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On the non-homogeneous central Morrey type spaces in $L^1(\mathbf{R}^n)$ and the weak boundedness of some operators

Katsuo MATSUOKA and Yoshihiro MIZUTA

Dedicated to the memory of Professor Yasuji Takahashi

Abstract

Our aim in this note is to discuss the weak boundedness of the maximal and generalized Riesz potential operators in the non-homogeneous central Morrey type space $M^{1,q,\nu}(\mathbf{R}^n)$ of all measurable functions f on $\mathbf{R}^n$ satisfying

$$\|f\|_{M^{1,q,\nu}(\mathbf{R}^n)} = \left(\int_1^\infty \left(r^{-\nu}\|f\|_{L^1(B(0,r))}\right)^q \frac{dr}{r}\right)^{1/q} < \infty$$

for $-\infty < \nu < \infty$ and $0 < q \le \infty$; when $q = \infty$, we apply a necessary modification. To do this, we consider the family $WM^{p,q,\nu}(\mathbf{R}^n)$ of all functions $f \in L^p_{loc}(\mathbf{R}^n)$ such that

$$\|f\|_{WM^{p,q,\nu}(\mathbf{R}^n)} = \sup_{\lambda>0} \int_1^\infty \left(r^{-\nu}\lambda|\{x \in B(0,r) : |f(x)| > \lambda\}|^{1/p}\right)^q \frac{dr}{r} < \infty,$$

where $1 \le p \le \infty$.

1 Introduction

In the n-dimensional Euclidean space $\mathbf{R}^n$, the space $B^p(\mathbf{R}^n)$ given by Beurling [5] is a special case of Herz spaces $K_p^{\alpha,r}(\mathbf{R}^n)$ introduced by Herz [13]. As an extension of the space $B^p(\mathbf{R}^n)$, Alvarez, Guzmán-Partida and Lakey [4] introduced the non-homogeneous central Morrey space $B^{p,\nu}(\mathbf{R}^n)$. Fu, Lin and Lu [11] proved the boundedness of the Riesz potential operator I_α on $B^{p,\nu}(\mathbf{R}^n)$, where $-n/p \le \nu < -\alpha$; see also [16].

It is well known that the maximal operator M is weakly bounded in the Lebesgue space $L^1(\mathbf{R}^n)$ (see [23]). We extend it to the non-homogeneous central Morrey type space $M^{1,q,\nu}(\mathbf{R}^n)$ consisting of all measurable functions f on $\mathbf{R}^n$ satisfying

$$\|f\|_{M^{1,q,\nu}(\mathbf{R}^n)} = \left(\int_1^\infty \left(r^{-\nu}\|f\|_{L^1(B(0,r))}\right)^q \frac{dr}{r}\right)^{1/q} < \infty$$

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when $0 < q < \infty$ and

$$\|f\|_{M^{1,\infty,\nu}(\mathbf{R}^n)} = \sup_{r>1} r^{-\nu}\|f\|_{L^1(B(0,r))} < \infty$$

when $q = \infty$, where $B(x, r)$ denotes the open ball centered at x of radius r.

There are several Morrey type spaces related to our non-homogeneous central Morrey type spaces; e.g., Morrey spaces by Adams-Xiao [1], local Morrey type spaces and complementary local Morrey type spaces by Burenkov and al. [3], [6, 7], [8], [9], grand Morrey spaces by Kokilashvili-Meskhi-Rafeiro [12] (see also [10], [17]), and Herz-Morrey spaces by the second author and Ohno [20, 21].

Our aim in this note is to establish the weak boundedness of the maximal and generalized Riesz potential operators in $M^{1,q,\nu}(\mathbf{R}^n)$.

2 Non-homogeneous central Morrey type spaces

DEFINITION 2.1 (**Non-homogeneous central Morrey type spaces**). *For $1 \le p \le \infty$, $0 < q \le \infty$ and $-\infty < \nu < \infty$, we define a non-homogeneous central Morrey type spaces $M^{p,q,\nu}(\mathbf{R}^n)$ of all measurable functions f on $\mathbf{R}^n$ such that*

$$\|f\|_{M^{p,q,\nu}(\mathbf{R}^n)} = \left(\int_1^\infty \left(r^{-\nu}\|f\|_{L^p(B(0,r))}\right)^q \frac{dr}{r}\right)^{1/q} < \infty$$

when $q < \infty$ and

$$\|f\|_{M^{p,\infty,\nu}(\mathbf{R}^n)} = \sup_{r>1} \; r^{-\nu}\|f\|_{L^p(B(0,r))} < \infty$$

when $q = \infty$.

Note that for $1 \le p \le \infty$,

(1) if $\nu = 0$, then $M^{p,\infty,\nu}(\mathbf{R}^n) = L^p(\mathbf{R}^n)$;

(2) if $\nu < 0$, then $M^{p,\infty,\nu}(\mathbf{R}^n) = \{0\}$;

(3) if $\nu > 0$, then $M^{p,\infty,\nu}(\mathbf{R}^n) \supset L^p(\mathbf{R}^n)$.

For fundamental properties of our Morrey type spaces, we have the following.

LEMMA 2.2. *Let $1 \le p \le \infty$ and $-\infty < \nu < \infty$. For $0 < q_1 < q_2 < \infty$,*

$$M^{p,q_1,\nu}(\mathbf{R}^n) \subset M^{p,q_2,\nu}(\mathbf{R}^n) \subset M^{p,\infty,\nu}(\mathbf{R}^n).$$

LEMMA 2.3. *If $1 \le p \le \infty$, $-\infty < \nu < \infty$ and $0 < q < \infty$, then*

$$\|f\|_{M^{p,q,\nu}(\mathbf{R}^n)} \sim \left(\sum_{j=1}^\infty \left(2^{-\nu j}\|f\|_{L^p(B(0,2^j))}\right)^q\right)^{1/q},$$

where the symbol $g \sim h$ means that $C^{-1}h \le g \le Ch$ for some constant $C > 0$.

3 Maximal functions

For a locally integrable function f on $\mathbf{R}^n$, the maximal function of f is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy,$$

where $|B(x,r)|$ denotes the Lebesgue measure of the open ball $B(x,r)$ centered at $x \in \mathbf{R}^n$ of radius r. It is well known that the maximal operator $M : f \longrightarrow Mf$ is weakly bounded in $L^1(\mathbf{R}^n)$, that is,

$$|\{x \in \mathbf{R}^n : Mf(x) > \lambda\}| \leq C\lambda^{-1} \int_{\mathbf{R}^n} |f(y)| dy$$

for all $\lambda > 0$ and $f \in L^1(\mathbf{R}^n)$.

Definition 3.1 (**Weak central Morrey type spaces**). *For $1 \leq p \leq \infty$, $0 < q \leq \infty$ and $-\infty < \nu < \infty$, we denote by $WM^{p,q,\nu}(\mathbf{R}^n)$ the family of all functions $f \in L^p_{loc}(\mathbf{R}^n)$ such that*

$$\|f\|_{WM^{p,q,\nu}(\mathbf{R}^n)} = \sup_{\lambda>0} \left(\int_1^\infty \left(r^{-\nu}\lambda|\{x \in B(0,r) : |f(x)| > \lambda\}|^{1/p}\right)^q \frac{dr}{r} \right)^{1/q} < \infty$$

when $q < \infty$ and

$$\|f\|_{WM^{1,\infty,\nu}(\mathbf{R}^n)} = \sup_{\lambda>0,\, r>1} r^{-\nu}\lambda|\{x \in B(0,r) : |f(x)| > \lambda\}|^{1/p} < \infty$$

when $q = \infty$.

In view of A. Almeida and D. Drihem [2], we know that the maximal operator M is bounded in $M^{p,q,\nu}(\mathbf{R}^n)$, when $1 < p < \infty$. The case $p = 1$ is treated in the following.

Theorem 3.2. *Let $0 \leq \nu < n$ and $0 < q \leq \infty$, or let $\nu = n$ and $q = \infty$. Then the maximal operator M is bounded from $M^{1,q,\nu}(\mathbf{R}^n)$ to $WM^{1,q,\nu}(\mathbf{R}^n)$, that is, there exists a constant $C > 0$ such that*

$$\|Mf\|_{WM^{1,q,\nu}(\mathbf{R}^n)} \leq C\|f\|_{M^{1,q,\nu}(\mathbf{R}^n)}$$

for $f \in M^{1,q,\nu}(\mathbf{R}^n)$.

Proof. We show only the case when $1 < q < \infty$, because the remaining case is easily obtained.

Let f be a measurable function on $\mathbf{R}^n$ such that $\|f\|_{M^{1,q,\nu}(\mathbf{R}^n)} \leq 1$. For $r > 1$, we write

$$f = f\chi_{B(0,2r)} + f\chi_{\mathbf{R}^n \setminus B(0,2r)} = f_1 + f_2,$$

where χ_E denotes the characteristic function of a measurable set $E \subset \mathbf{R}^n$. Note here that

$$\begin{aligned} Mf_2(x) &\leq \sup_{t \geq r} \frac{1}{|B(0,t)|} \int_{B(0,2t)\setminus B(0,2r)} |f(y)| dy \\ &\leq C \int_{\mathbf{R}^n \setminus B(0,2r)} |f(y)||y|^{-n} dy \end{aligned}$$

for $x \in B(0,R)$. Let $\lambda > 0$. Since

$$\begin{aligned}&\{x \in B(0,r) : Mf(x) > \lambda\}\\ &\qquad \subset \{x \in B(0,r) : Mf_1(x) > \lambda/2\} \cup \{x \in B(0,r) : Mf_2(x) > \lambda/2\},\end{aligned}$$

we have

$$\begin{aligned}&|\{x \in B(0,r) : Mf(x) > \lambda\}|\\ &\quad \le |\{x \in B(0,r) : Mf_1(x) > \lambda/2\}| + |\{x \in B(0,r) : Mf_2(x) > \lambda/2\}|\\ &\quad \le C\lambda^{-1} \int_{B(0,2r)} |f(y)|dy + C|B(0,r)|\lambda^{-1} \int_{\mathbf{R}^n \setminus B(0,2r)} |f(y)||y|^{-n}dy,\end{aligned}$$

so that

$$\begin{aligned}&r^{-\nu}\lambda|\{x \in B(0,r) : Mf(x) > \lambda\}|\\ &\quad \le Cr^{-\nu} \int_{B(0,2r)} |f(y)|dy + Cr^{n-\nu} \int_{\mathbf{R}^n \setminus B(0,2r)} |f(y)||y|^{-n}dy.\end{aligned}$$

Now it suffices to treat the Hardy type integral in the following:

$$\int_1^\infty \left(r^{n-\nu} \int_{\mathbf{R}^n \setminus B(0,2r)} |f(y)||y|^{-n}dy \right)^q \frac{dr}{r} < C.$$

In fact, for $\nu < \varepsilon < n$, we note by Hölder's inequality and Fubini's theorem

$$\begin{aligned}&\int_1^\infty \left(r^{n-\nu} \int_{\mathbf{R}^n \setminus B(0,2r)} |f(y)||y|^{-n}dy \right)^q \frac{dr}{r}\\ &\quad \le C \sum_j \left(2^{j(n-\nu)} \sum_{k \ge j} 2^{-kn} \int_{B(0,2^{k+1}) \setminus B(0,2^k)} |f(y)|dy \right)^q\\ &\quad \le C \sum_j 2^{j(n-\nu)q} \left(\left(\sum_{k \ge j} 2^{-k(n-\varepsilon)q'} \right)^{q/q'} \sum_{k \ge j} \left(2^{-k\varepsilon} \int_{B(0,2^{k+1}) \setminus B(0,2^k)} |f(y)|dy \right)^q \right)\\ &\quad \le C \sum_j 2^{-j(\nu-\varepsilon)q} \left(\sum_{k \ge j} \left(2^{-k\varepsilon} \int_{B(0,2^{k+1}) \setminus B(0,2^k)} |f(y)|dy \right)^q \right)\\ &\quad \le C \sum_k \left(\left(2^{-k\varepsilon} \int_{B(0,2^{k+1}) \setminus B(0,2^k)} |f(y)|dy \right)^q \sum_{j \le k} 2^{-j(\nu-\varepsilon)q} \right)\\ &\quad \le C \sum_k \left(2^{-k\nu} \int_{B(0,2^{k+1}) \setminus B(0,2^k)} |f(y)|dy \right)^q\\ &\quad \le C \int_1^\infty \left(r^{-\nu} \int_{B(0,r)} |f(y)|dy \right)^q \frac{dr}{r}\\ &\quad \le C,\end{aligned}$$

which proves the result. □

Meskhi [17] obtained the boundedness of the maximal operator in grand Morrey spaces. Here we define grand central Morrey spaces in $L^1(\mathbf{R}^n)$ in the following.

DEFINITION 3.3 (**Grand Lebesgue condition**). *For $1 \le p \le \infty$, $0 < q \le \infty$ and $-\infty < \nu < \infty$, we denote by $M^{p),q,\nu}(\mathbf{R}^n)$ the family of all measurable functions f on $\mathbf{R}^n$ such that*

$$\|f\|_{M^{p),q,\nu}(\mathbf{R}^n)} = \sup_{0<\varepsilon<1} \left(\int_1^\infty \left(r^{-\nu}\varepsilon \int_{B(0,r)} |f(y)|^{p-\varepsilon} dy \right)^q \frac{dr}{r} \right)^{1/q} < \infty$$

when $q < \infty$ and

$$\|f\|_{M^{p),\infty,\nu}(\mathbf{R}^n)} = \sup_{0<\varepsilon<1,\, r>1} r^{-\nu}\varepsilon \int_{B(0,r)} |f(y)|^{p-\varepsilon} dy < \infty$$

when $q = \infty$.

It will be expected that the maximal operator M is bounded from $M^{1,q,\nu}(\mathbf{R}^n)$ to $M^{1),q,\nu}(\mathbf{R}^n)$, but the only following weaker result can be proved.

THEOREM 3.4. *Let $\nu > 0$ and $0 < q < \infty$. Then there exist constants $A > 0$ and $C > 0$ such that*

$$\sup_{0<\varepsilon<1} \left(\int_1^\infty \left(r^{-\nu}\varepsilon \int_{\{x \in B(0,r): Mf(x) > 2A\}} \{Mf(x)\}^{1-\varepsilon} dx \right)^q \frac{dr}{r} \right)^{1/q} < C$$

for all f with $\|f\|_{M^{1,q,\nu}(\mathbf{R}^n)} \le 1$.

4 Generalized Riesz potentials

For $0 < \alpha < n$ and a positive integer k, we define the generalized Riesz potential $I_{\alpha,k}f$ of order α of a locally integrable function f on $\mathbf{R}^n$ by

$$\begin{aligned} I_{\alpha,k}f(x) &= \int_{B(0,1)} I_\alpha(x-y) f(y) dy \\ &\quad + \int_{\mathbf{R}^n \setminus B(0,1)} \left\{ I_\alpha(x-y) - \sum_{\{\lambda:|\lambda| \le k-1\}} \frac{x^\lambda}{\lambda!} (D^\lambda I_\alpha)(-y) \right\} f(y)\, dy, \end{aligned}$$

where $I_\alpha(x) = |x|^{\alpha-n}$ (cf. [14, 15]); recall that $I_{\alpha,0}f = I_\alpha f$. Here, $I_\alpha f$ is the Riesz potential of order α of a locally integrable function f on $\mathbf{R}^n$ defined by

$$I_\alpha f(x) = \int_{\mathbf{R}^n} I_\alpha(x-y) f(y) dy.$$

LEMMA 4.1 (cf. [18, 19]). (1) *If $2|x| < |y|$, then*

$$\left| I_\alpha(x-y) - \sum_{\{\lambda:|\lambda| \le k-1\}} \frac{x^\lambda}{\lambda!} (D^\lambda I_\alpha)(-y) \right| \le C|x|^k |y|^{\alpha-n-k}.$$

(2) *If $|x|/2 \le |y| \le 2|x|$, then*

$$\left| I_\alpha(x-y) - \sum_{\{\lambda:|\lambda| \le k-1\}} \frac{x^\lambda}{\lambda!} (D^\lambda I_\alpha)(-y) \right| \le C|x-y|^{\alpha-n}.$$

(3) *If* $1 \le |y| \le |x|/2$, *then*

$$\left| I_\alpha(x-y) - \sum_{\{\lambda:|\lambda|\le k-1\}} \frac{x^\lambda}{\lambda!}(D^\lambda I_\alpha)(-y) \right| \le C|x|^{k-1}|y|^{\alpha-n-(k-1)}.$$

Note that $I_{\alpha,k}f$ is finite a.e. on $\mathbf{R}^n$ if

$$\int_{\mathbf{R}^n} (1+|y|)^{\alpha-n-k}|f(y)|dy < \infty.$$

Let p^* denote the Sobolev exponent of p, i.e.,

$$1/p^* = 1/p - \alpha/n.$$

THEOREM 4.2. *Let* $\nu \ge 0$, $0 < q \le \infty$ *and* k *be a positive integer such that* $k-1 < \alpha-(n-\nu) < k$. *Then the generalized Riesz potential operator* $I_{\alpha,k} : f \longrightarrow I_{\alpha,k}f$ *is bounded from* $M^{1,q,\nu}(\mathbf{R}^n)$ *to* $WM^{1^*,q,\nu}(\mathbf{R}^n)$, *that is, there exists a constant* $C > 0$ *such that*

$$\|I_{\alpha,k}f\|_{WM^{1^*,q,\nu}(\mathbf{R}^n)} \le C\|f\|_{M^{1,q,\nu}(\mathbf{R}^n)}$$

for $f \in M^{1,q,\nu}(\mathbf{R}^n)$.

Proof. We treat the case $1 < q < \infty$ only.

Let f be a nonnegative measurable function on $\mathbf{R}^n$ such that $\|f\|_{M^{1,q,\nu}(\mathbf{R}^n)} \le 1$. For $r \ge 1$, we write

$$f = f\chi_{B(0,2r)} + f\chi_{\mathbf{R}^n\setminus B(0,2r)} = f_1 + f_2.$$

If $x \in B(0,r)$, then we have by Lemma 4.1

$$\begin{aligned} |I_{\alpha,k}f_2(x)| &\le C|x|^k \int_{\mathbf{R}^n\setminus B(0,2r)} |y|^{\alpha-n-k} f(y)dy \\ &\le Cr^k \int_{\mathbf{R}^n\setminus B(0,2r)} |y|^{\alpha-n-k} f(y)dy \equiv A_1. \end{aligned}$$

On the other hand, for $x \in B(0,r)$,

$$\begin{aligned} |I_{\alpha,k}f_1(x)| &\le C|x|^{k-1} \int_{B(0,|x|)} |y|^{\alpha-n-(k-1)} f(y)dy + C\int_{B(0,2r)} |x-y|^{\alpha-n} f(y)dy \\ &\le Cr^{k-1} \int_{B(0,r)} |y|^{\alpha-n-(k-1)} f(y)dy + C\int_{B(0,2r)} |x-y|^{\alpha-n} f(y)dy \\ &\equiv A_2 + CI_\alpha f_1(x), \end{aligned}$$

so that

$$|I_{\alpha,k}f(x)| \le (A_1 + A_2) + CI_\alpha f_1(x).$$

Now, we set $E = \{x \in B(0,r) : |I_{\alpha,k}f(x)| > \lambda\}$. If $\lambda > 2(A_1 + A_2)$, then

$$\begin{aligned} |E| &\le |\{x \in E : CI_\alpha f_1(x) > \lambda/2\}| \\ &\le C\lambda^{-1} \int_E I_\alpha f_1(x)dx \\ &\le C\lambda^{-1} \int_{\mathbf{R}^n} \left(\int_E I_\alpha(x-y)dx \right) f_1(y)dy \\ &\le C\lambda^{-1}|E|^{\alpha/n} \int_{\mathbf{R}^n} f_1(y)dy, \end{aligned}$$

so that

$$\lambda|E|^{1/1^*} \leq C \int_{B(0,2r)} f(y)dy,$$

which gives

$$\int_1^\infty (r^{-\nu}\lambda|\{x \in B(0,r) : |I_{\alpha,k}f(x)| > \lambda\}|^{1/1^*})^q \frac{dr}{r}$$
$$\leq C \int_1^\infty \left(r^{-\nu} \int_{B(0,2r)} f(y)dy \right)^q \frac{dr}{r} \leq C.$$

On the other hand, if $\lambda \leq 2(A_1 + A_2)$, then

$$r^{-\nu}\lambda|\{x \in B(0,r) : |I_{\alpha,k}f(x)| > \lambda\}|^{1/1^*} \leq r^{-\nu}2(A_1 + A_2)|B(0,r)|^{1/1^*}.$$

Hence, we treat

$$I_1 = \int_1^\infty \left(r^{-\nu+n/1^*} r^k \int_{\mathbf{R}^n \setminus B(0,2r)} |y|^{\alpha-n-k} f(y)dy \right)^q \frac{dr}{r}$$

and

$$I_2 = \int_1^\infty \left(r^{-\nu+n/1^*} r^{k-1} \int_{B(0,r)} |y|^{\alpha-n-(k-1)} f(y)dy \right)^q \frac{dr}{r}.$$

For I_1 we have

$$\begin{aligned}
I_1 &\leq C \sum_i \left(2^{i(-\nu+n-\alpha+k)} \sum_{j \geq i} 2^{j(\alpha-n-k)} \int_{B(0,2^{j+1}) \setminus B(0,2^j)} f(y)dy \right)^q \\
&\leq C \sum_i 2^{i(-\nu+n-\alpha+k)q} \left(\sum_{j \geq i} 2^{-j\varepsilon q'} \right)^{q/q'} \\
&\quad \times \left(\sum_{j \geq i} \left(2^{j(\alpha-n-k+\varepsilon)} \int_{B(0,2^{j+1}) \setminus B(0,2^j)} f(y)dy \right)^q \right) \\
&\leq C \sum_i 2^{i(-\nu+n-\alpha+k-\varepsilon)q} \left(\sum_{j \geq i} \left(2^{j(\alpha-n-k+\varepsilon)} \int_{B(0,2^{j+1}) \setminus B(0,2^j)} f(y)dy \right)^q \right) \\
&\leq C \sum_j \left(\left(2^{j(\alpha-n-k+\varepsilon)} \int_{B(0,2^{j+1}) \setminus B(0,2^j)} f(y)dy \right)^q \left(\sum_{i \leq j} 2^{i(-\nu+n-\alpha+k-\varepsilon)q} \right) \right) \\
&\leq C \sum_j \left(2^{-j\nu} \int_{B(0,2^{j+1}) \setminus B(0,2^j)} f(y)dy \right)^q \leq C
\end{aligned}$$

when $0 < \varepsilon < -\nu + n - \alpha + k$. Similarly, for I_2 we have

$$I_2 \leq C \sum_j \left(2^{-j\nu} \int_{B(0,2^{j+1}) \setminus B(0,2^j)} f(y)dy \right)^q \leq C$$

when $0 < \varepsilon < \nu - n + \alpha - k + 1$. Thus

$$\int_1^\infty (r^{-\nu}\lambda|\{x \in B(0,r) : |I_{\alpha,k}f(x)| > \lambda\}|^{1/1^*})^q \frac{dr}{r} \leq C,$$

which completes the proof. □

Remark 4.3. Suppose $p > 1$ and $1/p - \alpha/n > 0$ and $k - 1 < \alpha - (n - \nu)/p < k$. In view of [22, Theorem 4.5], one can find a constant $C > 0$ such that

$$\|I_{\alpha,k} f\|_{M^{p^*,\infty,\nu}(\mathbf{R}^n)} \leq C \|f\|_{M^{p,\infty,\nu}(\mathbf{R}^n)}$$

for all $f \in M^{p,\infty,\nu}(\mathbf{R}^n)$. The case $0 < q < \infty$ can be treated in a way similar to the above proof.

Remark 4.4. Let $\nu \geq 0$ and $0 < q < \infty$. Suppose $\alpha - (n - \nu)$ is a nonnegative integer $k - 1$. Then there exists a constant $C > 0$ such that

$$\sup_{\lambda>0,\, r>1} \left(\int_1^\infty \left(r^{-\nu} (\log(1+r))^{-1} \lambda |\{x \in B(0,r) : |I_{\alpha,k} f(x)| > \lambda\}|^{1/1^*} \right)^q \frac{dr}{r} \right)^{1/q} < C$$

for all f with $\|f\|_{M^{1,q,\nu}(\mathbf{R}^n)} \leq 1$.

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College of Economics
Nihon University
1-3-2 Misaki-cho Kanda Chiyoda-ku Tokyo 101-8360, Japan
E-mail : katsu.m@nihon-u.ac.jp
and
4-13-11 Hachi-Hom-Matsu-Minami
Higashi-Hiroshima 739-0144, Japan
E-mail : yomizuta@hiroshima-u.ac.jp