数理解析研究所講究録 第2095巻 2018年 213-221

Commutators of integral operators with a function in generalized Campanato spaces

新井 龍太郎 (茨城大学大学院理工学研究科)
Ryutaro Arai (Department of Mathematics, Ibaraki University)
中井 英一 (茨城大学理学部)
Eiichi Nakai (Department of Mathematics, Ibaraki University)

Dedicated to the memory of Professor Yasuji Takahashi

1 Introduction

This is an announcement of [1].

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and T be a Calderón-Zygmund singular integral operator. In 1976 Coifman, Rochberg and Weiss [3] proved that the commutator $[b,T] = bT - Tb$ is bounded on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$, that is,

$$\|[b,T]f\|_{L^p} = \|bTf - T(bf)\|_{L^p} \le C\|b\|_{\mathrm{BMO}}\|f\|_{L^p},$$

where C is a positive constant independent of b and f. For the fractional integral operator I_α, Chanillo [2] proved the boundedness of $[b, I_\alpha]$ in 1982. That is,

$$\|[b,I_\alpha]f\|_{L^q} \le C\|b\|_{\mathrm{BMO}}\|f\|_{L^p},$$

where $\alpha \in (0,n)$, $p,q \in (1,\infty)$ and $-n/p + \alpha = -n/q$. These results were extended to Morrey spaces by Di Fazio and Ragusa [4] in 1991.

In this talk we discuss the boundedness of the commutators $[b,T]$ and $[b,I_\rho]$ on generalized Morrey spaces with variable growth condition, where T is a Calderón-Zygmund operator, I_ρ is a generalized fractional integral operator and b is a function in generalized Campanato spaces with variable growth condition.

We denote by $B(x,r)$ the open ball centered at $x \in \mathbb{R}^n$ and of radius r, that is,

$$B(x,r) = \{y \in \mathbb{R}^n : |y - x| < r\}.$$

For a measurable set $G \subset \mathbb{R}^n$, we denote by $|G|$ and χ_G the Lebesgue measure of G and the characteristic function of G, respectively. For a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and a ball B, let

$$f_B = \fint_B f = \fint_B f(y)\,dy = \frac{1}{|B|}\int_B f(y)\,dy.$$

For a variable growth function $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ and a ball $B = B(x,r)$ we write $\varphi(B) = \varphi(x,r)$.

Definition 1.1. For $p \in [1,\infty)$ and $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, let $L^{(p,\varphi)}(\mathbb{R}^n)$ be the sets of all functions f such that the following functional is finite:

$$\|f\|_{L^{(p,\varphi)}(\mathbb{R}^n)} = \sup_B \left(\frac{1}{\varphi(B)} \fint_B |f(y)|^p \, dy \right)^{1/p},$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

Then $\|f\|_{L^{(p,\varphi)}(\mathbb{R}^n)}$ is a norm and $L^{(p,\varphi)}(\mathbb{R}^n)$ is a Banach space. If $\varphi_\lambda(x,r) = r^\lambda$ with $\lambda \in [-n, 0]$, then $L^{(p,\varphi_\lambda)}(\mathbb{R}^n)$ is the classical Morrey spaces. If $\lambda = -n$, then $L^{(p,\varphi_{-n})}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\lambda = 0$, then $L^{(p,\varphi_0)}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$.

Definition 1.2. For $p \in [1,\infty)$ and $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, let $\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)$ be the sets of all functions f such that the following functional is finite:

$$\|f\|_{\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)} = \sup_B \left(\frac{1}{\varphi(B)} \fint_B |f(y) - f_B|^p \, dy \right)^{1/p},$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

Then $\|f\|_{\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)}$ is a norm modulo constant functions and thereby $\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)$ is a Banach space. If $p = 1$ and $\varphi \equiv 1$, then $\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$. If $p = 1$ and $\varphi(r) = r^\alpha$ $(0 < \alpha \le 1)$, then $\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n) = \mathrm{Lip}_\alpha(\mathbb{R}^n)$.

A linear operator T from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ is said to be a Calderón-Zygmund operator if T is bounded on $L^2(\mathbb{R}^n)$ and there exists a standard kernel K such that, for $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) \, dy, \quad x \notin \operatorname{supp} f.$$

It is known that any Calderón-Zygmund operator T is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

For a function $\rho : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, we consider generalized fractional integral operators I_ρ defined by

$$I_\rho f(x) = \int_{\mathbb{R}^n} \frac{\rho(x, |x-y|)}{|x-y|^n} f(y) \, dy,$$

where we always assume that

$$\int_0^1 \frac{\rho(x,t)}{t} \, dt < \infty \quad \text{for each } x \in \mathbb{R}^n. \tag{1.1}$$

and that there exist positive constants C, K_1 and K_2 with $K_1 < K_2$ such that

$$\sup_{r \le t \le 2r} \rho(x,t) \le C \int_{K_1 r}^{K_2 r} \frac{\rho(x,t)}{t} \, dt \quad \text{for all } x \in \mathbb{R}^n \text{ and } r > 0. \tag{1.2}$$

If $\rho(x,r) = r^\alpha$, then I_ρ is the usual fractional integral operator I_α. It is known as the Hardy-Littlewood-Sobolev theorem that I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, if $\alpha \in (0,n)$, $p, q \in (1,\infty)$ and $-n/p + \alpha = -n/q$.

2 Main results

We say that θ is almost increasing (resp. almost decreasing) if there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r, s \in (0, \infty)$,

$$\theta(x,r) \le C\theta(x,s) \quad (\text{resp. } \theta(x,s) \le C\theta(x,r)), \quad \text{if } r < s.$$

In this talk we consider the following classes of φ:

Definition 2.1. (i) Let $\mathcal{G}^{dec}$ be the set of all functions $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ such that φ is almost decreasing, and that $r \mapsto \varphi(x,r)r^n$ is almost increasing. (ii) Let $\mathcal{G}^{inc}$ be the set of all functions $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ such that φ is almost increasing, and that $r \mapsto \varphi(x,r)/r$ is almost decreasing.

Let $\varphi \in \mathcal{G}^{dec}$. If φ satisfies

$$\lim_{r\to 0} \varphi(x,r) = \infty, \qquad \lim_{r\to\infty} \varphi(x,r) = 0, \tag{2.1}$$

then there exists $\tilde{\varphi} \in \mathcal{G}^{dec}$ such that $\varphi \sim \tilde{\varphi}$ and that $\varphi(x,\cdot)$ is continuous, strictly decreasing and bijective from $(0,\infty)$ to itself for each x.

We also consider the following conditions:
$\exists C > 0 \ \forall x, y \in \mathbb{R}^n \ \forall r \in (0,\infty)$,

$$\frac{1}{C} \le \frac{\theta(x,r)}{\theta(y,r)} \le C, \quad \text{if } |x-y| \le r. \tag{2.2}$$

$\exists C > 0 \ \forall x \in \mathbb{R}^n \ \forall r \in (0,\infty)$,

$$\int_r^\infty \frac{\varphi(x,t)}{t}\, dt \le C\varphi(x,r). \tag{2.3}$$

For functions f in Morrey spaces, we define $[b,T]f$ on each ball B by

$$[b,T]f(x) = [b,T](f\chi_{2B})(x) + \int_{\mathbb{R}^n \setminus 2B} (b(x) - b(y))K(x,y)f(y)\, dy, \quad x \in B.$$

Then we have the following theorem.

Theorem 2.1. *Let $1 < p \le q < \infty$ and $\varphi, \psi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that $\varphi \in \mathcal{G}^{dec}$ and $\psi \in \mathcal{G}^{inc}$. Let T be a Calderón-Zygmund operator.*

(i) *Assume that ψ satisfy (2.2), that φ satisfies (2.3), and that there exists a positive constant C_0 such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,*

$$\psi(x,r)\varphi(x,r)^{1/p} \le C_0\varphi(x,r)^{1/q}.$$

If $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, then $[b,T]f$ is well defined for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ and there exists a positive constant C, independent of b and f, such that

$$\|[b,T]f\|_{L^{(q,\varphi)}} \le C\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}.$$

(ii) *Conversely, assume that* φ *satisfies* (2.2) *and that there exists a positive constant* C_0 *such that, for all* $x \in \mathbb{R}^n$ *and* $r \in (0,\infty)$,

$$C_0 \psi(x,r)\varphi(x,r)^{1/p} \geq \varphi(x,r)^{1/q}.$$

If T *is a convolusion type such that*

$$Tf(x) = p.v. \int_{\mathbb{R}^n} K(x-y)f(y)\,dy$$

with homogeneous kernel K *satisfying* $K(x) = |x|^{-n}K(x/|x|)$, $\int_{S^{n-1}} K = 0$ *and* $K \in C^\infty(S^{n-1})$ *and* $K \not\equiv 0$, *and if* $[b,T]$ *is bounded from* $L^{(p,\varphi)}(\mathbb{R}^n)$ *to* $L^{(q,\varphi)}(\mathbb{R}^n)$, *then* $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ *and there exists a positive constant* C, *independent of* b, *such that*

$$\|b\|_{\mathcal{L}^{(1,\psi)}} \leq C \|[b,T]\|_{L^{(p,\varphi)} \to L^{(q,\varphi)}},$$

where $\|[b,T]\|_{L^{(p,\varphi)} \to L^{(q,\varphi)}}$ *is the opetator norm of* $[b,T]$ *from* $L^{(p,\varphi)}(\mathbb{R}^n)$ *to* $L^{(q,\varphi)}(\mathbb{R}^n)$.

In the above theorem, if $\psi \equiv 1$ and $\varphi(x,r) = r^{-n}$, then $\mathcal{L}^{(1,\psi)}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$ and $L^{(p,\varphi)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. This is the case of the theorem by Coifman, Rochberg and Weiss.

If $\psi(x,r) = r^\alpha$, $0 < \alpha \leq 1$, and $\varphi(x,r) = r^{-n}$, then $\mathcal{L}^{(1,\psi)}(\mathbb{R}^n) = \mathrm{Lip}_\alpha(\mathbb{R}^n)$, $L^{(p,\varphi)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^{(q,\varphi)}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$ with $-n/p + \alpha = -n/q$. That is,

$$\|[b,T]f\|_{L^q} \lesssim \|b\|_{\mathrm{Lip}_\alpha} \|f\|_{L^p}.$$

This is the case of Janson [5, Lemma 12].

Theorem 2.2. *Let* $1 < p < q < \infty$ *and* $\rho, \varphi, \psi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. *Assume that* $\varphi \in \mathcal{G}^{dec}$ *and* $\psi \in \mathcal{G}^{inc}$. *Assume also that* ρ *satisfies* (1.1) *and* (1.2). *Let* $\rho^*(x,r) = \int_0^r \frac{\rho(x,t)}{t}\,dt$.

(i) *Assume that* ρ, ρ^* *and* ψ *satisfy* (2.2), *that* φ *satisfies* (2.3) *and that there exist positive constants* ϵ, C_ρ, C_0, C_1 *and an exponent* $\tilde{p} \in (p,q]$ *such that, for all* $x,y \in \mathbb{R}^n$ *and* $r,s \in (0,\infty)$,

$$C_\rho \frac{\rho(x,r)}{r^{n-\epsilon}} \geq \frac{\rho(x,s)}{s^{n-\epsilon}}, \quad \text{if } r < s, \tag{2.4}$$

$$\left| \frac{\rho(x,r)}{r^n} - \frac{\rho(y,s)}{s^n} \right| \leq C_\rho (|r-s| + |x-y|) \frac{\rho^*(x,r)}{r^{n+1}}, \tag{2.5}$$

$$\text{if } \frac{1}{2} \leq \frac{r}{s} \leq 2 \text{ and } |x-y| < r/2,$$

$$\int_0^r \frac{\rho(x,t)}{t}\,dt\, \varphi(x,r)^{1/p} + \int_r^\infty \frac{\rho(x,t)\varphi(x,t)^{1/p}}{t}\,dt \leq C_0 \varphi(x,t)^{1/\tilde{p}}, \tag{2.6}$$

$$\psi(x,r)\varphi(x,r)^{1/\tilde{p}} \leq C_1 \varphi(x,r)^{1/q}. \tag{2.7}$$

If $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, *then* $[b, I_\rho]f$ *is well defined for all* $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ *and there exists a positive constant* C, *independent of* b *and* f, *such that*

$$\|[b, I_\rho]f\|_{L^{(q,\varphi)}} \le C\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}.$$

(ii) *Conversely, assume that* φ *satisfies* (2.2), *that* $\rho(x,r) = r^\alpha$, $0 < \alpha < n$, *and that*

$$C_0\psi(x,r)r^\alpha\,\varphi(x,r)^{1/p} \ge \varphi(x,r)^{1/q}. \qquad (2.8)$$

If $[b, I_\alpha]$ *is bounded from* $L^{(p,\varphi)}(\mathbb{R}^n)$ *to* $L^{(q,\varphi)}(\mathbb{R}^n)$, *then* $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ *and there exists a positive constant* C, *independent of* b, *such that*

$$\|b\|_{\mathcal{L}^{(1,\psi)}} \le C\|[b, I_\alpha]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}},$$

where $\|[b, I_\alpha]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}}$ *is the opetator norm of* $[b, I_\alpha]$ *from* $L^{(p,\varphi)}(\mathbb{R}^n)$ *to* $L^{(q,\varphi)}(\mathbb{R}^n)$.

3 Sketch of proof

We give a sketch of the proof of Theorem 2.2. To prove the theorem we use the following inequality and theorem:

$$M^\sharp([b, I_\rho]f)(x) \le C\|b\|_{\mathcal{L}^{(1,\psi)}}\Big(\big(M_{\psi^\eta}(|I_\rho f|^\eta)(x)\big)^{1/\eta} + \big(M_{(\rho^*\psi)^\eta}(|f|^\eta)(x)\big)^{1/\eta}\Big),$$

where $1 < \eta < \infty$, $\rho^*(x,r) = \int_0^r \rho(x,t)t^{-1}\,dt$ and

$$M^\sharp f(x) = \sup_{B\ni x} \fint_B |f(y) - f_B|\,dy, \quad M_\rho f(x) = \sup_{B\ni x} \rho(B) \fint_B |f(y)|\,dy.$$

Theorem 3.1 (Nakai, 2014). *Let* $p \in [1,\infty)$ *be a constant and* $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. *Assume that there exists a positive constant* C *such that,*

$$\varphi(x,r) \ge C\varphi(x,s) \text{ for all } x \in \mathbb{R}^n \text{and } r \in (0,s).$$

Then the operator M *is bounded from* $L^{(p,\varphi)}(\mathbb{R}^n)$ *to itself if* $p \in (1,\infty)$.

Theorem 3.2 (Nakai, 2014). *Let* $1 < p < q < \infty$ *and* $\rho, \varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. *Assume that* ρ *satisfies* (1.1) *and* (1.2) *and that* φ *is in* $\mathcal{G}^{\rm dec}$ *and satisfies* (2.1). *Assume also that there exists a positive constant* C *such that, for all* $x \in \mathbb{R}^n$ *and* $r \in (0,\infty)$,

$$\int_0^r \frac{\rho(x,t)}{t}\,dt\,\varphi(x,r)^{1/p} + \int_r^\infty \frac{\rho(x,t)\varphi(x,t)^{1/p}}{t}\,dt \le C\varphi(x,r)^{1/q}.$$

Then I_ρ *is bounded from* $L^{(p,\varphi)}(\mathbb{R}^n)$ *to* $L^{(q,\varphi)}(\mathbb{R}^n)$.

Theorem 3.3. *Let $1 < p < q < \infty$ and $\rho, \varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that φ is in $\mathcal{G}^{dec}$ and satisfies (2.1). Assume also that*

$$\rho(x,r)\varphi(x,r)^{1/p} \le C_0\varphi(x,r)^{1/q}. \tag{3.1}$$

Then M_ρ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$.

Proof. We may assume that $\varphi(x,\cdot)$ is continuous, strictly decreasing and bijective from $(0,\infty)$ to itself for each $x \in \mathbb{R}^n$.

We prove that, for $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ with $\|f\|_{L^{(p,\varphi)}(\mathbb{R}^n)} = 1$,

$$M_\rho f(x) \le CMf(x)^{p/q}, \quad x \in \mathbb{R}^n, \tag{3.2}$$

for some positive constant C independent of f and x. To prove (3.2) we show that, for any ball $B = B(x,r)$,

$$\rho(B) \fint_B |f| \le C_0 Mf(x)^{p/q}. \tag{3.3}$$

Choose $u > 0$ such that $\varphi(x,u) = Mf(x)^p$. If $r \le u$, then $\varphi(B) = \varphi(x,r) \ge Mf(x)^p$ and $\varphi(B)^{1/q-1/p} \le Mf(x)^{p/q-1}$. By (3.1) we have

$$\rho(B) \fint_B |f| \le C_0\varphi(B)^{1/q-1/p} \fint_B |f| \le C_0 Mf(x)^{p/q}.$$

If $r > u$, then $\varphi(B) = \varphi(x,r) < Mf(x)^p$ and $\varphi(B)^{1/q} < Mf(x)^{p/q}$. By (3.1) we have

$$\rho(B) \fint_B |f| \le \rho(B) \left(\fint_B |f|^p\right)^{1/p} \le \rho(B)\varphi(B)^{1/p} \le C_0\varphi(B)^{1/q} \le C_0 Mf(x)^{p/q}.$$

Then we have (3.3) and the conclusion. □

Proposition 3.4. *Let $1 \le p < \infty$ and $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Then, for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,*

$$\|f\|_{\mathcal{L}^{(p,\varphi)}} \le C\|M^\sharp f\|_{L^{(p,\varphi)}}, \tag{3.4}$$

where C is a positive constant independent of f.

Corollary 3.5. *Let $1 \le p < \infty$ and $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that $\varphi \in \mathcal{G}^{\mathrm{dec}}$ and that φ satisfies (2.3). For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, if $\lim_{r\to\infty} f_{B(0,r)} = 0$, then*

$$\|f\|_{L^{(p,\varphi)}} \le C\|M^\sharp f\|_{L^{(p,\varphi)}}, \tag{3.5}$$

where C is a positive constant independent of f.

Lemma 3.6 ([8, Theorem 2.1 and Remark 2.1])**.** *Let $p \in [1,\infty)$ and φ is in $\mathcal{G}^{dec}$ and satisfies (2.3). Then, for every $f \in \mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)$, $f_{B(0,r)}$ converges as $r \to \infty$ and*

$$\|f - \lim_{r\to\infty} f_{B(0,r)}\|_{L^{(p,\varphi)}} \sim \|f\|_{\mathcal{L}^{(p,\varphi)}},$$

For any cube $Q \subset \mathbb{R}^n$ centered at $a \in \mathbb{R}^n$ and with sidelength $2r > 0$, we denote by $\mathcal{Q}^{\mathrm{dy}}(Q)$ the set of all dyadic cubes with respect to Q.

For any cube $Q \subset \mathbb{R}^n$, let

$$M_Q^{\mathrm{dy}} f(x) = \sup_{R \in \mathcal{Q}^{\mathrm{dy}}(Q),\, x \in R \subset Q} \fint_Q |f(y)|\, dy,$$

$$M_Q^{\sharp,\mathrm{dy}} f(x) = \sup_{R \in \mathcal{Q}^{\mathrm{dy}}(Q),\, x \in R \subset Q} \fint_Q |f(y) - f_Q|\, dy.$$

Lemma 3.7 (Tsutsui, 2011 Komori, 2015)**.** *Let Q be a cube and $f \in L^1(Q)$. Then, for any $0 < \gamma \le 1$ and $\lambda > |f|_Q$,*

$$\begin{aligned} &|\{x \in Q :\, M_Q^{\mathrm{dy}} f(x) > 2\lambda, M_Q^{\sharp,\mathrm{dy}} f(x) \le \gamma\lambda\}| \\ &\qquad\qquad \le 2^n \gamma |\{x \in Q :\, M_Q^{\mathrm{dy}} f(x) > \lambda\}|. \end{aligned} \tag{3.6}$$

Lemma 3.8. *There exists a positive constant C, for any cube Q and any function $f \in L^1(Q)$,*

$$\|f - f_Q\|_{L^p(Q)} \le C \|M_Q^{\sharp,\mathrm{dy}} f\|_{L^p(Q)}.$$

Proof. By the good λ inequality (3.6) and the standard argument we have the following boundedness: There exists a positive constant C, for any cube Q and any function $f \in L^1(Q)$,

$$\|M_Q^{\mathrm{dy}} f\|_{L^p(Q)} \le C \left(\|M_Q^{\sharp,\mathrm{dy}} f\|_{L^p(Q)} + |Q|^{1/p} |f|_Q \right). \tag{3.7}$$

Actually, for any $L > 2|f|_Q$,

$$\begin{aligned} &\int_0^L p\lambda^{p-1} |\{x \in Q : M_Q^{\mathrm{dy}} f(x) > \lambda\}|\, d\lambda \\ &= \int_0^{2|f|_Q} p\lambda^{p-1} |\{x \in Q : M_Q^{\mathrm{dy}} f(x) > \lambda\}|\, d\lambda \\ &\qquad + \int_{2|f|_Q}^L p\lambda^{p-1} |\{x \in Q : M_Q^{\mathrm{dy}} f(x) > \lambda\}|\, d\lambda \\ &\le (2|f|_Q)^p |Q| + 2^p \int_{|f|_Q}^{L/2} p\lambda^{p-1} |\{x \in Q : M_Q^{\mathrm{dy}} f(x) > 2\lambda\}|\, d\lambda. \end{aligned}$$

By the good λ inequality (3.6) we have

$$\begin{aligned}
&2^p \int_{|f|_Q}^{L/2} p\lambda^{p-1} |\{x \in Q : M_Q^{\mathrm{dy}} f(x) > 2\lambda\}|\, d\lambda \\
&\le 2^{n+p}\gamma \int_{|f|_Q}^{L/2} p\lambda^{p-1} |\{x \in Q : M_Q^{\mathrm{dy}} f(x) > \lambda\}|\, d\lambda \\
&\qquad + 2^p \int_{|f|_Q}^{L/2} p\lambda^{p-1} |\{x \in Q : M_Q^{\sharp,\mathrm{dy}} f(x) > \gamma\lambda\}|\, d\lambda \\
&\le 2^{n+p}\gamma \int_0^{L} p\lambda^{p-1} |\{x \in Q : M_Q^{\mathrm{dy}} f(x) > \lambda\}|\, d\lambda \\
&\qquad + 2^p \gamma^{-p} \int_0^{\infty} p\lambda^{p-1} |\{x \in Q : M_Q^{\sharp,\mathrm{dy}} f(x) > \lambda\}|\, d\lambda.
\end{aligned}$$

Then, for small $\gamma > 0$,

$$\begin{aligned}
&(1 - 2^{n+p}\gamma) \int_0^{L} p\lambda^{p-1} |\{x \in Q : M_Q^{\mathrm{dy}} f(x) > \lambda\}|\, d\lambda \\
&\le (2|f|_Q)^p |Q| + 2^p \gamma^{-p} \int_0^{\infty} p\lambda^{p-1} |\{x \in Q : M_Q^{\sharp,\mathrm{dy}} f(x) > \lambda\}|\, d\lambda.
\end{aligned}$$

Letting $L \to \infty$, we have (3.7). Substitute $f - f_Q$ for f in (3.7). Then

$$\begin{aligned}
\|f - f_Q\|_{L^p(Q)} &\le \|M_Q^{\mathrm{dy}}(f - f_Q)\|_{L^p(Q)} \\
&\lesssim \|M_Q^{\sharp,\mathrm{dy}} f\|_{L^p(Q)} + |Q|^{1/p} ⨍_Q |f - f_Q| \\
&\le \|M_Q^{\sharp,\mathrm{dy}} f\|_{L^p(Q)} + |Q|^{1/p} \inf_{x \in Q} M_Q^{\sharp,\mathrm{dy}} f(x).
\end{aligned}$$

Since

$$\begin{aligned}
|Q|^{1/p} \inf_{x \in Q} M_Q^{\sharp,\mathrm{dy}} f(x) &= \left(\int_Q \left[\inf_{x \in Q} M_Q^{\sharp,\mathrm{dy}} f(x) \right]^p dy \right)^{1/p} \\
&\le \|M_Q^{\sharp,\mathrm{dy}} f\|_{L^p(Q)},
\end{aligned}$$

we have the conclusion. □

Proof of Proposition 3.4. For any ball $B = B(x, r)$, take the cube Q centered at x and with sidelength $2r$. Then $B \subset Q$. By Lemma 3.8 we have

$$\begin{aligned}
\left(\frac{1}{\varphi(B)} ⨍_B |f - f_B|^p \right)^{1/p} &\le \left(\frac{2}{\varphi(B)} \frac{|Q|}{|B|} ⨍_Q |f - f_Q|^p \right)^{1/p} \\
&\lesssim \left(\frac{1}{\varphi(B)} ⨍_Q (M_Q^{\sharp,\mathrm{dy}} f)^p \right)^{1/p} \\
&\lesssim \|M^{\sharp} f\|_{L^{(p,\varphi)}(\mathbb{R}^n)}.
\end{aligned}$$

This shows the conclusion. □

References

[1] R. Arai and E. Nakai, Commutators of Calderón-Zygmund and generalized fractional integral operators on generalized Morrey spaces, Rev. Mat. Complut. published online.

[2] S. Chanillo, A note on commutators, Indiana Univ. Math. J. 31 (1982), no. 1, 7–16.

[3] R. R. Coifman, R. Rochberg and G. Weiss, Factorization theorems for Hardy spaces in several variables, Ann. of Math. (2) 103 (1976), no. 3, 611–635.

[4] G. Di Fazio and M. A. Ragusa, Commutators and Morrey spaces. Boll. Un. Mat. Ital. A (7) 5 (1991), no. 3, 323–332.

[5] S. Janson, Mean oscillation and commutators of singular integral operators. Ark. Mat. 16 (1978), no. 2, 263–270.

[6] Y. Komori-Furuya, Local good-λ estimate for the sharp maximal function and weighted Morrey space. J. Funct. Spaces 2015, Art. ID 651825, 4 pp.

[7] E. Nakai, *Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces*, Math. Nachr. 166 (1994), 95–103.

[8] E. Nakai, *The Campanato, Morrey and Hölder spaces on spaces of homogeneous type*, Studia Math. 176 (2006), no. 1, 1–19.

[9] E. Nakai, *Orlicz-Morrey spaces and the Hardy-Littlewood maximal function*, Studia Math. 188 (2008), no 3, 193–221.

[10] E. Nakai, *Generalized fractional integrals on generalized Morrey spaces*, Math. Nachr. 287 (2014), no. 2-3, 339–351.

[11] E. Nakai and H. Sumitomo, On generalized Riesz potentials and spaces of some smooth functions, Sci. Math. Jpn. 54 (2001), no. 3, 463–472.

[12] E. Nakai and T. Yoneda, Bilinear estimates in dyadic BMO and the Navier-Stokes equations, J. Math. Soc. Japan 64 (2012), no. 2, 399–422.

[13] Y. Tsutsui, Sharp maximal inequalities and its application to some bilinear estimates, J. Fourier Anal. Appl., 17 (2011), 265–289.