

Attractive Point and Weak Convergence Theorems for Two Commutative Nonlinear Mappings in Banach Spaces

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Abstract. In this article, using the class of generalized nonspreading mappings in Banach spaces which covers generalized hybrid mappings in a Hilbert space, we prove an attractive point theorem. Furthermore, we prove a nonlinear mean convergence theorem of Baillon's type and a weak convergence theorem of Mann's type for generalized nonspreading mappings in a Banach space. Using these theorems, we obtain new attractive point theorems, mean convergence theorems and weak convergence theorems in Hilbert spaces and Banach spaces.

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1 Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H . Let T be a mapping of C into H . Then we denote by $F(T)$ the set of *fixed points* of T and by $A(T)$ the set of *attractive points* [33] of T , i.e.,

- (i) $F(T) = \{z \in C : Tz = z\}$;
- (ii) $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|, \forall x \in C\}$.

We know from [33] that $A(T)$ is closed and convex. This property is important for proving our main theorems. In 2010, Kocourek, Takahashi and Yao [17] defined a broad class of nonlinear mappings in a Hilbert space: Let H be a real Hilbert space and let C be a nonempty subset of H . A mapping $T : C \rightarrow H$ is called *generalized hybrid* [17] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{1.1}$$

for all $x, y \in C$. Such a mapping T is called (α, β) -generalized hybrid. We also know the following mapping: For $\lambda \in \mathbb{R}$, $U : C \rightarrow H$ is called λ -hybrid [2] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + 2(1 - \lambda)\langle x - Ux, y - Uy \rangle \tag{1.2}$$

for all $x, y \in C$. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a $(1,0)$ -generalized hybrid mapping is nonexpansive. It is nonspreading [21, 22] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also hybrid [31] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [14]. We know that λ -hybrid mappings are contained in the class of generalized hybrid mappings; see [7]. In 1975, Baillon [3] proved the following nonlinear ergodic theorem in a Hilbert space:

Theorem 1.1 ([3]). *Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H . Let $T : C \rightarrow C$ be a nonexpansive mapping such that the set $F(T)$ of fixed points of T is nonempty. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to a point of $F(T)$.

This theorem for nonexpansive mappings has been extended to Banach spaces by many authors; see, for example, [4, 5, 6, 23, 24]. On the other hand, Kocourek, Takahashi and Yao [17] extended this theorem to generalized hybrid mappings in a Hilbert space. Recently, Kohsaka [19] also proved the following theorem:

Theorem 1.2 ([19]). *Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H . Let S and T be commutative λ and μ -hybrid mappings of C into itself such that the set $F(S) \cap F(T)$ of common fixed points of S and T is nonempty. Then, for any $x \in C$,*

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^n \sum_{l=0}^n S^k T^l x$$

converges weakly to a point of $F(S) \cap F(T)$.

We also know Mann's iteration [26] introduced in 1953. Let C be a nonempty, closed and convex subset of a Banach space E . A mapping $T : C \rightarrow C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. For an initial guess $x_1 \in C$, an iteration process $\{x_n\}$ is defined recursively by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

In this article, using the class of generalized nonspreading mappings in Banach spaces which covers generalized hybrid mappings in a Hilbert space, we prove an attractive point theorem. Furthermore, we prove a nonlinear mean convergence theorem of Baillon's type and a weak convergence theorem of Mann's type for generalized nonspreading mappings in a Banach space. Using these theorems, we obtain new attractive point theorems, mean convergence theorems and weak convergence theorems in Hilbert spaces and Banach spaces.

2 Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the topological dual space of E . We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E , we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \rightarrow x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon \right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let C be a nonempty subset of a Banach space E . A mapping $T : C \rightarrow E$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \rightarrow E$ is quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$, where $F(T)$ is the set of fixed points of T . If C is a nonempty, closed and convex subset of a strictly convex Banach space E and $T : C \rightarrow E$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [15]. Let E be a Banach space. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In this case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^* . We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. It is also said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. A Banach space E is called uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then J is uniformly norm-to-weak* continuous on each bounded subset of E , and if the norm of E is Fréchet differentiable, then J is norm-to-norm continuous. If E is uniformly smooth, J is uniformly norm-to-norm continuous on each bounded subset of E . For more details, see [29, 30]. The following result is also well known.

Lemma 2.1 ([29]). *Let E be a smooth Banach space and let J be the duality mapping on E . Then, $\langle x-y, Jx-Jy \rangle \geq 0$ for all $x, y \in E$. Further, if E is strictly convex and $\langle x-y, Jx-Jy \rangle = 0$, then $x = y$.*

Let E be a smooth Banach space. The function $\phi : E \times E \rightarrow (-\infty, \infty)$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for $x, y \in E$, where J is the duality mapping of E ; see [1] and [16]. We have from the definition of ϕ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle \tag{2.2}$$

for all $x, y, z \in E$. From $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \geq 0$. Furthermore, we can obtain the following equality:

$$2\langle x - y, Jz - Jw \rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w) \quad (2.3)$$

for $x, y, z, w \in E$. If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \iff x = y. \quad (2.4)$$

The following lemmas are in Xu [36] and Kamimura and Takahashi [16].

Lemma 2.2 ([36]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \rightarrow [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $x, y \in B_r$ and λ with $0 \leq \lambda \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 2.3 ([16]). *Let E be smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \rightarrow \mathbb{R}$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \phi(x, y)$$

for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let E be a smooth Banach space and let C be a nonempty subset of E . Then a mapping $T : C \rightarrow E$ is called generalized nonexpansive [10] if $F(T) \neq \emptyset$ and

$$\phi(Tx, y) \leq \phi(x, y)$$

for all $x \in C$ and $y \in F(T)$. Let D be a nonempty subset of a Banach space E . A mapping $R : E \rightarrow D$ is said to be sunny if

$$R(Rx + t(x - Rx)) = Rx$$

for all $x \in E$ and $t \geq 0$. A mapping $R : E \rightarrow D$ is said to be a retraction or a projection if $Rx = x$ for all $x \in D$. A nonempty subset D of a smooth Banach space E is said to be a generalized nonexpansive retract (resp. sunny generalized nonexpansive retract) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto D ; see [9, 10] for more details. The following results are in Ibaraki and Takahashi [10].

Lemma 2.4 ([10]). *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E . Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.5 ([10]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then the following hold:*

- (i) $z = Rx$ if and only if $\langle x - z, Jy - Jz \rangle \leq 0$ for all $y \in C$;
- (ii) $\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.

In 2007, Kohsaka and Takahashi [20] proved the following results:

Lemma 2.6 ([20]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E . Then the following are equivalent:*

- (a) C is a sunny generalized nonexpansive retract of E ;
- (b) C is a generalized nonexpansive retract of E ;
- (c) JC is closed and convex.

Lemma 2.7 ([20]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed sunny generalized nonexpansive retract of E . Let R be the sunny generalized nonexpansive retraction from E onto C and let $(x, z) \in E \times C$. Then the following are equivalent:*

- (i) $z = Rx$;
- (ii) $\phi(x, z) = \min_{y \in C} \phi(x, y)$.

Ibaraki and Takahashi [13] also obtained the following result concerning the set of fixed points of a generalized nonexpansive mapping.

Lemma 2.8 ([13]). *Let E be a smooth, strictly convex and reflexive Banach space and let T be a generalized nonexpansive mapping from E into itself. Then, $F(T)$ is closed and $JF(T)$ is closed and convex.*

The following lemma is a direct consequence of Lemmas 2.6 and 2.8.

Lemma 2.9 ([13]). *Let E be a smooth, strictly convex and reflexive Banach space and let T be a generalized nonexpansive mapping from E into itself. Then, $F(T)$ is a sunny generalized nonexpansive retract of E .*

Using Lemma 2.6, we have the following result.

Lemma 2.10. *Let E be a smooth, strictly convex and reflexive Banach space and let $\{C_i : i \in I\}$ be a family of sunny generalized nonexpansive retracts of E such that $\bigcap_{i \in I} C_i$ is nonempty. Then $\bigcap_{i \in I} C_i$ is a sunny generalized nonexpansive retract of E .*

3 Attractive Point and Fixed Point Theorem

Kocourek, Takahashi and Yao [18] extended the concept of generalized hybrid mappings [17] in a Hilbert space to that in a Banach space. Let E be a smooth Banach space and let C be a nonempty subset of E . Then a mapping $T : C \rightarrow E$ is called *generalized nonspreading* [18] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\phi(Tx, Ty) + (1 - \alpha)\phi(x, Ty) + \gamma\{\phi(Ty, Tx) - \phi(Ty, x)\} \\ \leq \beta\phi(Tx, y) + (1 - \beta)\phi(x, y) + \delta\{\phi(y, Tx) - \phi(y, x)\} \end{aligned}$$

for all $x, y \in C$. We call such a mapping $(\alpha, \beta, \gamma, \delta)$ -generalized nonspreading. Let E be a smooth Banach space. Let C be a nonempty subset of E and let T be a mapping of C into E . We denote by $A(T)$ the set of *attractive points* of T , i.e., $A(T) = \{z \in E : \phi(z, Tx) \leq \phi(z, x), \forall x \in C\}$; see [25].

Lemma 3.1 ([25]). *Let E be a smooth Banach space and let C be a nonempty subset of E . Let T be a mapping from C into E . Then $A(T)$ is a closed and convex subset of E .*

We prove the following lemma.

Lemma 3.2. *Let E be a smooth, strictly convex and reflexive Banach space with the duality mapping J and let C be a nonempty subset of E . Let S and T be mappings of C into itself. Let $\{x_n\}$ be a bounded sequence of E and let μ be a mean on l^∞ . Suppose that*

$$\mu_n \phi(x_n, Sy) \leq \mu_n \phi(x_n, y) \text{ and } \mu_n \phi(x_n, Ty) \leq \mu_n \phi(x_n, y)$$

for all $y \in C$. Then $A(S) \cap A(T)$ is nonempty. Additionally, if C is closed and convex and $\{x_n\} \subset C$, then $F(S) \cap F(T)$ is nonempty.

Using Lemma 3.2, we can prove an attractive point and fixed point theorem for commutative generalized nonspreading mappings in a Banach space.

Theorem 3.3 ([34]). *Let C be a nonempty subset of a smooth, strictly convex and reflexive Banach space E and let S and T be commutative generalized nonspreading mappings of C into itself. Suppose that there exists an element $z \in C$ such that $\{S^k T^l z : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded. Then $A(S) \cap A(T)$ is nonempty. Additionally, if C is closed and convex, then $F(S) \cap F(T)$ is nonempty.*

4 Nonlinear Ergodic Theorems of Baillon's Type

Now, using the technique developed by [28], we prove a mean convergence theorem of Baillon's type for generalized nonspreading mappings in a Banach space. For proving it, we need the following lemma.

Lemma 4.1. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E . Let S and T be commutative generalized nonspreading mappings of C into itself. If $\{S^k T^l x : k, l \in \mathbb{N} \cup \{0\}\}$ for some $x \in C$ is bounded and*

$$S_n x = \frac{1}{(1+n)^2} \sum_{k=0}^n \sum_{l=0}^n S^k T^l x$$

for all $n \in \mathbb{N} \cup \{0\}$, then every weak cluster point of $\{S_n x\}$ is a point of $F(S) \cap F(T)$.

Let E be a smooth Banach space. Let C be a nonempty subset of E and let T be a mapping of C into E . We denote by $B(T)$ the set of *skew-attractive points* [25] of T , i.e., $B(T) = \{z \in E : \phi(Tx, z) \leq \phi(x, z), \forall x \in C\}$. Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E . Let T be a mapping of C into E . Define a mapping T^* as follows:

$$T^* x^* = JTJ^{-1} x^*, \quad \forall x^* \in JC,$$

where J is the duality mapping on E and J^{-1} is the duality mapping on E^* . A mapping T^* is called the duality mapping of T ; see also [35] and [8]. It is easy to show that if T is a mapping of C into itself, then T^* is a mapping of JC into itself. In fact, for $x^* \in JC$, we have $J^{-1} x^* \in C$ and hence $TJ^{-1} x^* \in C$. So, we have

$$T^* x^* = JTJ^{-1} x^* \in JC.$$

Then, T^* is a mapping of JC into itself. The following result is in Lin and Takahashi [25].

Lemma 4.2 ([25]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E . Let T be a mapping of C into E and let T^* be the duality mapping of T . Then, the following hold:*

- (1) $JB(T) = A(T^*);$
- (2) $JA(T) = B(T^*).$

In particular, $JB(T)$ is closed and convex.

Let $D = \{(k, l) : k, l \in \mathbb{N} \cup \{0\}\}$. Then D is a directed set by the binary relation:

$$(k, l) \leq (i, j) \quad \text{if } k \leq i \text{ and } l \leq j.$$

Now, we can prove the following nonlinear ergodic theorem for generalized nonspreading mappings in a Banach space.

Theorem 4.3 ([32]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty, closed and convex sunny generalized nonexpansive retract of E . Let S and T be commutative generalized nonspreading mappings of C into itself with $F(S) \cap F(T) \neq \emptyset$ such that $\phi(Sx, u) \leq \phi(x, u)$ and $\phi(Tx, v) \leq \phi(x, v)$ for all $x \in C$ and $u \in F(S)$ and $v \in F(T)$, respectively. Let R be the sunny generalized nonexpansive retraction of E onto $F(S) \cap F(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^n \sum_{l=0}^n S^k T^l x$$

converges weakly to an element q of $F(S) \cap F(T)$, where $q = \lim_{(k,l) \in D} RS^k T^l x$.

Using Theorem 4.3, we obtain the two following theorems.

Theorem 4.4. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let $S, T : E \rightarrow E$ be commutative $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ -generalized nonspreading mappings with $F(S) \cap F(T)$ such that $\alpha > \beta$ and $\gamma \leq \delta$ and $\alpha' > \beta'$ and $\gamma' \leq \delta'$, respectively. Assume that $F(S) \cap F(T) \neq \emptyset$ and let R be the sunny generalized nonexpansive retraction of E onto $F(S) \cap F(T)$. Then, for any $x \in E$,*

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^n \sum_{l=0}^n S^k T^l x$$

converges weakly to an element q of $F(S) \cap F(T)$, where $q = \lim_{(k,l) \in D} RS^k T^l x$.

Theorem 4.5. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . Let $S, T : C \rightarrow C$ be commutative generalized hybrid mappings with $F(S) \cap F(T) \neq \emptyset$ and let P be the metric projection of H onto $F(S) \cap F(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^n \sum_{l=0}^n S^k T^l x$$

converges weakly to an element p of $F(S) \cap F(T)$, where $p = \lim_{(k,l) \in D} PS^k T^l x$.

5 Weak Convergence Theorems of Mann's Type

In this section, we prove a weak convergence theorem of Mann's type iteration for generalized nonspreading mappings in a Banach space. For proving it, we need the following lemma.

Lemma 5.1. *Let E be a smooth and uniformly convex Banach space and let C be a nonempty and closed subset of E such that J_C is closed and convex. Let S and T be commutative generalized nonspreading mappings of C into itself such that $F(S) \cap F(T) \neq \emptyset$, $\phi(Sx, u) \leq \phi(x, u)$ and $\phi(Tx, v) \leq \phi(x, v)$ for for all $x \in C$ and $u \in F(S)$ and $v \in F(T)$. Let R be a sunny generalized nonexpansive retraction of E onto $F(S) \cap F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and let $\{x_n\}$ be a sequence in C generated by $x_1 = x \in C$ and*

$$x_{n+1} = R_C \left(\alpha_n x_n + (1 - \alpha_n) \frac{1}{(n+1)^2} \sum_{k=0}^n \sum_{l=0}^n S^k T^l x_n \right), \quad \forall n \in \mathbb{N},$$

where R_C is a sunny generalized nonexpansive retraction of E onto C . Then $\{R x_n\}$ converges strongly to a point z of $F(S) \cap F(T)$.

Using Lemma 5.1 and the technique developed by [11], we prove the following theorem.

Theorem 5.2 ([32]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed convex sunny generalized nonexpansive retract of E . Let S and T be commutative generalized nonspreading mappings of C into itself such that $F(S) \cap F(T) \neq \emptyset$, $\phi(Sx, u) \leq \phi(x, u)$ and $\phi(Tx, v) \leq \phi(x, v)$ for for all $x \in C$ and $u \in F(S)$ and $v \in F(T)$. Let R be the sunny generalized nonexpansive retraction of E onto $F(S) \cap F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$. Then, a sequence $\{x_n\}$ generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \frac{1}{(n+1)^2} \sum_{k=0}^n \sum_{l=0}^n S^k T^l x_n, \quad \forall n \in \mathbb{N}$$

converges weakly to $z \in F(S) \cap F(T)$, where $z = \lim_{n \rightarrow \infty} R x_n$.

Using Theorem 5.2, we can prove the following two weak convergence theorems.

Theorem 5.3. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let $S, T : E \rightarrow E$ be commutative $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ -generalized nonspreading mappings such that $\alpha > \beta$ and $\gamma \leq \delta$ and $\alpha' > \beta'$ and $\gamma' \leq \delta'$, respectively. Assume that $F(S) \cap F(T) \neq \emptyset$ and let R be the sunny generalized nonexpansive retraction of E onto $F(S) \cap F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$. Then, a sequence $\{x_n\}$ generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \frac{1}{(n+1)^2} \sum_{k=0}^n \sum_{l=0}^n S^k T^l x_n x_n, \quad \forall n \in \mathbb{N}$$

converges weakly to $z \in F(S) \cap F(T)$, where $z = \lim_{n \rightarrow \infty} R x_n$.

Theorem 5.4. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . Let $S, T : C \rightarrow C$ be commutative generalized hybrid mappings with $F(S) \cap F(T) \neq \emptyset$ and let P be the metric projection of H onto $F(S) \cap F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers*

such that $0 \leq \alpha_n < 1$ and $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$. Then, a sequence $\{x_n\}$ generated by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \frac{1}{(n+1)^2} \sum_{k=0}^n \sum_{l=0}^n S^k T^l x_n x_n, \quad \forall n \in \mathbb{N}$$

converges weakly to $z \in F(S) \cap F(T)$, where $z = \lim_{n \rightarrow \infty} P x_n$.

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