} [\vec{V}^j(t_n^j, \cdot + x_n^j) + \vec{W}^j(-t_n^j, \cdot + x_n^j)] + \vec{\gamma}_n^J(0) \\ &= \sum_{1\leqslant j\leqslant J_0} (\phi(\cdot + x_n^j), 0) + \sum_{J_0<j<J} \pi_1 \vec{S}_0(t_n^j)(2\phi_j, 2\psi_j)(\cdot + x_n^j) + \vec{\gamma}_n^J(0)\end{aligned} \tag{55}$$

For the nonlinear evolution we claim that

$$\begin{aligned}u_n(t) = \sum_{j=1}^{J_0} U_j(t, \cdot + x_n^j) + \sum_{J_0<j<J} [V^j(t - t_2 + t_n^j, \cdot + x_n^j) + W^j(t - t_2 - t_n^j, \cdot + x_n^j)] \\ + \gamma_n^J(t - t_2) + \eta_n(t) \quad \text{for all } \ t \in I\end{aligned} \tag{56}$$

where U_j are the solutions of (22) with data $(\phi_j, 0)$ at time t_2. The error satisfies $\eta_n(t) = o(1)$ in $\mathcal{H}$ as $n \to \infty$, uniformly in $t \in I$. This is obtained by means of Lemma 2.19 as before. The only difference is that $\sum_{j=1}^{J_0} U_j(t, \cdot + x_n^j)$ is close to a solution to the nonlinear equation due to the large separation between the translations. From this we deduce the crucial orthogonality property

$$\begin{aligned}\|\partial_t u_n\|^2_{L^2(I,L^2(\mathbb{R}^d))} = \sum_{1\leqslant j\leqslant J_0} \|\partial_t U_j\|^2_{L^2(I,L^2(\mathbb{R}^d))} + \sum_{J_0<j<J} \|\partial_t U_j^n\|^2_{L^2(I,L^2(\mathbb{R}^d))} \\ + \|\partial_t \gamma_n^J(\cdot - t_2)\|^2_{L^2(I,L^2(\mathbb{R}^d))} + o(1)\end{aligned}$$

with essentially the same proof as before. Passing to the limit $n \to \infty$ implies that $\partial_t U_j = 0$ for all $1 \leqslant j \leqslant J_0$ so that each $U_j = \phi_j$ is in fact a stationary solution. From the orthogonality of the free energy we conclude from here that

$$\sum_{j=1}^{J_0} \|\phi_j\|^2_{H^1} \leqslant M$$

Since the H^1 norm of any stationary solutions satisfies $\|\phi_j\|_{H^1} \geqslant \varepsilon_0 > 0$ with some absolute ε_0 that only depends on the nonlinearity, it follows that $J_0 \leqslant M\varepsilon_0^{-2}$ is uniformly bounded. In place of the decomposition (30) we have

$$u_n(t) = \sum_{j=1}^{J_0} \phi_j(\cdot + x_n^j) + \eta_n(t), \qquad \|\vec{\eta}_n\|_{L^\infty(I,\mathcal{H})} = o(1) \tag{57}$$

The error $\eta_n \neq 0$ solves,

$$\partial_{tt}\eta_n - \Delta\eta_n + \eta_n - 3\sum_{j=1}^{J_0} \phi_j^2(\cdot + x_n^j)\eta_n = 3\Phi_n\eta_n^2 + \eta_n^3 + \Psi_n\eta_n + R_n \tag{58}$$

with

$$\begin{aligned}\Phi_n &= \sum_{j=1}^{J_0} \phi_j(\cdot + x_n^j) \\ \Psi_n &= 3\sum_{j\neq k} \phi_j(\cdot + x_n^j)\phi_k(\cdot + x_n^k)\end{aligned} \tag{59}$$

as well as the error

$$R_n = \Big(\sum_{j=1}^{J_0} \phi_j(\cdot + x_n^j)\Big)^3 - \sum_{j=1}^{J_0} \phi_j(\cdot + x_n^j)^3$$

Since $\|R_n\|_2 + \|\Psi_n\|_\infty \to 0$ as $n \to \infty$, the solution of (58) satisfies $\|\eta_n\|_S \to 0$ where S is any admissible Strichartz norm, cf. (33). Indeed, note that the linear evolution of the left-hand side of (58) obeys the usual local in time Strichartz estimates since the potential term can be moved perturbatively to the right-hand side.

To obtain a contradiction to (31) we apply Lemma 5 to the time derivative $\psi_n = \partial_n \eta_n$ which solves the equation

$$\partial_{tt}\psi_n - \Delta\psi_n + \eta_n - 3\sum_{j=1}^{J_0} \phi_j^2(\cdot + x_n^j)\psi_n = 6\Phi_n\eta_n\psi_n + 3\eta_n^2\psi_n + \Psi_n\psi_n =: G_n$$

It is essential that the R_n here drops out, since it does not depend on time. We can now proceed exactly as for (34) treating the right-hand side G_n perturbatively. We thus obtain from (44) that

$$\|\psi_n\|_{L^\infty(I,L^2)} \leqslant C\|\psi_n\|_{L^2(I,L^2(\mathbb{R}^d))}$$

contradicting (31). □

Finally, the arguments extends to the entire subcritical range as in the nonradial case. The arguments following (37) do not use any radial assumption.

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