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On lattice points which become vertices of Heronian triangles *

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1 Introduction

Heronian triangles are triangles whose side lengths and area are all integers. For example, the triangles with side lengths $3, 4$ and 5 (this triangle has area 6) , and with side lengths $5, 5$ and 6 (this triangle has area 12), etc. are Heronian triangles. Specially the triangle whose side lengths $3, 4$ and 5 is *Pythagorean triangle*, that is, a right triangle whose side lengths are all integers. We discuss the lattice points which become vertices of Heronian triangles.

In section 2, we introduce classical results. We state on the existence of circles containing a finite number of lattice points, and a finite number of points whose distance between any pair of them is integer. Finally, we state on the formula of Pythagorean triple.

In section 3, we state our main theorems. We claim that there exists n lattice points on a circle such that the distance between any pair of them is integer. Next, using about the formula of Pythagorean triple, we show one of the sets of lattice points which become vertices of Heronian triangles.

2 Classical results

2.1 rational points on a circle and lattice points on a circle

A point $(x, y) \in \mathbb{R}^2$ is *rational point* if x and y are rational numbers, and a point (x, y) is a *lattice point* if x and y are integers.

First, we address Pick's Theorem. Consider a polygon P whose vertices lie at lattice points. As mentioned above, Pick's Theorem allows us to determine

*This paper is a preliminary version and a final version will be submitted to elsewhere.

the area of P from the number of lattice points on the boundary of P, $B(P)$, and the number of lattice points in the interior of P, $I(P)$. More specifically, Pick's Theorem states the following,

Proposition 2.1 (Pick's theorem, 1899) *Let P be a polygon in the plane with its vertices at lattice point. Then the area of P, $A(P)$, is given by*

$$A(P) = \frac{1}{2}B(P) + I(P) - 1,$$

where $B(P)$ is the number of lattice points on the boundary of P and $I(P)$ is the number of lattice points in the interior of P.

By this proposition, we see that the area of a polygon in the plane with its vertices at lattice point is rational number. Now suppose that there exists a regular triangle such that its vertices at lattice point, for a side length a, it has area $\sqrt{3}a^2/4$, where a^2 is a positive integer, hence this is a irrational number, a contradiction. Therefore the regular triangle cannot be embedded in the lattice. Containing in case other regular n-gon, we state the following proposition.

Proposition 2.2 *If n lattice points form a regular n-gon ($n > 2$), then $n = 4$; thus the square is the only regular polygon that can be embedded in the lattice.*

Pythagorean triangles are right triangles whose sides have integral lengths. This proposition implies the following proposition.

Proposition 2.3 *If α is an acute angle of a Pythagorean triangle, the number α/π is irrational number.*

Proposition 2.4 *The number of rational points on a circle is 0, 1, 2, or ∞.*

Corollary 2.5 *For any integer $n > 0$, there exists at least one circle which contains more than n lattice points.*

2.2 rational distance and integral distance

In Euclidean geometry, Ptolemy's theorem is a relation between the four sides and two diagonals of a cyclic quadrilateral, that is, a quadrilateral whose vertices lie on a common circle.

Proposition 2.6 (Ptolemy's theorem) *If $ABCD$ is a cyclic quadrilateral,* *then*

$$AB \cdot CD + AD \cdot BC = AC \cdot BD.$$

This proposition implies the following proposition that was known to Euler.

Proposition 2.7 [1, Proposition 1.1],[2, Theorem 1.14] *There exist infinite points on a circle such that the distance between any pair of them is rational number.*

Corollary 2.8 *For any integer $n > 0$, there exist n points on a circle such that the distance between any pair of them is integer.*

2.3 Pythagorean triples

A *Pythagorean triple* is a triple a, b, c of natural numbers that satisfy the equation $a^2 + b^2 = c^2$. A triangle whose side lengths form a Pythagorean triple is a Pythagorean triangle, and is necessarily a right triangle.

Proposition 2.9 *Let a, b, c be a solution of the equation*

$$a^2 + b^2 = c^2$$

in relatively prime integers. Then a or b must be even. On the assumption that a is even, a, b, c has the form

$$a = 2uv, \quad b = u^2 - v^2, \quad c = u^2 + v^2,$$

where u and v are relatively prime natural numbers with $u+v$ odd and $u > v$. Every such pair u, v corresponds to exactly one primitive Pythagorean triple a, b, c, that is, greatest common divisor of a, b, c is equal to 1.

3 Main theorems

In connection with Proposition 2.4 and Proposition 2.7, we get the following proposition. The basic idea of following proof is due to [2].

Proposition 3.1 *There exist circles which contain infinite rational points such that the distance between any pair of them is rational number.*

Outline of the proof. Let a, b, c be a Pythagorean triple, now we consider a circumcircle Γ of isosceles triangle with side lengths c, c and $2a$. Let α be the angle subtended at the center by the chord with side length c, we consider the circumcircle Γ with its center at the origin. Let P_0 be a intersection point of circumcircle Γ and y-axis, and we put $P_i (i = 1, 2, \cdots)$ be points on the circumference equally spaced with central angle α. The points $P_i (i = 0, 1, \cdots)$ are all rational points and the distance between any pair of them is rational number(see Figure A). ■

Lemma 3.2 *If there exist three lattice points such that the distance between any pair of them is integer, then the area of triangle with three lattice points is integer, that is, they become vertices of Heronian triangle.*

By these arguments, we get the following theorem.

Theorem 3.3 *There exist circles which contain n lattice points such that the distance between any pair of them is integer. Specially, in case $n = 3$, there are three lattice points on circles which become vertices of Heronian triangles.*

Next, using classical formula of Pythagorean triples, Proposition 2.9, we get the following theorem.

Theorem 3.4 *Not on the line three points which belong to the following set consists of lattice points*

$$\{(0, \pm y), (0, 0), (\pm x_1, 0), \cdots, (\pm x_n, 0)\}$$

become vertices of Heronian triangles (see Figure B), where let p_i $(i = 1, \cdots, n)$ be odd prime number satisfying $p_i > p_j$ (for $i < j$), and $p_1 > 2p_2 \cdots p_n$, we can write $x_k = (p_1 \cdots p_k)^2 - 4(p_{k+1} \cdots p_n)^2$ $(k = 1, \cdots, n)$, $y = 4p_1 \cdots p_n$.

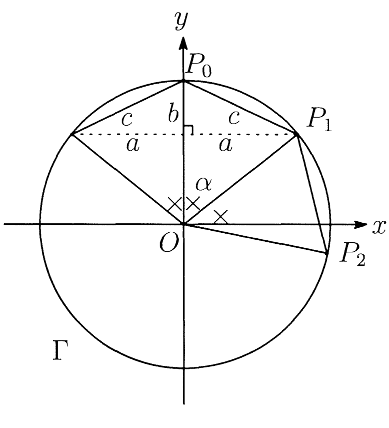

Figure A

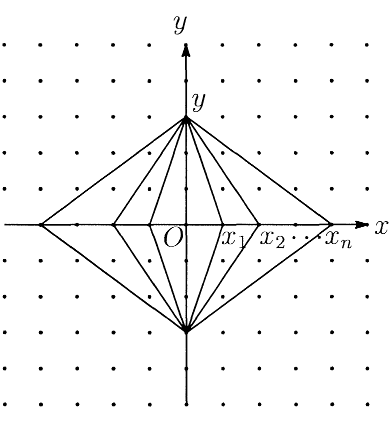

Figure B

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