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MONOTONICITY ESTIMATE AND GLOBAL EXISTENCE FOR THE P-HARMONIC FLOW

(p 調和写像流に対する単調性評価と大域存在)

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1 Introduction

Let $\mathcal{N}$ be a $n-$dimensional smooth compact Riemannian manifold without boundary and isometrically embedded in $\mathbb{R}^l$ $(l>n)$. For a map u from $\mathbb{R}^m_\infty := (0,\infty)\times\mathbb{R}^m$ to $\mathbb{R}^l$ we consider the $p-$harmonic flow

$$\begin{cases} \partial_t u - \operatorname{div}\left(|Du|^{p-2}Du\right) + |Du|^{p-2}A(u)(Du,Du) = 0 \\ u \in \mathcal{N} \end{cases}$$

where $p\geq 2$, $u(t,x) = (u^i(t,x))$, $i=1,\ldots,l$, is a vector-valued function, defined for $(t,x)\in\mathbb{R}^m_\infty$ with values into $\mathbb{R}^l$. $D_\alpha = \partial/\partial x_\alpha$, $\alpha=1,\ldots,m$, $Du = (D_\alpha u^i)$ is the spatial gradient of a map u, $|Du|^2 = \sum_{\alpha=1}^m\sum_{i=1}^l (D_\alpha u^i)^2$ and $\partial_t u$ is the derivative on time t. The second fundamental form $A(u)(Du,Du)$ of $\mathcal{N}\subset\mathbb{R}^l$ is on the orthogonal complement of the tangent space $\mathcal{T}_u\mathcal{N}$ (if necessary, the manifold $\mathcal{N}$ is assumed to be orientable). Since $u=u(t,x)$, $(t,x)\in\mathbb{R}^m_\infty$, moves on the manifold $\mathcal{N}$, $\partial_t u\in\mathcal{T}_u\mathcal{N}$, and thus, $\partial_t u\cdot A(u)(Du,Du)=0$ and, by multiplying the equation by $\partial_t u$ and the divergence theorem

$$|\partial_t u|^2 - \operatorname{div}(|Du|^{p-2}Du\cdot\partial_t u) + \partial_t\frac{1}{p}|Du|^p = 0,$$

$$E(u) := \int_{\mathbb{R}^m}\frac{1}{p}|Du|^p\,dx, \quad \frac{d}{dt}E(u(t)) = -\|\partial_t u(t)\|_2^2$$

and thus, $E(u(t))\searrow 0$ and $u(t)$ may converge to a constant map as $t\nearrow\infty$.

Theorem 1 (A global existence and regularity for the $p-$harmonic flow) *Let $p>2$ and let u_0 be a smooth map defined on $\mathbb{R}^m$ with values to $\mathcal{N}$, satisfying $E(u_0)<\infty$. Then, there exists a global weak solution u of the Cauchy problem for the $p-$harmonic flow with initial data u_0, satisfying the energy inequality*

$$\|\partial_t u\|^2_{L^2(\mathbb{R}^m_\infty)} + \sup_{0<t<\infty} E(u(t)) \leq E(u_0).$$

Moreover, the solution u is partial regular in the following sense : For any positive number γ_0, $2<\gamma_0<p$, there exists a relatively closed set $\mathcal{S}$ in $\mathbb{R}^m_\infty$ such that u and its gradient Du are locally in time-space continuous in the complement $\mathbb{R}^m_\infty\setminus\mathcal{S}$, and the size of $\mathcal{S}$ is also estimated by the Hausdorff measure : The set $\mathcal{S}$ is of at most locally zero $m-$dimensional Hausdorff measure with respect to the time-space metric $|t|^{1/\gamma_0}+|x|$, and, furthermore, for any positive time $\tau<\infty$, the $(m-\gamma_0)-$dimensional Hausdorff measure of $\{\tau\}\times\mathcal{S}$ with respect to the usual Euclidean metric is locally zero.

Remark. The exponent γ_0 can be as close to p as possible.

In this note we report on the global existence of a partial regular weak solution of the Cauchy problem for $p-$harmonic flow. We use the so-called penalty approximating equation for the $p-$harmonic flow, and devise new monotonicity type formulas of a local scaled

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energy and establish a uniform local regularity estimate for regular solutions of those equation. The regularity criterion obtained is almost optimal, comparing with that of the corresponding stationary case.

2 Penalty approximation

In this section we explain the approximation scheme for the $p-$harmonic flow. We will approximate the $p-$harmonic flow by the solutions of the gradient flow for the so-called penalized functional, introduced in [3] for the harmonic flow case $p = 2$.

Since the manifold $\mathcal{N}$ is smooth and compact, there exists a tubular neighborhood $\mathcal{O}_{2\delta_{\mathcal{N}}}$ with width $2\delta_{\mathcal{N}}$ of $\mathcal{N}$ in $\mathbb{R}^l$ such that any point $u \in \mathcal{O}_{2\delta_{\mathcal{N}}}$ has a unique nearest point $\pi_{\mathcal{N}}(u) \in \mathcal{N}$ satisfying $\text{dist}\,(u, \mathcal{N}) = |u - \pi_{\mathcal{N}}(u)|$ for the Euclidean distance $|\cdot|$ in $\mathbb{R}^l$, where the projection $\pi_{\mathcal{N}} : \mathcal{O}_{2\delta_{\mathcal{N}}} \to \mathcal{N}$ is smooth, since the manifold $\mathcal{N}$ is smooth. The distance function $\text{dist}(u, \mathcal{N})$ is Lipschitz continuous on $u \in \mathcal{O}_{2\delta_{\mathcal{N}}}$.

Let χ be a smooth, non-decreasing real-valued function defined on $[0, \infty)$ such that $\chi(s) = s$ for $s \le (\delta_{\mathcal{N}})^2$ and $\chi(s) = 2(\delta_{\mathcal{N}})^2$ for $s \ge 4(\delta_{\mathcal{N}})^2$. Then, the function $\chi\left(\text{dist}^2(u, \mathcal{N})\right)$ is smooth on $u \in \mathbb{R}^l$ (for the proof we refer to the recent study of the squared distance function to manifold, due to Ambrosio et al. [1, Theorem 2.1]). Its gradient at $u \in \mathcal{O}_{2\delta_{\mathcal{N}}}$ is computed as

$$\begin{aligned} D_u\chi\left(\text{dist}^2(u, \mathcal{N})\right) &= 2\chi'\left(\text{dist}^2(u, \mathcal{N})\right)\text{dist}(u, \mathcal{N})D_u\text{dist}(u, \mathcal{N}) \quad ; \\ D_u\text{dist}(u, \mathcal{N}) &= \frac{u - \pi_{\mathcal{N}}(u)}{|u - \pi_{\mathcal{N}}(u)|} \end{aligned}$$

parallel to the vector field $u - \pi_{\mathcal{N}}(u)$ and orthogonal to $\mathcal{T}_{\pi_{\mathcal{N}}(u)}\mathcal{N}$. We also have that, for any $u \in \mathcal{N}$ and any tangent vector $\tau \in \mathcal{T}_u\mathcal{N}$,

$$\left|\tau^i\tau^j D_{u^i} D_{u^j}\text{dist}(u, \mathcal{N})\right| \le C(\mathcal{N})|\tau|^2$$

(See [1, Theorem 2.2]).

For positive parameters $1 \le K \nearrow \infty$ and $1 > \epsilon \searrow 0$, we consider the Cauchy problem in $\mathbb{R}^m_\infty$ with initial data u_0 for the gradient flow, called the *penalized equation*,

$$(2.1) \qquad \begin{cases} \partial_t u - \Delta_{p,\epsilon} u + C_0\, K\,\chi'\left(\text{dist}^2(u, \mathcal{N})\right)\text{dist}(u, \mathcal{N})D_u\text{dist}(u, \mathcal{N}) = 0 \\ u(0) = u_0 \end{cases}$$

associated with the *penalized functional*, defined by

$$(2.2) \qquad F_{K,\epsilon}(u) := E_\epsilon(u) + C_0\,\frac{K}{2}\int_{\mathbb{R}^m}\chi\left(\text{dist}^2(u, \mathcal{N})\right)\,dx,$$

where the positive constant C_0 will be stipulated later, depending only on p, m and $\mathcal{N}$ (See Lemma 8). The partial differential operator $\Delta_{p,\epsilon}$ and its corresponding energy, called the regularized $p-$Laplace operator and the regularized $p-$energy, respectively, are defined as

(2.3)

$$\Delta_{p,\epsilon} u := \text{div}\left(\left(\epsilon + |Du|^2\right)^{\frac{p-2}{2}} Du\right) \quad ; \quad E_\epsilon(u) := \int_{\mathbb{R}^m}\frac{1}{p}\left(\epsilon + |Du|^2\right)^{\frac{p}{2}}\,dx.$$

We have the global existence for (2.1), by the usual Galerkin method and monotonicity of the $p-$Laplace operator (refer to [2]). The regularity of solutions are obtained from Hölder regularity estimates for the evolutionary $p-$Laplace operator, with a boundedness of the derivative of the penalty term, the last term in (2.1).

Lemma 2 (Existence for the penalty approximation) *Let $p > 2$ and let u_0 be a smooth map defoned on $\mathbb{R}^m$ with values to $\mathcal{N}$, satisfying $E(u_0) < \infty$. For each positive numbers K and ϵ, there exists a weak solution $u = u_{K,\epsilon}$ of the Cauchy problem for the penalized equation (2.1) such that $u = u_{K,\epsilon}$ satisfies the energy inequality*

$$\|\partial_t u\|^2_{L^2(\mathbb{R}^m_\infty)} + \sup_{0<t<\infty} F_{K,\epsilon}(u) \le E_\epsilon(u_0) \tag{2.4}$$

and, that u, Du, $\partial_t u$ and $D^2 u$ are locally (Hölder) continuous on time and space (with some Hölder exponent) in $\mathbb{R}^m_\infty$ and u satisfies the penalized equation everywhere in $\mathbb{R}^m_\infty$.

We will call a solution having the regularity properties as in Lemma 2, a regular solution.

3 Uniform regularity estimate

In this section we show some regularity estimates for solutions $u = u_{K,\epsilon}$ of the penalized equations (2.1).

Lemma 3 (Energy inequality) *Let u_0 be a smooth map on $\mathbb{R}^m$ with values to $\mathcal{N}$, satisfying $E(u_0) < \infty$, and $u = u_{K,\epsilon}$ be a regular solution of (2.1). Then, (2.4) holds.*

Proof. The energy inequality (2.4) is shown to be valid in the proof of Lemma 2. However, as a priori estimates for regular solutions of (2.1), we naturally multiply (2.1) by $\partial_t u$ and integrate by parts on space variable in $\mathbb{R}^m_T$ for any $T > 0$. □

Lemma 4 (Boundedness) *Let $u = u_{K,\epsilon}$ be a regular solution of (2.1). Then it holds that $\sup_{\mathbb{R}^m_\infty} |u| \le H$, where the positive number H is so large that $B(H) \supset \mathcal{O}_{2\delta_\mathcal{N}}(\mathcal{N})$ in $\mathbb{R}^l$, where $B(H) = B(H, 0)$ is a ball in $\mathbb{R}^l$ of radius H with center of origin 0.*

Proof. We multiply (2.1) by $u(|u|^2 - H^2)_+$ and integrate in $\mathbb{R}^m_\infty$, where $(f)_+$ is the positive part of a function f. Since the support of χ' is in $\mathcal{O}_{2\delta_\mathcal{N}}(\mathcal{N}) \subset B(H)$, $\chi'(\mathrm{dist}^2(u, \mathcal{N}))$ is zero in $\mathbb{R}^l \setminus B(H)$. Also $u_0 \in \mathcal{N} \subset B(H)$. Hence, we have

$$\begin{aligned}
&\frac{1}{4}\int_{\mathbb{R}^m} \left(|u(t)|^2 - H^2\right)_+ dx \\
&\qquad + \int_{\mathbb{R}^m_t} \left(\epsilon + |Du|^2\right)^{\frac{p-2}{2}} \left(\frac{1}{2}\left|D\left(|u|^2 - H^2\right)_+\right|_g^2 + |Du|^2\left(|u|^2 - H^2\right)_+\right) dz \ = 0 \quad ; \\
&\frac{1}{4}\int_{\mathbb{R}^m} \left(|u(t)|^2 - H^2\right)_+^2 dx \le 0
\end{aligned}$$

and thus, $|u(t)| \le H$ in $\mathbb{R}^m$ and any $t \ge 0$. □

The partial regularity is based on the so-called *small energy regularity estimate* (refer to [9, Theorems 5.1, 5.3, 5.4 ; their proofs, pp. 491-494]). The small energy regularity estimate for the $p-$harmonic flow in the case $p > 2$ has been recently established in [7, 8]. Our main assertion here is that the small energy regularity estimate holds uniformly for solutions of the penalized equations.

Let us denote the penalized energy density for a map u by

$$e_{K,\epsilon}(u) := \frac{1}{p}\left(\epsilon + |Du|^2\right)^{\frac{p}{2}} + \frac{K}{2}\chi\left(\mathrm{dist}^2(u, \mathcal{N})\right) . \tag{3.1}$$

Theorem 5 (Small energy regularity estimate) *Let $p > 2$. Let λ_0, B_0 and a_0 be positive numbers satisfying the conditions*

$$\frac{6p-4}{p+2} < \lambda_0 = B_0 < p \quad ; \quad \frac{\lambda_0 - 2}{p-2} < a_0 \leq 1. \tag{3.2}$$

Let $u = u_{K,\epsilon}$ be a regular solution of (2.1) on $\mathbb{R}^m_T = (0, T) \times \mathbb{R}^m$ for a positive $T < \infty$, satisfying the energy bound

$$\|\partial_t u\|^2_{L^2(\mathbb{R}^m_T)} + \sup_{0<t<T} F_{K,\epsilon}(u) \leq C \tag{3.3}$$

for a positive number C depending only on m, p and $\mathcal{N}$. Then, there exists a small positive numeber $R_0 < 1$, depending only on m, $\mathcal{N}$, p, B_0 and a_0, and the following holds true : Let γ_0 be any positive number satisfying

$$2 < \gamma_0 < \frac{B_0(p+2) - 4p}{p-2}.$$

If, for some small positive $R < \min\{R_0, T^{1/\lambda_0}\}$,

$$\limsup_{r \searrow 0} r^{\gamma_0 - m} \int_{\{t = T - R^{\lambda_0}\} \times B(r,0)} e_{K,\epsilon}(u(t, x))\, dx \leq 1 \tag{3.4}$$

then, there holds

$$\sup_{(T-(R/4)^{\lambda_0}, T) \times B(R/4, 0)} e_{K,\epsilon}(u(t, x)) \leq C\, R^{-a_0 p}, \tag{3.5}$$

where the positive constant C depends only on γ_0, λ_0, B_0, a_0, p, m and $\mathcal{N}$.

Remark. The positive number γ_0 can be as close to p as possible, if B_0 is close to p.

The novelty here is a new *monotonicity* type estimate of a *localized* scaled energy, which may be of its own interest. Let us define our localized scaled energy in the following way: Let $T \geq 0$ and $X \in \mathbb{R}^m$ be given, and (t_0, x_0) in the parabolic like envelope

$$\{(t, x) \in \mathbb{R}^m_\infty : t - T \geq |x - X|^{\lambda_0}\} \quad ; \quad \lambda_0 > 2.$$

Hereafter the notation of double sign correspondence is used. The localized scaled energy is defined by

$$\begin{aligned} E_\pm(r) &= \frac{1}{\Lambda^p} \int_{\{t = t_0 \pm \Lambda^{2-p} r^2\} \times \mathbb{R}^m} \bar{e}_{K,\epsilon}(u(t, x))\, \mathcal{B}_\pm(t_0, x_0; t, x)\, \mathcal{C}^q(t, x)\, dx \quad ; \\ \bar{e}_{K,\epsilon}(u) &:= \frac{1}{p}\left(\epsilon + |Du|^2\right)^{\frac{p}{2}} + C_0\, \frac{K}{2} \chi\left(\mathrm{dist}^2(u, \mathcal{N})\right) \end{aligned} \tag{3.6}$$

and $\Lambda = \Lambda(r)$ is a function of a scale radius r, defined as

$$\Lambda = \Lambda(r) = r^{\frac{B_0 - 2}{2-p}} \quad ; \quad B_0 > \frac{6p-4}{p+2} \tag{3.7}$$

for any $r > 0$. The *forward* or *backward* in time Barenblatt like function, denoted by $\mathcal{B}_+$ and $\mathcal{B}_-$, respectively, are defined by

$$\mathcal{B}_\pm(t_0, x_0\,; t, x) = \frac{1}{(\mp t_0 \pm t)^{\frac{m}{B_0}}} \left(1 - \left(\frac{|x - x_0|}{2\,(\mp t_0 \pm t)^{\frac{1}{B_0}}}\right)^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}, \quad \mp t < \mp t_0. \tag{3.8}$$

The localized function $\mathcal{C}$ is defined and used as

$$\mathcal{C}(t,x) := \Big((t-T)^{1/\lambda_0} - |x-X|\Big)_+ \quad ; \quad q > 2. \tag{3.9}$$

We call $E_+(r)$ and $E_-(r)$ the forward and backward localized scaled $p-$energy, respectively.

Our main ingredient is the following monotonicity type estimate of a scaled energy.

Lemma 6 (Monotonicity estimate for the backward localized scaled $p-$energy) *Let $p > 2$ and $q > 2$. Suppose that $t_0 - T \leq 1$. For any regular solution to (2.1) the following estimate holds for all positive numbers r, ρ, $r^{B_0} = \Lambda(r)^{2-p} r^2 < \rho^{B_0} = \Lambda(\rho)^{2-p}\rho^2 \leq \min\{1,\, (t_0-T)/2\}$,*

$$\begin{aligned} E_-(r) \quad &\leq \quad E_-(\rho) + C\,(\rho^\mu - r^\mu) \\ &+ C \int_{t_0-\rho^{B_0}}^{t_0-r^{B_0}} \|\mathcal{C}^{\tilde q}(t)\, \bar e_{K,\epsilon}(u(t))^2\|_{L^\infty\left(B((t_0-t)^{1/B_0},\, x_0)\right)}\, dt, \end{aligned} \tag{3.10}$$

where $\tilde q = \min\{q-2,\, q(p-1)/p\}$, B_0 as in (3.7), and the positive exponent μ depends only on $\mathcal{N}$, m, p and B_0, and the positive constant C depends only on the same ones as μ and q.

Lemma 7 (Monotonicity estimate for the forward localized scaled $p-$energy) *Let $p > 2$ and $q > 2$. Suppose that $t_0 - T \leq 1$. For any regular solution to (2.1) the following estimate holds for all positive numbers r, ρ, $r^{B_0} = \Lambda(r)^{2-p} r^2 < \rho^{B_0} = \Lambda(\rho)^{2-p}\rho^2 \leq 1$*

$$\begin{aligned} E_+(\rho) \quad &\leq \quad \left(1 + r^{-c_0\, B_0}\right) E_+(r) + C\,(\rho^\mu - r^\mu) \\ &+ C \int_{t_0+r^{B_0}}^{t_0+\rho^{B_0}} \|\mathcal{C}^{\tilde q}(t)\, \bar e_{K,\epsilon}(u(t))^2\|_{L^\infty\left(B((t-t_0)^{1/B_0},\, x_0)\right)}\, dt, \end{aligned} \tag{3.11}$$

where c_0 is a positive number satisfying $c_0 > 2(p-B_0)/B_0(p-2)$, which can be as close to $2(p-B_0)/B_0(p-2)$ as possible, $\tilde q = \min\{q-2,\, q(p-1)/p\}$, B_0 as in (3.7), and the positive constants μ and C have the same dependence as those in Lemma 6.

Remark. In Lemma 7, the positive number c_0 can be as close to 0 as possible, if B_0 is close to p.

We need the so-called Bochner type estimate for the penalized energy density. Here the positive constant C_0 in (2.1) is appropriately chosen.

Lemma 8 (Bochner type estimate) *Let $p > 2$ and $u = u_{K,\epsilon}$ be a regular solution to (2.1). For brevity, put $e(u) = e_{K,\epsilon}(u)$. Then, it holds in $\mathbb{R}^m_\infty$ that*

$$\begin{aligned} &\partial_t e(u) - \sum_{\alpha,\,\beta=1}^{m} D_\alpha\left(\left(\epsilon + |Du|^2\right)^{\frac{p-2}{2}} \mathcal{A}^{\alpha\beta} D_\beta e(u)\right) \\ &\qquad + C_1\left(\epsilon + |Du|^2\right)^{\frac{p-2}{2}} \left|D^2 u\right|^2 + C_2 \left|2^{-1} K\, D_u \chi\Big(\mathrm{dist}^2(u,\mathcal{N})\Big)\right|^2 \\ &\leq C_3\left(1 + e(u)^{\frac{2}{p}}\right) e(u)^{2\left(1-\frac{1}{p}\right)}, \end{aligned} \tag{3.12}$$

where

$$\mathcal{A}^{\alpha\beta} := \delta^{\alpha\beta} + (p-2)\frac{D_\alpha u \cdot D_\beta u}{\epsilon + |Du|^2},$$

the positive constants C_i $(i = 1,2,3)$ depend on m, p and $\mathcal{N}$.

4 Passing to the limit

In this section we present the proof of Theorem 1, based on Theorem 5.

Let $\{\epsilon_k\}$ and $\{K_k\}$ be sequences such that $\epsilon_k \searrow 0$ and $K_k \nearrow \infty$ as $k \to \infty$. Let u_{K_k,ϵ_k}, $k = 1, 2, \ldots$, be a sequence of solutions of the Cauchy problem with initial data u_0 for the penalized equations (2.1) with approximating numbers $\epsilon = \epsilon_k$ and $K = K_k$, obtained in Lemma 2. Hereafter we put $u_k = u_{K_k,\epsilon_k}$ $e_k(u_k) = e_{K_k,\epsilon_k}(u_{K_k,\epsilon_k})$, for brevity.

By the energy inequality (2.4), there exist a subsequence of $\{u_k\}$, denoted by the same notation, and the limit map u such that, as $k \to \infty$,

$$\text{(4.1)} \qquad u_k \longrightarrow u \quad \text{weakly} * \text{ in } L^\infty\Big(0, \infty\,;\, W^{1,p}(\mathbb{R}^m, \mathbb{R}^l)\Big),$$

$$\text{(4.2)} \qquad \partial_t u_k \longrightarrow \partial_t u \quad \text{weakly in } L^2\left(\mathbb{R}^m_\infty, \mathbb{R}^l\right),$$

$$\text{(4.3)} \qquad Du_k \longrightarrow Du \quad \text{weakly in } L^p_{\text{loc}}\left(\mathbb{R}^m_\infty, \mathbb{R}^{ml}\right),$$

$$\text{(4.4)} \qquad \chi(\text{dist}^2(u_k, \mathcal{N})) \longrightarrow 0 \quad \text{strongly in } L^2_{\text{loc}}\left(\mathbb{R}^m_\infty, \mathbb{R}^l\right),$$

$$\text{(4.5)} \qquad u_k \longrightarrow u \quad \text{strongly in } L^q_{\text{loc}}\left(\mathbb{R}^m_\infty, \mathbb{R}^l\right) \text{ for any } q,\ 1 \le q < \tfrac{mp}{(m-p)_+},$$

where the strong convergence in (4.5) follows from (4.1) and (4.2) (see [2, Lemma 1.4, p. 28]). Thus, furthermore, for a subsequence $\{u_k\}$ denoted by the same notation,

$$\text{(4.6)} \qquad u_k \longrightarrow u, \quad \text{dist}(u_k, \mathcal{N}) \longrightarrow 0 \quad \text{almost everywhere in } \mathbb{R}^m_\infty.$$

We demonstrate that the limit map u is a *partial regular* weak solution of the $p-$harmonic flow, as in the statement of Theorem 1. The proof is divided to several steps and proceeded.

Size estimate of the singular set Let R_0 be a sufficient small positive number, determined in Theorem 5. For τ, $0 < \tau < \infty$, and R, $0 < R < \min\{R_0, \tau^{1/\lambda_0}\}$, we put two subsets in $\mathbb{R}^m$ as

$$\mathcal{S}(\tau, R) := \Big\{x_0 \in \mathbb{R}^m \,:\, \limsup_{k\to\infty}\Big(\limsup_{r\searrow 0} r^{\gamma_0 - m} \int_{\{t=\tau-R^{\lambda_0}\}\times B(r,x_0)} e_k(u_k(t, x))\,dx\Big) \ge 1\Big\} \quad ;$$

$$\mathcal{T}(\tau, R) := \bigcap_{l=1}^{\infty}\bigcup_{k=l}^{\infty}\Big\{x_0 \in \mathbb{R}^m \,:\, \limsup_{r\searrow 0} r^{\gamma_0 - m} \int_{\{t=\tau-R^{\lambda_0}\}\times B(r,x_0)} e_k(u_k(t, x))\,dx > 1/2\Big\}.$$

(4.7)

From the definition of limit supremum on k and (4.7), we see that, for every τ, $0 < \tau < \infty$, and R, $0 < R < \min\{R_0, \tau^{1/\lambda_0}\}$,

$$\text{(4.8)} \qquad \mathcal{S}(\tau, R) \subset \mathcal{T}(\tau, R).$$

Here we have the estimation of size (see [5, Theorem 2.2 ; its proof, pp. 101-103] for the proof) : It holds that, for every τ, $0 < \tau < \infty$, and R, $0 < R < \min\{R_0, \tau^{1/\lambda_0}\}$,

$$\mathcal{H}^{m-\gamma_0}(\mathcal{T}(\tau, R)) = 0$$

and so, by (4.8),

$$\mathcal{H}^{m-\gamma_0}(\mathcal{S}(\tau, R)) = 0 \quad ; \quad \mathcal{H}^{m-\gamma_0}\Big(\bigcap_{0<R<\min\{R_0,\tau^{1/\lambda_0}\}} \mathcal{S}(\tau, R)\Big) = 0.$$

Let us define the *singular set* as

$$\mathcal{S} = \bigotimes_{0<\tau<\infty} \bigcap_{0<R<\min\{R_0,\,\tau^{1/\lambda_0}\}} \mathcal{S}(\tau,\, R), \tag{4.9}$$

where $\bigotimes_{0<\tau<\infty}$ means the direct product of sets on positive time $\tau < \infty$. Then, for any positive $T < \infty$ and any open set K compactly contained in $\mathbb{R}^m$, letting $K_T = (0,\, T) \times K$, with respect to the time-space metric $|t|^{1/\gamma_0} + |x|$,

$$\mathcal{H}^m\Big(\mathcal{S} \bigcap K_T\Big) = \int_0^T \mathcal{H}^{m-\gamma_0}\Big(\bigcap_{0<R<R_0} \mathcal{S}(\tau,\, R) \bigcap K\Big)\, d\tau = 0.$$

Regularity of the limit map We now show the regularity of limit map u in the complement of $\mathcal{S}$. Let $(t_0,\, x_0)$ be in the complement of $\mathcal{S}$. Thus, there exist a positive $R < \min\{R_0,\, (t_0)^{1/\lambda_0}\}$ and an infinite family $\{u_k\}$ of regular solutions such that

$$\limsup_{r\searrow 0} r^{\gamma_0 - m} \int_{\{t=t_0-R^{\lambda_0}\}\times B(r,\, x_0)} e_k(u_k(t,\, x))\, dx < 1.$$

Then we can apply Theorem 5 for each u_k above to obtain

$$\sup_{\left(t_0-(R/4)^{\lambda_0},\, t_0\right)\times B(R/4,\, x_0)} e(u_k) \le C\, R^{-a_0 p}, \tag{4.10}$$

where the positive constant C depends only on λ_0, B_0, m, p and $\mathcal{N}$.

Now we will show the uniform continuity of $\{u_k\}$ in $Q := \Big(t_0 - (R/8)^{\lambda_0},\, t_0\Big) \times B(R/8,\, x_0)$. For this purpose we will have a local L^2 estimate of derivative of the penalty term. For any smooth function ϕ of compact support in Q, we multiply the Bochner type estimate (3.12) by ϕ^2 and integrate by parts in Q to have, letting $K = K_k$, $u = u_k$ and $e(u) = e_k(u_k)$,

(4.11)

$$\begin{aligned} &\int_Q \phi^2 \left(\frac{C_1}{2}\, (\epsilon + |Du|^2)^{\frac{p-2}{2}}\, \left|D^2 u\right|^2 + \frac{C_2}{2} \left| \frac{K}{2} D_u \chi\big(\mathrm{dist}^2(u,\, \mathcal{N})\big) \right|^2 \right) dz \\ &\le \int_Q \left(\phi\, |\partial_t \phi|\, e(u) + |D\phi|^2 \left(\frac{2p}{C_1}\, e(u) + \frac{2}{C_2}\, e(u)^{\frac{2}{p}} \right) + C_3\, \phi\, 2 \left(1 + e(u)^{\frac{2}{p}}\right) e(u)^{2\left(1-\frac{1}{p}\right)} \right) dz, \end{aligned}$$

where we use the Cauchy inequality in the first inequality.

Let $(t_0,\, x_0) \subset Q$ be any point and $r \le R/8$ be any positive number, and $Q(r) = (t_0 - r^q,\, t_0) \times B(r,\, x_0)$ with $q > 1$. In (4.11) we choose a smooth function ϕ such that $0 \le \phi \le 1$, $\phi = 1$ in $Q(r)$, $\phi = 0$ outside $Q(2r)$, and $|D\phi| \le C/r$ and $|\partial_t \phi| \le C/r^q$. Thus we have, by (4.10),

$$\begin{aligned} &\int_{Q(r)} \left(\frac{C_1}{2}\, (\epsilon + |Du|^2)^{\frac{p-2}{2}}\, \left|D^2 u\right|^2 + \frac{C_2}{2} \left| \frac{K}{2} D_u \chi\big(\mathrm{dist}^2(u,\, \mathcal{N})\big) \right|^2 \right) dz \\ &\le C\, \left(r^m + r^{m+q-2} + r^{m+q}\right) \le C\, r^m. \end{aligned} \tag{4.12}$$

We also need the Poincaré inequality of parabolic type (refer to [6]) : Let $u = u_k$. There exists a positive constant C, depending only on m and p, such that, for any $Q(r) \subset Q$,

$$\begin{aligned} \|u - \bar{u}_{Q(r)}\|^2_{L^2(Q(r))} \le C\, \Big(& r^2\, \|Du\|^2_{L^2(Q(r))} + r^{-m+q-2}\, \big\|\left(\epsilon + |Du|^2\right)^{1/2}\big\|^{2(p-1)}_{L^{p-1}(Q(r))} \\ &+ r^{2q}\, \|2^{-1} K D_u \chi(^2(u,\, \mathcal{N}))\|^2_{L^2(Q(r))}\Big), \end{aligned} \tag{4.13}$$

where $\bar{u}_{Q(r)}$ is the integral mean of u in $Q(r)$.

Substituting (4.10) and (4.12) into (4.13), we have, for any $(t_0, x_0) \subset Q$, any positive $r \le R/8$, and $Q(r) = (t_0 - r^q, t_0) \times B(r, x_0)$,

$$\|u - \bar{u}_{Q(r)}\|^2_{L^2(Q(r))} \le C\left(r^{m+q+2} + r^{m+3q-2} + r^{m+2q}\right) \tag{4.14}$$

and thus, choosing $q > 1$ in (4.14), we obtain from Campanato's isomorphism theorem (refer to [5]) that $\{u_k\}$ is uniformly Hölder continuous in Q with exponent $\min\{1, q-1, \frac{q}{2}\}$, uniformly on u_k. Thus, we see that $\{u_k\}$ is equicontinuous, and uniformly bounded in Q by Lemma 4. Therefore, by Arzela-Ascoli theorem we find for a subsequence denoted by the same notation $\{u_k\}$ and the limit map u that, as $k \to \infty$,

$$u_k \longrightarrow u \quad \text{uniformly in } Q \tag{4.15}$$

and that the limit map u is uniformly continuous in Q. From (4.10) and (4.15), we obtain that, as $k \to \infty$,

$$\chi(\mathrm{dist}^2(u_k, \mathcal{N})) \le C/K_k \longrightarrow 0 \quad \text{uniformly in } Q \quad \Longrightarrow u \in \mathcal{N} \quad \text{in } Q \tag{4.16}$$

Now we will show that the limit map u satisfies the $p-$harmonic flow equation in Q. From (4.10) and (4.11) we also see that $\left\{(K_k/2)\, D_u\chi\big(\mathrm{dist}^2(u, \mathcal{N})\big|_{u=u_k}\right\}$ is bounded in $L^2(Q, \mathrm{I\!R}^l)$ and thus, there exists a vector-valued function $\nu \in L^2(Q, \mathrm{I\!R}^l)$ such that, as $k \to \infty$,

$$(K_k/2)D_u\chi\big(\mathrm{dist}^2(u, \mathcal{N})\big|_{u=u_k} \longrightarrow \nu \quad \text{weakly in } L^2(Q). \tag{4.17}$$

By the continuity of u in Q the image $u(Q)$ of Q is an open subset of $\mathcal{N}$. Let$\mathcal{P}_{\mathcal{N}}(u(Q))$ be a neighborhood of $u(Q)$ in $\mathcal{N}$. Let $\tau(v)$ be any smooth tangent vector field of $\mathcal{N}$ on $\mathcal{P}_{\mathcal{N}}(u(Q))$, $\tau(v) \in \mathcal{T}_v\mathcal{N}$ for any $v \in \mathcal{P}_{\mathcal{N}}(u(Q))$. By (4.15), we can choose a sufficiently large k_0 such that, for any $k \ge k_0$, $u_k \in \mathcal{O}_{\delta_{\mathcal{N}}}$ in Q, and $\pi_{\mathcal{N}}(u_k) \in \mathcal{P}_{\mathcal{N}}(u(Q)) \subset \mathcal{N}$ and $\tau(\pi_{\mathcal{N}}(u_k)) \in \mathcal{T}_{\pi_{\mathcal{N}}(u_k)}\mathcal{N}$ in Q, where $\mathcal{O}_{\delta_{\mathcal{N}}}$ is a tubular neighborhood of $\mathcal{N}$ with width $\delta_{\mathcal{N}}$, and $\pi_{\mathcal{N}}$ is the nearest point projection to $\mathcal{N}$ from the tubular neighborhood of $\mathcal{N}$. Thus, we have that

$$\begin{aligned} D_u\chi\big(\mathrm{dist}^2(u, \mathcal{N})\big)\big|_{u=u_k} \cdot \tau(\pi_{\mathcal{N}}(u_k)) &= 2\chi'\mathrm{dist}(u_k, \mathcal{N})\; D_u\mathrm{dist}(u, \mathcal{N})|_{u=u_k} \cdot \tau(\pi_{\mathcal{N}}(u_k)) \\ &= 0 \quad \text{in } Q, \end{aligned}$$

because $D_u\mathrm{dist}(u, \mathcal{N})|_{u=u_k}$ is orthogonal to $\mathcal{T}_{\pi_{\mathcal{N}}(u_k))}\mathcal{N}$ for any $z \in Q$, and thus,

$$\int_Q \frac{K_k}{2}\, D_u\chi\big(\mathrm{dist}^2(u, \mathcal{N})\big)\big|_{u=u_k} \cdot \tau(\pi_{\mathcal{N}}(u_k))\, dz = 0. \tag{4.18}$$

By (4.15) and (4.17), we can take the limit as $k \to \infty$ in (4.18) to have, for any smooth tangent vector field $\tau(v)$ of $\mathcal{N}$ on $\mathcal{P}_{\mathcal{N}}(u(Q)) \subset \mathcal{N}$, as $k \to \infty$,

$$\begin{aligned} 0 = \int_Q \frac{K_k}{2}\, D_u\chi\big(\mathrm{dist}^2(u, \mathcal{N})\big)\big|_{u=u_k} \cdot \tau(\pi_{\mathcal{N}}(u_k))\, dz &\longrightarrow \int_Q \nu \cdot \tau(u)\, dz \\ \Longrightarrow \int_Q \nu \cdot \tau(u)\, dz = 0 & \\ \Longleftrightarrow \nu(z) \perp \mathcal{T}_{u(z)}\mathcal{N} \quad \text{for any } z \in Q. & \end{aligned} \tag{4.19}$$

and, thus, $\nu(z)$ is a normal vector field along $u(z)$ for any $z \in Q$. In the weak form of (2.1), for any smooth map ϕ with compact support in Q,

$$\int_Q \left(\partial_t u_k \cdot \phi + \left(\epsilon_k + |Du_k|^2\right)^{\frac{p-2}{2}} Du_k \cdot D\phi + \frac{K_k}{2}\, D_u\chi\big(\mathrm{dist}^2(u, \mathcal{N})\big)\big|_{u=u_k} \cdot \phi\right) dz = 0,$$

we pass to the limit as $k \to \infty$ to find that the limit map u satisfies

$$\int_Q \left(\partial_t u \cdot \phi + |Du|^{p-2} Du \cdot D\phi + \nu \cdot \phi\right) dz = 0, \tag{4.20}$$

where we use the convergence in the 1st line of (4.19) and, the strong convergence of gradients $\{Du_k\}$, obtained from (2.1) with the convergence (4.1), (4.2) and (4.17) (see [2, Theorem 2.1, pp. 31-33]). Therefore, we have that

$$\partial_t u - \Delta_p u + \nu = 0 \quad \text{almost everywhere in } Q \text{ as } L^2(Q)\text{−map.} \tag{4.21}$$

We now observe that

$$|\nu(z)| = -|Du(z)|^{p-2} Du(z) \cdot \left(Du(z) \cdot D_u\gamma(u)|_{u=u(z)}\right) \quad \text{almost every } z \in Q. \tag{4.22}$$

Let $\bar{z} = (\bar{t}, \bar{x}) \in Q$ be arbitrarily taken and fixed. Let $\gamma(v)$ be a smooth unit normal vector field of $\mathcal{N}$ in $u(Q) \subset \mathcal{N}$ such that $\gamma(v) \in (\mathcal{T}_v\mathcal{N})^{\perp}$, $|\gamma(v)| = 1$ for any $v \in u(Q)$ and $\gamma(u(\bar{z})) = \nu(\bar{z})/|\nu(\bar{z})|$. We take the composite map $\gamma(u)$ of $\gamma(\cdot)$ and the limit map u, and use a test function $\gamma(u)\,\eta$ for any smooth real-valued function η with compact support in Q to have

$$\int_Q \left(\partial_t u \cdot \gamma(u)\eta + |Du|^{p-2} Du \cdot \left(D\gamma(u)\,\eta + \gamma(u) D\eta\right) + \nu \cdot \gamma(u)\,\eta\right) dz = 0 \quad ;$$
$$\int_Q \left(|Du|^{p-2} Du \cdot D\gamma(u) + \nu \cdot \gamma(u)\right) \eta\, dz = 0,$$
$$\Longrightarrow \nu \cdot \gamma(u) = -|Du|^{p-2} Du \cdot D\gamma(u) \quad \text{almost everywhere in } Q,$$

where, in the 2nd line, we use that $\partial_t u, D_\alpha u \in \mathcal{T}_u\mathcal{N}$, $\alpha = 1, \ldots, m$, and $\gamma(u) \in (\mathcal{T}_u\mathcal{N})^{\perp}$ in Q. The last line yields, at $z = \bar{z}$,

$$|\nu(\bar{z})| = -|Du(\bar{z})|^{p-2} Du(\bar{z}) \cdot \left(\, Du(\bar{z}) \cdot D_u\gamma(u)|_{u=u(\bar{z})}\right).$$

Furthermore. we find that, for a positive constant C depending only on bounds of curvature of $\mathcal{N}$,

$$|\nu| \le C\,|Du|^p \quad \text{almost everywhere in } Q. \tag{4.23}$$

In fact, from (4.22) we obtain

$$|\nu(z)| \le C \max_{v \in u(Q)} |D_v\gamma(v)|\,|Du(z)|^p \quad \text{for almost every } z \in Q.$$

Finally, we have by (4.23) and (4.10) that

$$\begin{aligned} &\partial_t u - \Delta_p u = -\nu \in L^\infty(Q) \quad \text{almost everywhere in } Q \\ &\Longrightarrow Du \text{ is locally Hölder continuous in } Q, \end{aligned} \tag{4.24}$$

where, for the last statement of gradient continuity, we refer to [4, Theorem 1.1, p. 245 ; Sect 4, p. 291 ; Sect. 1 -(ii), pp. 217-218]. The use of convergence (4.3) and (4.2) in the energy boundedness (2.4) for u_k also yields

$$\|\partial_t u\|^2_{L^2(\mathbb{R}^m_\infty)} + \sup_{0<t<\infty} E(u(t)) \le E(u_0). \tag{4.25}$$

Closedness of $\mathcal{S}$ $\mathcal{S}$ is actually closed set in $\mathcal{M}_\infty$. For any $z_0 = (t_0, x_0)$ in the complement of $\mathcal{S}$, we can take a positive $R \leq R_0$ and an neighborhood of z_0, $Q' := (t_0 - (R/4)^{\lambda_0}, t_0) \times B(R/4)(x_0)$, and an infinite family $\{u_k\}$ of regular solutions of (2.1), and have the uniform boundedness in Q' of gradients as in (4.10). Thus, we have that, for any solution u_k, and any $z' = (t', x')$ in $Q := (t_0 - (R/8)^{\lambda_0}, t_0) \times B(R/8)(x_0)$ and all small positive $r < R/8$,

$$r^{\gamma_0 - m} \int_{\{t = t' - (R/8)^{\lambda_0}\} \times B(r, x')} e(u_k(t, x))\, dx \leq C\, R^{-p a_0}\, r^{\gamma_0} \tag{4.26}$$

and thus, for any $z' = (t', x')$ in Q,

$$\limsup_{k \to \infty} \Big(\limsup_{r \searrow 0} r^{\gamma_0 - m} \int_{\{t = t' - R^{\lambda_0}\} \times B(r, x')} e(u_k(t, x))\, dx \Big) = 0,$$

which implies that Q is a subset of the complement of $\mathcal{S}$. Therefore, we see that the complement of $\mathcal{S}$ is open and thus, $\mathcal{S}$ is closed.

Weak solution of the $p-$harmonic flow The proof is based on the size estimate of singular set $\mathcal{S}$ above. A covering argument is applied for the singular set $\mathcal{S}$, by use of a family of parabolic cylinders under an intrinsic scaling, depending on a size of gradient of solution. For the details see [8]. □

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