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Diagrams of numerical semigroups whose general members are non-Weierstrass [1]

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Abstract

We construct diagrams consisting of an infinite number of numerical semigroups through dividing by two whose general members are non-Weierstrass where the bottom of the diagram is some Weierstrass numerical semigroup.

1 Introduction

Let $\mathbb{N}_0$ be the additive monoid of non-negative integers. A submonoid H of $\mathbb{N}_0$ is called a *numerical semigroup* if the complement $\mathbb{N}_0 \backslash H$ is finite. The cardinality of $\mathbb{N}_0 \backslash H$ is called the *genus* of H, denoted by $g(H)$. In this article H always stands for a numerical semigroup. We set

$$c(H) = \min\{c \in \mathbb{N}_0 \mid c + \mathbb{N}_0 \subseteqq H\},$$

which is called the *conductor* of H. It is known that $c(H) \leqq 2g(H)$. H is said to be *symmetric* if $c(H) = 2g(H)$. H is said to be *quasi-symmetric* if $c(H) = 2g(H) - 1$. We are interested in the case $c(H) = 2g(H) - 2$.

A *curve* means a complete non-singular irreducible algebraic curve over an algebraically closed field k of characteritic 0. For a pointed curve (C, P) we set

$$H(P) = \{n \in \mathbb{N}_0 \mid \exists f \in k(C) \text{ such that } (f)_\infty = nP\},$$

where $k(C)$ is the field of rational functions on C. Then $H(P)$ is a numerical semigroup of genus $g(C)$ where $g(C)$ is the genus of C.

We set $d_2(H) = \{h' \in \mathbb{N}_0 \mid 2h' \in H\}$, which is a numerical semigroup. Let $\pi : \tilde{C} \longrightarrow C$ be a double covering of a curve with a ramification point $\tilde{P}$. Then $d_2(H(\tilde{P})) = H(\pi(\tilde{P}))$. H is said to be *Weierstrass* if there exists a pointed curve (C, P) with $H(P) = H$. H is said to be *of double covering type* (abbreviated to *DC*) if there exists a double covering $\pi : C \longrightarrow C'$ with a ramification point P such that $H = H(P)$. If H is *DC*, then both H and $d_2(H)$ are Weierstrass. For positive integers $a_1, \ldots, a_s$ we denote by $\langle a_1, \ldots, a_s \rangle$ the monoid generated by $a_1, \ldots, a_s$. For example, $H = \langle 2, 2g+1 \rangle$ is DC with $d_2(H) = \mathbb{N}_0$. Indeed, let π be a double covering from a curve of genus g to the projective line $\mathbb{P}^1$ and P be any ramification point. Then $H(P) = H$. The following is an open problem:

[1] This paper is an extended abstract and the details will appear elsewhere.
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Problem. ([4] and [1]) *What is the proportion of non-Weierstrass numerical semigroups in the whole set of numerical semigroups ?*

Our purpose in this article is to construct diagrams consisting of an infinite number of numerical semigroups through the map d_2 whose general members in the diagram are non-Weierstrass where the bottom of the diagram is a Weierstrass numerical semigroup H with $c(H) = 2g(H) - 2$.

2 Towers of symmetric numerical semigroups

We set $m(H) = \min\{h \in H \mid h > 0\}$, which is called the *multiplicity* of H.

Remark 2.1 *(i) Let n be an odd integer. Then $2H + n\mathbb{N}_0$ is a numerical semigroup.*
(ii) Let n be an odd integer with $n \geqq c(H) + m(H) - 1$. Then we have $d_2(2H + n\mathbb{N}_0) = H$.

We have the following result for the above numerical semigroups:

Theorem 2.2 (Komeda-Ohbuchi [5]) *Let n be an odd integer with*

$$n \geqq \max\{c(H) + m(H) - 1, 2g(H) + 1\}.$$

If H is Weierstrass, then $2H + n\mathbb{N}_0$ is DC, hence Weierstrass.

We have towers consisting of symmetric numerical semigroups which are DC.

Theorem 2.3 *Let H_0 be a symmetric Weierstrass numerical semigroup. For each $i \geqq 1$ let us take an odd integer*

$$n_i \geqq \max\{c(H_{i-1}) + m(H_{i-1}) - 1, 2g(H_{i-1}) + 1\}$$

where we set $H_i = 2H_{i-1} + n_i\mathbb{N}_0$ for $i \geqq 1$. Then we have towers of symmetric numerical semigroups which are DC as follows:

$$\begin{array}{c} H_{i+1} \\ \downarrow d_2 \\ H_i \end{array} \quad \text{for } i \geqq 1.$$

3 Towers of quasi-symmetric numerical semigroups

Lemma 3.1 ([2]) *Let H and $\tilde{H}$ be quasi-symmetric numerical semigroups with $d_2(\tilde{H}) = H$. Then we obtain $g(\tilde{H}) = 2g(H) - 1$.*

By the above lemma and Riemann-Hurwitz Formula we get the following:

Theorem 3.2 *Let H and $\tilde{H}$ be quasi-symmetric numerical semigroups with $d_2(\tilde{H}) = H$. Then $\tilde{H}$ is not DC.*

Proposition 3.3 ([2]) *Let H' be a quasi-symmetric numerical semigroup. We set*

$$n = \min\{h' \in H' \mid h' \text{ is odd}\} \text{ and } s_i = \min\{h' \in H' \mid h' \equiv i \bmod n\}$$

for all $i = 1, \ldots, n-1$. We set

$$\{s_1, \ldots, s_{n-1}\} = \{s^{(1)} < \cdots < s^{(n-1)}\}$$

and

$$H = \langle n, 2s^{(1)}, \ldots, 2s^{(\frac{n-3}{2})}, 2s^{(\frac{n-1}{2})} - n, \ldots, 2s^{(n-1)} - n\rangle.$$

Then H is a quasi-symmetric numerical semigroup of genus $2g(H') - 1$ with $d_2(H) = H'$.

Example. Let $H_0 = \langle 3, 4, 5\rangle$. For each odd $m \geqq 1$ (resp. even $m \geqq 2$) we set

$$H_m = \langle 3, 3m+2, 3 \cdot 2m + 1\rangle \text{ (resp. } H_m = \langle 3, 3m+1, 3(2m-1)+2\rangle).$$

Then we have towers of quasi-symmetric numerical semigroups which are not DC as follows:

$$\begin{array}{c} H_{i+1} \\ \downarrow {\scriptstyle d_2} \\ H_i \end{array} \quad \text{for } i \geqq 0.$$

4 Diagrams of numerical semigroups with $c(H) = 2g(H) - 2$

We set

$$PF(H) = \{\gamma \in \mathbb{N}_0 \backslash H \mid \gamma + h \in H, \text{ all } h \in H > 0\},$$

whose elements are called *pseudo-Frobenius numbers* of H. We have $c(H) - 1 \in PF(H)$. We set $t(H) = \sharp PF(H)$, which is called the *type* of H.

Remark 4.1 *We have $c(H) + t(H) \leqq 2g(H) + 1$. (For example, see [6].)*

H is said to be *almost symmetric* if the equality $c(H) + t(H) = 2g(H) + 1$ holds.

Remark 4.2 *i) H is symmetric if and only if $t(H) = 1$. In this case H is almost symmetric.*
ii) If H is quasi-symmetric, then $t(H) = 2$. The converse does not hold. In this case H is also almost symmetric.
iii) If $c(H) = 2g(H) - 2$, then $t(H) = 2$ or 3.

We set $PF^*(H) = PF(H) \backslash \{c(H) - 1\}$.

Proposition 4.3 ([3]) *If H is almost symmetric, then we have an automorphism of $PF^*(H)$ sending γ to $c(H) - 1 - \gamma$.*

Corollary 4.4 *If $c(H) = 2g(H) - 2$ and $t(H) = 3$, we have $PF^*(H) = \{\gamma, 2g(H) - 3 - \gamma\}$ for some $\gamma \in \mathbb{N}_0 \backslash H$.*

Example. Let $H = \langle 4, 6, 4l + 1, 4l + 3\rangle$ for $l \geqq 1$. Then we have $c(H) = 4l = 2g(H) - 2$ and $PF^*(H) = \{2, 4l - 1 - 2\}$, hence $t(H) = 3$, i.e., H is almost symmetric.

Example. Let $H = \langle 4, 4l+1, 4(2l-1)+3\rangle$ for $l \geqq 1$. Then we have $c(H) = 12l-4 = 2g(H)-2$ and $PF^*(H) = \{4 \cdot 2l - 2\}$, hence $t(H) = 2$, i.e., H is not almost symmetric.

Remark 4.5 *Let n be an odd integer with*

$$n \geqq \max\{c(H) + m - 1, 2m\}.$$

Then we have $g(2H + n\mathbb{N}_0) = 2g(H) + \dfrac{n-1}{2}$ with $d_2(2H + n\mathbb{N}_0) = H$.

By the definition of $PF(H)$ we get the following:

Lemma 4.6 *Let $d_2(\tilde{H}) = H$ and $n = \min\{\tilde{h} \in \tilde{H} \mid \tilde{h}$ is odd$\}$. Then the following are equivalent:*
i) $g(\tilde{H}) = 2g(H) + \dfrac{n-1}{2} - 1$.
ii) $\tilde{H} = 2H + \langle n, n + 2f\rangle$ for some $f \in PF(H)$.

Theorem 4.7 *Assume that $c(H) = 2g(H) - 2$ and $t(H) = 3$. Let $PF^*(H) = \{f_1, f_2\}$. We set $\tilde{H}_i = 2H + \langle n, n + 2f_i\rangle$ for $i = 1, 2$. Then one of the following holds:*
i) H is Weierstrass and $\tilde{H}_1$, $\tilde{H}_2$ are DC.
ii) H is Weierstrass and renumbering 1 and 2 $\tilde{H}_1$ is DC and $\tilde{H}_2$ is not DC.
iii) H is non-Weierstrass.
If $n >> 0$, then both $\tilde{H}_1$ and $\tilde{H}_2$ are non-Weierstrass.

Proof. For i) and ii) see [2]. Applying [7] we get iii). □

For $1 \leqq i \leqq m(H) - 1$ we define s_i by $\min\{h \in H \mid h \equiv i \bmod m(H)\}$. We set

$$S(H) = \{m(H)\} \cup \{s_i \mid i = 1, \ldots, m(H) - 1\},$$

which is called the *standard basis* for H.

Remark 4.8 *([3]) We have $PF(H) = \{s_i - m(H) \mid s_i + s_j \notin S(H)$ for all $j\}$.*

Theorem 4.9 *([2]) Assume that $c(H) = 2g(H) - 2$. Let $f = s_i - m(H) \in PF^*(H)$. Let n be an odd number with*

$$n \geqq 4((2m(H) - 1)(s_i - m(H)) + 1 - g(H)) + 1.$$

We set $\tilde{H} = 2H + \langle n, n + 2f\rangle$. Then we have the following:
i) $c(\tilde{H}) = 2g(\tilde{H}) - 2$ and $t(\tilde{H}) = 3$.
ii) Assume $(2i + 1, m(H)) = 1$. For odd $N \geqq n + 2(2g(H) - 3 + m(H))$ we obtain that

$$\tilde{\tilde{H}} = 2\tilde{H} + \langle N, N + 2(2s_i - 2m(H))\rangle$$

is not DC.

Using the above theorem we get our main result in this article.

Corollary 4.10 *Let H be a numerical semigroup with $c(H) = 2g(H) - 2$. Assume that $m(H)$ is a power of 2. Then we can construct a diagram of numerical semigroups whose general members are non-Weierstrass such that the bottom of the diagram is H. Here, general members mean all members in the interior of the diagram except finite ones.*

References

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