数理解析研究所講究録 第2095巻 2018年 180-187

Solutions of the stationary Navier-Stokes equations in Besov and Triebel-Lizorkin spaces

Hiroyuki Tsurumi
Department of Mathematics, Faculty of Science and Engineering, Waseda University

1 Introduction

We consider the stationary Navier-Stokes equations in $\mathbb{R}^n$, $n \geq 3$;

$$\begin{cases} -\Delta u + u \cdot \nabla u + \nabla \pi = f & \text{in } x \in \mathbb{R}^n, \\ \nabla \cdot u = 0 \quad \text{in } x \in \mathbb{R}^n, \end{cases} \tag{SNS}$$

where $u = u(x) = (u_1(x), u_2(x), \ldots, u_n(x))$ and $\pi = \pi(x)$ denote the unknown velocity vector and the unknown pressure of the fluid at the point $x \in \mathbb{R}^n$, respectively, while $f = f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$ is the given external force. Here $u \cdot \nabla u \equiv \sum_{m=1}^{n} u_m \frac{\partial u}{\partial x_m}$. Until now, there have been various studies on existence, uniqueness, and regularity of strong solutions to (SNS). For instance, Leray[8] and Ladyzhenskaya[7] showed the existence of solutions to (SNS), and later on, Heywood[3] constructed the solution of (SNS) as a limit of solutions of the non-stationary Navier-Stokes equations. Moreover, Chen[2] proved that for every smooth external force which is small in $\dot{H}^{-1,\frac{n}{2}}$, there exists a unique solution of (SNS) in $L^n \cap \dot{H}^{1,\frac{n}{2}}$ which is small in $\dot{H}^{1,\frac{n}{2}}$. Here $\dot{H}^{s,r}$ denotes the homogeneous Sobolev space with the norm $\|f\|_{\dot{H}^{s,r}} \equiv \|(-\Delta)^{\frac{s}{2}} f\|_{L^r}$. In this way, it seems to be important to find more general spaces such that every small external force in these spaces yields a unique solution of (SNS), and to find more regularity of solutions.

Recently, the above existence, uniqueness, and regularity problems in the homogeneous Besov spaces were well studied by Kaneko-Kozono-Shimizu[5]. They proved that for every small external force in $\dot{B}_{p,q}^{-3+\frac{n}{p}}$ with $1 \leq p < n$ and $1 \leq q \leq \infty$, there exists a unique small solution of (SNS) in $\dot{B}_{p,q}^{-1+\frac{n}{p}}$. We should note here that $\dot{B}_{p,q}^{-3+\frac{n}{p}}$ and $\dot{B}_{p,q}^{-1+\frac{n}{p}}$ are scaling invariant for external forces and velocities in (SNS), respectively. Indeed, since the scaling transform is $\{u, \pi, f\}\{u_\lambda, \pi_\lambda, f_\lambda\}$ with $u_\lambda(x) \equiv \lambda u(\lambda x)$, $\pi_\lambda(x) \equiv \lambda^2 \pi(\lambda x)$, $f_\lambda(x) \equiv \lambda^3(\lambda x)$, we see that $\|f_\lambda\|_{\dot{B}_{p,q}^{-3+\frac{n}{p}}} = \|f\|_{\dot{B}_{p,q}^{-3+\frac{n}{p}}}$ and $\|u_\lambda\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} = \|u\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}}$ for all $\lambda > 0$. Moreover, they showed that if the small external force f in $\dot{B}_{p,q}^{-3+\frac{n}{p}}$ has more regularity, so does the above solution u. More precisely, if such an external force f belongs to $\dot{H}^{s-2,r}$ with $1 < r < \infty$ and $s \geq 0$ satisfying

$$q \leq r, \quad (n/r) - n + 1 < s < \min\{n/p, n/r\}, \tag{1.1}$$

then the solution u belongs to $\dot{H}^{s,r}$. Using the embedding $\dot{H}^{-1,\frac{n}{2}} \hookrightarrow \dot{B}^{-3+\frac{n}{p}}_{p,\frac{n}{2}}$ for $n/2 < p < n$, we can see that their result includes that of Chen[2], by taking $n/2 < p < n$, $q = r = n/2$, and $s = 1$.

We first discuss similar problems on (SNS) in homogeneous Triebel-Lizorkin spaces. We show that as for existence and uniqueness, the similar result to Kaneko-Kozono-Shimizu[5] can be obtained. More precisely, we prove that for every small external force in $\dot{F}^{-3+\frac{n}{p}}_{p,q}$, there exists a unique small solution in $\dot{F}^{-1+\frac{n}{p}}_{p,q}$, provided $1 < p < n$ and $1 \leq q \leq \infty$, and provided $p = n$ and $1 \leq q \leq 2$. They are of course scaling invariant spaces for external forces and velocities. We can prove these existence and uniqueness by similar methods to Kaneko-Kozono-Shimizu[5]. Indeed, we make use of the boundedness of the Riesz transform, the para-product formula, and the embedding theorem in homogeneous Triebel-Lizorkin spaces. For the additional regularity of solutions, we prove that if a small external force in the above scaling invariant Triebel-Lizorkin spaces with $p < n$ also belongs to $\dot{H}^{s-2,r}$ with $s > 0$ and $1 < r < \infty$, or with $s = 0$ and $n/(n-1) < r < \infty$, then the solution belongs to $\dot{H}^{s,r}$. Although Kaneko-Kozono-Shimizu[5] showed a similar result, some additional restrictions for s, r are required in the case of Besov spaces as mentioned above. Such difference seems to stem from the fact that Sobolev spaces are closely related to Triebel-Lizorkin spaces rather than Besov spaces.

Secondly, we also focus on the well-posedness problem of (SNS). In fact, by the boundedness of the Riesz transform and the bilinear form $(u, v) \mapsto (-\Delta)^{-1}P(u \cdot \nabla v)$, Kaneko-Kozono-Shimizu[5] and our result as above guarantee the uniquely existence of solutions in $\dot{B}^{-1+\frac{n}{p}}_{p,q}$ ($\dot{F}^{-1+\frac{n}{p}}_{p,q}$) dependent continuously on given external forces $\dot{B}^{-3+\frac{n}{p}}_{p,q}$ ($\dot{F}^{-3+\frac{n}{p}}_{p,q}$) when $n \geq 3$, $p < n$, $1 \leq q \leq \infty$. However, we can see that once we take $p = \infty$, then this continuity is broken. More precisely, we can find a sequence of external forces which converges to zero in $\dot{B}^{-3}_{\infty,q}$ ($\dot{F}^{-3}_{\infty,q}$) and yields a sequence of solutions of (SNS) which does not converge to zero in $\dot{B}^{-1}_{\infty,q}$ ($\dot{F}^{-1}_{\infty,q}$). For the proof of our theorem, we apply the sequence of initial data proposed by Bourgain[1] and Sawada[9], which studied the ill-posedness problem on non-stationary Navier-Stokes equations, to (SNS) as the external force with some modifications.

2 Main Results

First of all, let us define some spaces of functions and distributions. We denote by $\mathcal{S}$ the space of rapidly decreasing functions, and $\mathcal{S}'$ denotes the dual space of $\mathcal{S}$, which is called the space of tempered distributions.

For $f \in \mathcal{S}$ and $s \in \mathbb{R}$, we define the Riesz potential $(-\Delta)^{\frac{s}{2}}$ by

$$(-\Delta)^{\frac{s}{2}} f \equiv \mathcal{F}^{-1}|\xi|^s \mathcal{F} f,$$

where $\mathcal{F}$ denotes the Fourier transform. Then we define the homogeneous Sobolev space $\dot{H}^{s,r}$ for $s \in \mathbb{R}$ and $1 \leq r \leq \infty$ as

$$\dot{H}^{s,r} \equiv \left\{ f \in \mathcal{S}'/\mathcal{P};\ \|f\|_{\dot{H}^{s,r}} \equiv \|(-\Delta)^{\frac{s}{2}} f\|_{L^r} < \infty \right\},$$

where $\mathcal{S}'/\mathcal{P}$ denotes the quotient space with the polynomials space $\mathcal{P}$.

We next introduce the Littlewood-Paley decomposition. First, we take $\phi \in \mathcal{S}$ such that

$$\operatorname{supp} \phi \subset \left\{\xi \in \mathbb{R}^n;\ \frac{1}{2} \le |\xi| \le 2\right\}, \quad \sum_{j\in\mathbb{Z}} \phi(2^{-j}\xi) = 1\ (\xi \neq 0). \tag{2.1}$$

Then, we define a family $\{\varphi_j\}_{j\in\mathbb{Z}} \subset \mathcal{S}$ of functions as

$$\mathcal{F}\varphi_j(\xi) = \phi(2^{-j}\xi), \quad j \in \mathbb{Z}. \tag{2.2}$$

By (2.1), (2.2), and boundedness of $\mathcal{F}$ and $\mathcal{F}^{-1}$ in $\mathcal{S}'$, we see that every $f \in \mathcal{S}'$ can be decomposed as $f = \sum_{j\in\mathbb{Z}} \varphi_j * f$.

Associated with $\{\varphi_j\}_{j\in\mathbb{Z}}$ above, we define the homogeneous Besov spaces $\dot{B}^s_{p,q}$ and Triebel-Lizorkin spaces $\dot{F}^s_{p,q}$ by

$$\dot{B}^s_{p,q} \equiv \left\{f \in \mathcal{S}'/\mathcal{P};\ \|f\|_{\dot{B}^s_{p,q}} < \infty\right\}, \quad s \in \mathbb{R},\ \ 1 \le p \le \infty,\ \ 1 \le q \le \infty,$$

$$\dot{F}^s_{p,q} \equiv \left\{f \in \mathcal{S}'/\mathcal{P};\ \|f\|_{\dot{F}^s_{p,q}} < \infty\right\}, \quad s \in \mathbb{R},\ \ 1 \le p < \infty,\ \ 1 \le q \le \infty$$

with the norms

$$\|f\|_{\dot{B}^s_{p,q}} \equiv \begin{cases} \left(\sum_{j\in\mathbb{Z}} (2^{js}\|\varphi_j * f\|_{L^p})^q\right)^{\frac{1}{q}}, & q < \infty, \\ \sup_{j\in\mathbb{Z}} (2^{js}\|\varphi_j * f\|_{L^p}), & q = \infty, \end{cases}$$

$$\|f\|_{\dot{F}^s_{p,q}} \equiv \begin{cases} \left\|\left\{\sum_{j=1}^{\infty} (2^{sj}|\varphi_j * f(\cdot)|)^q\right\}^{\frac{1}{q}}\right\|_{L^p}, & q < \infty, \\ \left\|\sup_{j\in\mathbb{Z}} 2^{js}|\varphi_j * f(\cdot)|\right\|_{L^p}, & q = \infty. \end{cases}$$

Although $\dot{F}^s_{\infty,q}$ is defined in a different way for $1 \le q < \infty$, we do not treat such a space in this paper. It is known that this definition is independent of choice of a function ϕ satisfying (2.1).

Let us rewrite (SNS) to the generalized form so that we can apply successive approximation. First, we note that since $\nabla \cdot u = 0$, there holds

$$u \cdot \nabla u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(u_i u) = \nabla \cdot (u \otimes u),$$

where $u \otimes v$ denotes the tensor product with $(u \otimes v)_{ij} \equiv u_i v_j$, $1 \le i, j \le n$. We next introduce the projection $P : L^p \to L^p_\sigma \equiv \overline{\{f \in C_0^\infty;\ \nabla \cdot f = 0\}}^{\|\cdot\|_{L^p}}$. In $\mathbb{R}^n$, P is defined as a matrix-valued operator $P = (P_{jk})_{1\le j,k\le n}$ with $P_{jk} \equiv \delta_{jk} + R_j R_k$, where $R_j = \frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}}$, $j = 1, 2, \ldots, n$, denotes the Riesz transform. Applying P to (SNS), we obtain

$$-\Delta u + P\nabla \cdot (u \otimes u) = Pf,$$

implied by $P(\nabla\pi) = 0$ and $Pu = u$, since $\nabla \cdot u = 0$. Hence, the solution u of (SNS) can be expressed as

$$\begin{aligned} u &= (-\Delta)^{-1}Pf - (-\Delta)^{-1}P\nabla \cdot (u \otimes u) \qquad \text{(rSNS)} \\ &\equiv Lf + K(u \otimes u), \end{aligned}$$

where $Lf \equiv (-\Delta)^{-1}Pf$ and $Kg \equiv -(-\Delta)^{-1}P\nabla \cdot g$ (g is a matrix function).

Our main theorems now read as follows. First, we state existence and uniqueness of solutions of (rSNS) for small external forces.

Theorem 2.1. *Let $n \geq 3$, and suppose that the exponents p and q satisfy the following either* (i) *or* (ii);

(i) $1 < p < n$, $1 \leq q \leq \infty$,

(ii) $p = n$, $1 \leq q \leq 2$.

Then there is a constant $\delta = \delta(n,p,q)$ such that if $f \in \dot{F}_{p,q}^{-3+\frac{n}{p}}$ satisfies $\|f\|_{\dot{F}_{p,q}^{-3+\frac{n}{p}}} < \delta$, then there exists a solution $u \in P\dot{F}_{p,q}^{-1+\frac{n}{p}}$ of (rSNS). *Moreover, there exists a constant $\eta = \eta(n,p,q) > 0$ such that the above solution u is unique provided $\|u\|_{\dot{F}_{p,q}^{-1+\frac{n}{p}}} < \eta$.*

Next, we show more regularity of solutions under some additional assumption on external forces as follows.

Theorem 2.2. *Let $n \geq 3$, and suppose that the exponents $p, q, r,$ and s satisfy the following either* (i), (ii), *or* (iii);

(i) $s > 0$, $1 < r < \infty$, *p and q satisfy either* (i) *or* (ii) *of Theorem 2.1,*

(ii) $s = 0$, $n/(n-1) < r < \infty$, *p and q satisfy* (i) *of Theorem 2.1,*

(iii) $s = 0$, $r = n$, *p and q satisfy* (ii) *of Theorem 2.1.*

Then there exists a constant $\delta' = \delta'(n,s,p,q,r)$ such that if $f \in \dot{F}_{p,q}^{-3+\frac{n}{p}} \cap \dot{H}^{s-2,r}$ satisfies $\|f\|_{\dot{F}_{p,q}^{-3+\frac{n}{p}}} < \delta'$, then the solution u of (rSNS) given by Theorem 2.1 belongs to $\dot{H}^{s,r}$.

Remark 2.1. (i) In Theorems 2.1 and 2.2, the spaces $\dot{F}_{p,q}^{-1+\frac{n}{p}}$ for solutions u and $\dot{F}_{p,q}^{-3+\frac{n}{p}}$ for external forces f are both scaling invariant with respect to (SNS).
(ii) Theorem 2.2 means that a smooth external force whose scaling invariant Triebel-Lizorkin norm is small enough yields a smooth solution of (E). We here note that the $\dot{H}^{s-2,r}$ norm of an external force do not have to be small. Moreover, in Theorem 2.2, we can take $s \geq 0$ arbitrary large (compare with the case of Besov spaces, (1.1)).
(iii) If we let $p > n/2$ and $1 \leq q \leq \infty$, then we have $\dot{H}^{-1,\frac{n}{2}} \hookrightarrow \dot{F}_{p,q}^{-3+\frac{n}{p}}$. Therefore, Theorems 2.1 and 2.2 include that of Chen[2], provided $p > n/2$, $1 \leq q \leq \infty$, $s = 1$, and $r = n/2$.
(iv) It is seen from Theorem 2.1 with $p = n, q = 2$ that a small external force f in $\dot{H}^{-2,n} \cong \dot{F}_{n,2}^{-2}$ yields an unique solution $u \in L^n \cong \dot{F}_{n,2}^{0}$ of (E). Moreover, if this f also belongs to L^n, then it holds from Theorem 2.2 with $s = 2$ and $r = n$ that u also belongs to $\dot{H}^{2,n}$. Hence u belongs to the inhomogeneous Sobolev space $H^{2,n} = L^n \cap \dot{H}^{2,n}$, which implies that u satisfies the original equation (SNS) almost everywhere in $\mathbb{R}^n$.

Now let us discuss the well-posedness problem on (SNS). Suppose that E and S are spaces such that either $(E,S) = (\dot{B}_{p,q}^{-3+\frac{n}{p}}, P\dot{B}_{p,q}^{-1+\frac{n}{p}})$ with $1 \le p < n$ and $1 \le q \le \infty$, or $(E,S) = (\dot{F}_{p,q}^{-3+\frac{n}{p}}, P\dot{F}_{p,q}^{-1+\frac{n}{p}})$ under the condition of Theorem 2.1. In addition, let $B_E(\delta) \equiv \{f \in E; \|f\|_E < \delta\}$ and $B_S(\eta) \equiv \{u \in S; \|u\|_S < \eta\}$ with small $\delta, \eta > 0$. Then by Kaneko-Kozono-Shimizu[5] and Theorem 2.1, we can define the solution map $f \in (B_E(\delta), \|\cdot\|_E) \mapsto u \in (B_S(\eta), \|\cdot\|_S)$, which is actually continuous. However, the following claim holds.

Theorem 2.3. *Let $n \ge 3$, and let $E \equiv \dot{B}_{\infty,1}^{-3}$, $S \equiv P\dot{B}_{\infty,1}^{-1}$. Then for any $\delta > 0$ and $\eta > 0$, there exists a constant $\varepsilon > 0$ and a sequence $\{f_N\}_{N=1}^{\infty} \subset BUC^2 \cap B_E(\delta)$ of external forces satisfying both* (i) *and* (ii) *as follows*:

(i) $\|f_N\|_S \to 0$, *as* $N \to \infty$,

(ii) *For each f_N, there exists a solution $u_N \in BUC^2 \cap B_S(\eta)$ of* (rSNS), *which also satisfies the original equation* (SNS) *pointwise with a constant pressure π, i.e., there holds*

$$\begin{cases} -\Delta u_N(x) + (u_N \cdot \nabla u_N)(x) = f_N(x) \quad \forall x \in \mathbb{R}^n, \\ (\nabla \cdot u)(x) = 0 \quad \forall x \in \mathbb{R}^n. \end{cases}$$

Moreover, it holds that $\|u_N\|_{\dot{B}_{\infty,\infty}^{-1}} > \varepsilon$ for every $N \in \mathbb{N}$.

Remark 2.2. (i) Theorem 2.3 shows the ill-posedness of (SNS) from $\dot{B}_{\infty,q}^{-3}$ to $\dot{B}_{\infty,q}^{-1}$ for all $1 \le q \le \infty$ in the sense that the solution map $f \in \dot{B}_{\infty,q}^{-3} \mapsto u \in \dot{B}_{\infty,q}^{-1}$ is, even if it is well-defined, not continuous in each norm. We should note here that the solution u above is not necessarily unique one.
(ii) Since $\dot{B}_{\infty,1}^{-3} \hookrightarrow \dot{F}_{\infty,1}^{-3}$ and $\dot{B}_{\infty,\infty}^{-1} \cong \dot{F}_{\infty,\infty}^{-1}$, Theorem 2.3 also holds for Triebel-Lizorkin spaces with the same indices.
(iii) Combining this result with that of Kaneko-Kozono-Shimizu[5], we can find that an open problem is whether or not (SNS) is well-posed from $\dot{B}_{p,q}^{-3+\frac{n}{p}}$ to $\dot{B}_{p,q}^{-1+\frac{n}{p}}$ for $n \le p < \infty$.

3 Outline of the proof of Theorem 2.1-2.2

For the proof of Theorem 2.1-2.2, it suffices to show four lemmata as follows.

Lemma 3.1. *Let $n \ge 2$, $s \in \mathbb{R}$ and let $1 \le p, q \le \infty$. Then the operator $L \equiv (-\Delta)^{-1}P$ is bounded from $\dot{F}_{p,q}^{s-2}$ onto $P\dot{F}_{p,q}^{s}$ with the estimate*

$$\|Lf\|_{\dot{F}_{p,q}^{s}} \le C\|f\|_{\dot{F}_{p,q}^{s-2}},$$

where $C = C(n,s,p,q)$ is a constant.

Lemma 3.2. *Let $n \geq 2$, $s \in \mathbb{R}$, and let $1 \leq p, q \leq \infty$. Then the operator $K \equiv -(-\Delta)^{-1}P\nabla\cdot$ is bounded from $\dot{F}^{s-1}_{p,q}$ onto $P\dot{F}^{s}_{p,q}$ with the estimate*

$$\|Kg\|_{\dot{F}^{s}_{p,q}} \leq C\|g\|_{\dot{F}^{s-1}_{p,q}},$$

where $C = C(n, s, p, q)$ is a constant.

Lemma 3.3. *Let $n \geq 3$. Suppose that there holds either $1 < p < n$, $1 \leq q \leq \infty$, or $p = n$, $1 \leq q \leq 2$. Then for $u, v \in \dot{F}^{-1+\frac{n}{p}}_{p,q}$, we have $u \otimes v \in \dot{F}^{-2+\frac{n}{p}}_{p,q}$ with the estimate*

$$\|u \otimes v\|_{\dot{F}^{-2+\frac{n}{p}}_{p,q}} \leq C\|u\|_{\dot{F}^{-1+\frac{n}{p}}_{p,q}}\|v\|_{\dot{F}^{-1+\frac{n}{p}}_{p,q}},$$

where $C = C(n, p, q)$ is a constant.

Lemma 4.4. *Let $n \geq 2$, and suppose that p, q, r, and s satisfy either* (i), (ii), *or* (iii) *of Theorem 2.2. Then for $u, v \in \dot{F}^{-1+\frac{n}{p}}_{p,q} \cap \dot{H}^{s,r}$, we have $u \otimes v \in \dot{H}^{s-1,r}$ with the estimate*

$$\|u \otimes v\|_{\dot{H}^{s-1,r}} \leq C\left(\|u\|_{\dot{F}^{-1+\frac{n}{p}}_{p,q}}\|v\|_{\dot{H}^{s,r}} + \|u\|_{\dot{H}^{s,r}}\|v\|_{\dot{F}^{-1+\frac{n}{p}}_{p,q}}\right),$$

where $C = C(n, s, p, q, r)$ is a constant.

We can show Lemma 3.1-3.2 by the isomorphism $(-\Delta)^{\frac{s}{2}} : \dot{F}^{s_0}_{p,q} \to \dot{F}^{s_0-s}_{p.q}$, and by the boundedness of the Riesz transform $R_j = \frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}}$ $(j = 1, 2, \ldots, n)$ from $\dot{F}^{s}_{p,q}$ onto itself for any $s \in \mathbb{R}$ and $1 \leq p < \infty$, $1 \leq q \leq \infty$. Indeed, since

$$\varphi_j * f = (\varphi_{j-1} + \varphi_j + \varphi_{j+1}) * \varphi_j * f, \ \forall j \in \mathbb{Z},$$

and since $R_k\varphi_j(x) = 2^{nj}R_k\varphi_0(2^j x)$ is in $\mathcal{S}$, we can see the boundedness of R_k on $\dot{F}^{s}_{p,q}$ by the theory of vector-valued maximal functions. On the other hand, we can prove Lemma 3.3-3.4 by the following propositions.

Proposition 3.1. (Jawerth[4]) (1) *Let $s_1 > s_2$, and let $1 \leq p_1 < p_2 < \infty$, $1 \leq q, r \leq \infty$. If $s_1 - n/p_1 = s_2 - n/p_2$, then there holds*

$$\dot{F}^{s_1}_{p_1,q} \hookrightarrow \dot{F}^{s_2}_{p_2,r}.$$

(2) *Let $s \in \mathbb{R}$, and let $1 < p < \infty$. Then there holds $\dot{F}^{s}_{p,2} \cong \dot{H}^{s.p}$.*

Proposition 3.2. (Kozono-Shimada[6]) *Let $s, \alpha > 0$, $1 < p < \infty$, and let us take $1 < p_1, p_2 < \infty$ so that $1/p = 1/p_1 + 1/p_2$. Then there is a constant $C = C(s, \alpha, p, p_1, p_2)$ such that for every $f, g \in \dot{F}^{s+\alpha}_{p_1,\infty} \cap \dot{F}^{-\alpha}_{p_2,\infty}$, there holds $f \cdot g \in \dot{F}^{s}_{p,\infty}$ with the estimate*

$$\|f \cdot g\|_{\dot{F}^{s}_{p,\infty}} \leq C\left(\|f\|_{\dot{F}^{s+\alpha}_{p_1,\infty}}\|g\|_{\dot{F}^{-\alpha}_{p_2,\infty}} + \|f\|_{\dot{F}^{-\alpha}_{p_2,\infty}}\|g\|_{\dot{F}^{s+\alpha}_{p_1,\infty}}\right).$$

Using Lemma 3.1-3.4, we can show Theorem 2.1-2.2 by a similar method to that of Kaneko-Kozono-Shimizu[5]. We should note here that in Besov spaces, Proposition 3.1 (1) holds only if $q \leq r$, and (2) does not hold. This difference seems to cause that of assumptions for the results of additional regularity, (1.1) and (i)-(iii) of Theorem 2.2.

4 Outline of the proof of Theorem 2.3

We take a parametrized external force as

$$f_{Q,r}(x) \equiv Qr^2 \{e_2 \cos(rx_1) + e_3 \cos(rx_1 - x_2)\}, \quad x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n,$$

where $e_2 \equiv (0,1,0,0,\dots,0)$ and $e_3 \equiv (0,0,1,0,\dots,0)$ are unit vectors in $\mathbb{R}^n$, while $Q > 0$ and $r \in \mathbb{N}$ are parameters. This function is similar to the parameterized initial data proposed by Bourgain[1] and Sawada[9] on the topic of ill-posedness of non-stationary Navier-Stokes equations. Following the classical methods, we define the approximative sequence $\{u_j\}_{j\in\mathbb{N}}$ to the solution u of (rSNS) as

$$\begin{cases} u_1 \equiv Lf_{Q,r}, \\ u_j \equiv u_1 + K(u_{j-1} \otimes u_{j-1}), \quad j \geq 2. \end{cases}$$

Moreover, we rewrite these u_j as forms of series in accordance with Sawada[9]. Let

$$\begin{cases} v_1 \equiv u_1, \\ v_2 \equiv K(u_1 \otimes u_1) = K(v_1 \otimes v_1), \\ v_k \equiv K(u_{k-1} \otimes u_{k-1}) - K(u_{k-2} \otimes u_{k-2}), \quad k \geq 3. \end{cases} \tag{4.1}$$

Obviously, it holds

$$u_j = \sum_{k=1}^{j} v_k, \quad j \geq 1. \tag{4.2}$$

As for $f_{Q,r}$ and $\{v_k\}_{k\in\mathbb{N}}$, we can obtain the following estimates by easy calculation.

Proposition 3.3. *There exists a constant $C = C(n) > 0$ such that*

(i) $v_k \in BUC^2 \cap \dot{B}^{-1}_{\infty,1}, \ \nabla \cdot v_k = 0, \quad \forall k \geq 1,$

(ii) $\|f_{Q,r}\|_{\dot{B}^{-3}_{\infty,1}}, \|v_1\|_{\dot{B}^{-1}_{\infty,1}} \leq C\dfrac{Q}{r},$

(iii) $C^{-1}Q^2 \leq \|v_2\|_{\dot{B}^{-1}_{\infty,\infty}} \leq \|v_2\|_{\dot{B}^{-1}_{\infty,1}} \leq CQ^2, \quad if \ \ r \gg Q,$

(iv) $\|v_k\|_{L^\infty}, \|v_k\|_{\dot{B}^{-1}_{\infty,1}} \leq CQ^2 \left(\dfrac{Q}{r}\right)^{k-2}, \quad \forall k \geq 3, \quad if \ \ r \gg Q.$

Hence, it holds $f_{Q,r} \to 0$ in $\dot{B}^{-3}_{\infty,1}$ as $r \to \infty$ for each fixed $Q > 0$. Moreover, by fixing the parameters as $r \gg Q$, we can see that there exists a limit function $u_{Q,r} = \lim_{j\to\infty} u_j = \sum_{k=1}^{\infty} v_k$ in $BUC^2 \cap \dot{B}^{-1}_{\infty,1}$. Actually, this $u_{Q,r}$ satisfies (ii) of Theorem 2.3 provided $Q \ll \eta$ and $N = r \gg Q$. Indeed, we can see from Proposition 3.3 that there exists a constant $\varepsilon > 0$ such that $\|u_{Q,r}\|_{\dot{B}^{-1}_{\infty,\infty}} > \varepsilon$ for every $r \gg Q$. Furthermore, there holds

$$\lim_{j\to\infty} K(u_j \otimes u_j) = K(u_{Q,r} \otimes u_{Q,r}) \text{ in } L^\infty \text{ and } \dot{B}^{-3}_{\infty,1}$$

by the theorem of termwise differentiation .

Acknowledgements. The author are grateful to Prof. Katsuo Matsuoka, Nihon University, for kind invitation to the conference of RIMS and giving the opportunity to make a presentation there. The author would also like to thank Prof. Yoshihiro Sawano, Tokyo Metropolitan University, for advice and comments on the boundedness of Riesz transforms on Triebel-Lizorkin spaces.

References

[1] J. Bourgain, N. Pavlović: J. Funct. Anal. **255**, pp.2233-2247 (2008)

[2] Z. Chen: Pacific J. Math. **158**, pp.293-303 (1993)

[3] J. G. Heywood: Arch. Rational. Mech. Anal. **37**, pp.48-60 (1970)

[4] B. Jawerth, Math. Scand. **40**, pp.94104. (1977)

[5] K. Kaneko, H. Kozono, S. Shimizu: to appear in Indiana Univ. Math. Journal.

[6] H. Kozono, Y. Shimada: Math. Nachr. **276**, pp.6374 (2004)

[7] O. A. Ladyzhenskaya: Uspehi Mat. Nauk **14**, pp.57-97 (1959)

[8] J. Leray: J. Math. Pures Appl. **12**, pp.1-82 (1933)

[9] O. Sawada: Harmonic analysis and nonlinear partial differential equations, RIMS Kôkyûroku Bessatsu, **B33**, pp.59-85. Res. Inst. Math. Sci. (RIMS), Kyoto (2012)

Department of Mathematics, Faculty of Science and Engineering
Waseda University
Tokyo 169-8555
Japan
E-mail adress: bf-hanpan@fuji.waseda.jp