数理解析研究所講究録 第2095巻 2018年 106-113

CONVERGENCE THEOREMS OF ITERATIVE SEQUENCES FOR NONLINEAR OPERATORS

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ABSTRACT. In this paper, we study an implicit iterative procedure for extended generalized hybrid mappings in a Banach space and study weak convergence theorems for such mappings in a Banach space satisfying Opial's condition. We also give some weak convergence theorems for nonlinear mappings.

1. INTRODUCTION

Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to H$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. For a mapping $T : C \to H$, we denote by $F(T)$ the set of *fixed points* of T. An important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping. A mapping is said to be firmly nonexpansive mapping

$$\|Fx - Fy\| \leq \langle x - y, Fx - Fy\rangle$$

for all $x, y \in C$ (see, for instance, Browder [7] and Goebel and Kirk [10]). It is known that a firmly nonexpansive mapping F can be deduced from an equilibrium problem in a Hilbert space (see, for instance, [6, 8]). Kohsaka and Takahashi [17], and Takahashi [24] introduced the following nonlinear mappings which are deduced from a firmly nonexpansive mapping in a Hilbert space. A mapping $T : C \to H$ is called nonspreading [17] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. A mapping $T : C \to H$ is called hybrid [24] if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

2010 *Mathematics Subject Classification.* Primary 47H09, 47H10.

Key words and phrases. Fixed point, iteration, nonexpansive mapping, nonexpansive semigroup, hybrid mapping, generalized hybrid mapping, strong convergence.

for all $x, y \in C$. They proved fixed point theorems for such mappings (see also [18, 14, 26]). Aoyama, Iemoto, Kohsaka and Takahashi [1] introduced the class of λ-hybrid mappings in a Hilbert space. This class contains the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings in a Hilbert space. They proved fixed point theorems and mean convergence theorems for such mappings. Following [1], we say that a mapping $T : C \to C$ is λ-hybrid if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2(1 - \lambda)\langle x - Tx, y - Ty\rangle$$

for all $x, y \in C$. In general, nonspreading and hybrid mappings are not continuous mappings. Kocourek, Takahashi and Yao [15] introduced a more broad class of nonlinear mappings than the class of λ-hybrid mappings in Hilbert spaces. They called such a class the class of *generalized hybrid* mapping and proved a mean convergence theorem for generalized hybrid mappings which generalizes Baillon's nonlinear ergodic theorem [5]. Hsu, Takahashii and Yao [12] extended this class in a Hilbert space to that of a Banach space. Further, they proved fixed point theorems for such mappings in a Banach space (see also [16]). A mapping $T : C \to E$ is called *generalized hybrid* [15, 12] if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. Hojo and Takahashi [11] introduce a more broad class of nonlinear mappings in a Banach space which covers generalized hybrid mappings. They proved fixed point and weak convergence theorem of Mann's type for such mappings in a Banach space satisfying Opial's condition.

On the other hand, Xu and Ori [28] studied the following implicit iterative procedure for finite nonexpansive mappings $T_1, T_2, \cdots, T_r$ in a Hilbert space: $x_0 = x \in C$,

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \ \forall n \geq 1, \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $T_n = T_{n+r}$. And they proved a weak convergence of the iterates defined by (1.1) in a Hilbert space (see also [21]). In this paper, motivated by [11, 28], we study an implicit iterative procedure for extended generalized hybrid mappings in a Banach space and study weak convergence theorems for such mappings in a Banach space satisfying Opial's condition (see also [27]). We also give some weak convergence theorems for nonlinear mappings.

2. Preliminaries and notations

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{Z}^+$ the set of all positive integers and the set of all nonnegative integers, respectively. We also denote by $\mathbb{R}$ the set of all real numbers. Let E be a real Banach space with norm $\|\cdot\|$. We denote by B_r the set $\{x \in E : \|x\| \leq r\}$. Let E^* be the dual space of a Banach space E. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$. Let C be a closed subset of a Banach space and let T be a mapping of C into itself. We denote by $F(T)$ the set $\{x \in C : x = Tx\}$.

The duality mapping J from E into 2^{E^*} is defined by

$$J(x) = \{y^* \in E^* : \langle x, y^* \rangle = \|x\|^2 = \|y^*\|^2\}, x \in E.$$

From the Hahn-Banach theorem, we see that $J(x) \neq \emptyset$ for all $x \in E$. We say that a Banach space E satisfies *Opial's condition* [20] if for each sequence $\{x_n\}$ in E which converges weakly to x,

$$\varliminf_{n\to\infty} \|x_n - x\| < \varliminf_{n\to\infty} \|x_n - y\| \tag{2.1}$$

for each $y \in E$ with $y \neq x$. If E is reflexive Banach space with weakly sequentially continuous duality mapping, then E satisfies Opial's condition. Each Hilbert space and the sequence spaces ℓ^p with $1 < p < \infty$ satisfy Opial's condition (see [20]). Though an L^p-space with $p \neq 2$ does not usually satisfy Opial's condition, each separable Banach space can be equivalently renormed so that it satisfies Opial's condition (see [?, 20]).

Banach space E is said to be smooth if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each x and y in S_1, where $S_1 = \{u \in E : \|u\| = 1\}$. The norm of E is said to be uniformly Gâteaux differentiable if for each y in S_1, the limit is attained uniformly for x in S_1. We know that if E is smooth, then the duality mapping is single-valued and norm to weak star continuous and that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping is single-valued and norm to weak star, uniformly continuous on each bounded subset of E.

Every weakly compact convex subset of a Banach space satisfying Opial's condition has normal structure (see [19]). We note that closed convex subset C of a Banach space E is said to have the fixed point property for nonexpansive mappings if for every bounded closed convex subset K of C, every nonexpansive mapping on K, has a fixed point.

Following [1], we say that a mapping $T : C \to C$ is λ-hybrid if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2(1 - \lambda)\langle x - Tx, y - Ty\rangle$$

for all $x, y \in C$. It is obvious that T is 1-hybrid if and only if T is nonexpansive; T is 0-hybrid if and only if T is nonspreading [17]; T is 1/2-hybrid if and only if T is hybrid [24]); In general, nonspreading and hybrid mappings are not continuous mappings. A mapping $T : C \to C$ is called *quasi-nonexpansive* if $F(T)$ is nonempty and $\|w - Tx\| \leq \|w - x\|$ for all $w \in F(T)$ and $x \in C$. By Dotson [9, Theorem 1] and Itoh and Takahashi [13, Corollary 1], we know that $F(T)$ is closed convex whenever T is quasi-nonexpansive. Every λ-hybrid mapping with a fixed point is clearly quasi-nonexpansive. Thus, the set of fixed points of each λ-hybrid mapping is closed convex.

3. Weak convergence theorems

In this section, we study an implicit iterative procedure for nonlinear mappings and prove weak convergence theorems for extended generalized hybrid mappings in a Banach space satisfying Opial's condition (see also [11, 28]). We also give some weak convergence theorem for nonlinear mappings. A mapping $T : C \to E$ is called *extended generalized hybrid* [11] if there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha + \beta + \gamma + \delta \geq 0, \alpha + \beta > 0$ and

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2$$

for all $x, y \in C$. Now, we get the following weak convergence theorems for extended generalized hybrid mappings in a Banach space satisfying Opial's condition (see [3]).

Theorem 3.1 ([3]). *Let E be a uniformly convex Banach space which satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and let T be a $(\alpha, \beta, \gamma, \delta)$-extended generalized hybrid mapping of C into itself such that $\beta \leq 0$ and $\gamma \leq 0$. Let $\{\gamma_n\}$ be a sequence of real numbers such that $0 < a \leq \gamma_n \leq b < 1$ for some $a, b \in \mathbb{R}$ and define a sequence $\{x_n\}$ on C as follows: $x_1 = x \in C$ and*

$$x_n = \gamma_n x_{n-1} + (1 - \gamma_n)Tx_n \qquad \textit{for } n \in \mathbb{N}.$$

If $F(T), \neq \emptyset$, then $\{x_n\}$ converges weakly to some element $z \in F(T)$.

Theorem 3.2 ([3]). *Let E be a uniformly convex Banach space which satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and let T be a $(\alpha, \beta, \gamma, \delta)$-extended generalized hybrid mapping of C into itself such that $\beta \leq 0$ and $\gamma \leq 0$. Let $\{\gamma_n\}$ be a sequence in $(0, 1]$ such that $\lim_{n\to\infty} \gamma_n = 0$ and define a sequence $\{x_n\}$ on C as follows: $x_1 = x \in C$ and*

$$x_n = \gamma_n x_{n-1} + (1 - \gamma_n)Tx_n \qquad \text{for } n \in \mathbb{N}.$$

If $F(T), \neq \emptyset$, then $\{x_n\}$ converges weakly to some element $z \in F(T)$.

From Theorem 3.1, we get the following weak convergence theorem.

Theorem 3.3 ([3]). *Let E be a uniformly convex Banach space which satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let $\alpha, \beta \in \mathbb{R}$ and let T be a (α, β)-generalized hybrid mapping of C into itself such that $\alpha \geq 1$ and $\beta \geq 0$. Let $\{\gamma_n\}$ be a sequence of real numbers such that $0 < a \leq \gamma_n \leq b < 1$ for some $a, b \in \mathbb{R}$ and define a sequence $\{x_n\}$ on C as follows: $x_1 = x \in C$ and*

$$x_n = \gamma_n x_{n-1} + (1 - \gamma_n)Tx_n \qquad \text{for } n \in \mathbb{N}.$$

If $F(T), \neq \emptyset$, $\{x_n\}$ converges weakly to some element $z \in F(T)$.

From Theorem 3.2, we get the following weak convergence theorem.

Theorem 3.4 ([3]). *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let α, β and let T be a (α, β)-generalized hybrid mapping of C into itself such that $\alpha \geq 1$ and $\beta \geq 0$. Let $\{\gamma_n\}$ be a sequence in $(0, 1]$ such that $\lim_{n\to\infty} \gamma_n = 0$ and define a sequence $\{x_n\}$ on C as follows: $x_1 = x \in C$ and :*

$$x_n = \gamma_n x_{n-1} + (1 - \gamma_n)Tx_n \qquad \text{for } n \in \mathbb{N}.$$

If $F(T), \neq \emptyset$, then $\{x_n\}$ converges weakly to some element $z \in F(T)$.

From Theorem 3.1, we get the following weak convergence theorems.

Theorem 3.5 ([3]). *Let E be a uniformly convex Banach space which satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let T be a hybrid mapping of C into itself. Let $\{\gamma_n\}$ be a sequence of real numbers such that $0 < a \leq \gamma_n \leq b < 1$ for some $a, b \in \mathbb{R}$ and define a sequence $\{x_n\}$ on C as follows: $x_1 = x \in C$ and*

$$x_n = \gamma_n x_{n-1} + (1 - \gamma_n)Tx_n \qquad \text{for } n \in \mathbb{N}.$$

If $F(T), \neq \emptyset$, $\{x_n\}$ converges weakly to some element $z \in F(T)$.

Theorem 3.6 ([3]). *Let E be a uniformly convex Banach space which satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let T be a nonspreading mapping of C into itself. Let $\{\gamma_n\}$ be a sequence of real numbers such that $0 < a \leq \gamma_n \leq b < 1$ for some $a, b \in \mathbb{R}$ and define a sequence $\{x_n\}$ on C as follows: $x_1 = x \in C$ and*

$$x_n = \gamma_n x_{n-1} + (1 - \gamma_n) T x_n \qquad \text{for } n \in \mathbb{N}.$$

If $F(T), \neq \emptyset$, $\{x_n\}$ converges weakly to some element $z \in F(T)$.

Theorem 3.7 ([3]). *Let E be a uniformly convex Banach space which satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let T be a nonexpansive mapping of C into itself. Let $\{\gamma_n\}$ be a sequence of real numbers such that $0 < a \leq \gamma_n \leq b < 1$ for some $a, b \in \mathbb{R}$ and define a sequence $\{x_n\}$ on C as follows: $x_1 = x \in C$ and*

$$x_n = \gamma_n x_{n-1} + (1 - \gamma_n) T x_n \qquad \text{for } n \in \mathbb{N}.$$

If $F(T), \neq \emptyset$, $\{x_n\}$ converges weakly to some element $z \in F(T)$.

References

1. K. Aoyama, S. Iemoto, F. Kohsaka and W. Takahashi, *Fixed point and ergodic theorems for λ-hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **11** (2010), 335–343.
2. S. Atsushiba, *Strong convergence to common attractive points of uniformly asymptotically regular nonexpansive semigroups*, J. Nonlinear Convex Anal. **16** (2015), 69-78.
3. S. Atsushiba, *Weak convergence theorems for nonlinear hybrid mappings in Banach spaces*, submitted.
4. S. Atsushiba, and W. Takahashi, *Nonlinear ergodic theorems without convexity for nonexpansive semigroups in Hilbert spaces*, J. Nonlinear Convex Anal. **14** (2013), 209-219.
5. J.-B. Baillon, *Un theoreme de type ergodique pour les contractions non lineaires dans un espace de Hilbert*, C. R. Acad. Sei. Paris Ser. A-B **280** (1975), 1511 - 1514.
6. E.Blum and W.Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), 123 145.
7. F.E.Browder, *Convergence theorems for sequences of nonlinear operators in Banach spaces*, Math. Z. **100** (1967), 201 225.
8. P.L. Combettes and S.A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal. **6** (2005), 117 136.
9. W. G. Dotson. Jr., *Fixed points of quasi-nonexpansive mappings.*, J. Austral. Math. Soc. **13** (1972), 167–170.

10. K.Goebel and W.A.Kirk, Topics in metric fixed point theory. Cambridge Studies in Advanced Mathematics, 28. Cambridge University Press, Cambridge, 1990.
11. M. Hojo and W. Takahashi, *Fixed point and weak convergence theorems for nonlinear hybrid mappings in Banach spaces*, Linear Nonlinear Anal. **3** (2017), 61 72.
12. M.-H. Hsu, W. Takahashi and J.-C.Yao, *Generalized hybrid mappings in Hilbert spaces and Banach spaces*, Taiwanese J. Math. **16** (2012), 129 149. 47H10 (47H05)
13. S. Itoh and W. Takahashi *The common fixed point theory of single-valued mappings and multivalued mappings.*, Pacific J. Math., **79** (1978), 493–508.
14. S. Iemoto and W. Takahashi, *Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space*, Nonlinear Anal. **71** (2009), 2082 2089.
15. P. Kocourek, W. Takahashi, and J.-C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese J. Math. **14** (2010), 2497–2511.
16. P. Kocourek, W. Takahashi, and J.-C. Yao, *Fixed point theorems and ergodic theorems for nonlinear mappings in Banach spaces*, Adv. Math. Econ. **15** (2011), 67–88.
17. F. Kohsaka and W. Takahashi *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. (Basel) **91** (2008), 166–177.
18. F. Kohsaka and W. Takahashi Existence and Approximation of Fixed Points of Firmly Nonexpansive-Type Mappings in Banach Spaces, SIAM J. Optim., **19** (2008) 824 835.
19. J.P. Gossez and E. Lami Dozo, *Some geometric properties related to the fixed point theory for nonexpansive mappings*, Pacific J. Math. **40** (1972), 565 - 573.
20. Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597.
21. Z.H.Sun, C.He and Y.Q.Ni, *Strong convergence of an implicit iteration process for nonexpansive mappings in Banach spaces*, Nonlinear Funct. Anal. Appl. **8** (2003), 595 602.
22. W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space,* Proc. Amer. Math. Soc. **81** (1981), 253-256.
23. W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama, 2000.
24. W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal., **11** (2010), 79-88.
25. W. Takahashi and Y. Takeuchi, *Nonlinear ergodic theorem without convexity for generalized hybrid mappings in a Hilbert space*, J. Nonlinear Convex Anal. **12** (2011), 399–406.
26. W. Takahashi and J.-C. Yao, *Fixed point theorems and ergodic theorems for nonlinear mappings in Hilbert spaces,* Taiwanese J. Math. **15** (2011), 457 472.
27. W. Takahashi and J.-C. Yao, *Weak convergence theorems for generalized hybrid mappings in Banach spaces*, J. Nonlinear Anal. Optim. **2** (2011), 155 166.

28. H.K.Xu and R.G. Ori, *An implicit iteration process for nonexpansive mappings*, Numer. Funct. Anal. Optim. **22** (2001), 767 773.

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