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Primitivity of group rings of groups with non-trivial center

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In [2], we consider the following condition $(*)$: for each subset M of G consisting of a finite number of elements not equal to 1, and for any positive integer m, there exist distinct a, b, and c in G so that if $(x_1^{-1}g_1x_1)\cdots(x_m^{-1}g_mx_m) = 1$, where g_i is in M and x_i is equal to a, b, or c for all i between 1 and m, then $x_i = x_{i+1}$ for some i. This condition is often satisfied by a non-noetherian group which has a non-abelian free subgroup and the trivial center. For a such group G, we have proved that the group ring RG of G over a domain R is primitive if G satisfies $(*)$ and $|R| \leq |G|$. However as long as we deal with groups satisfying $(*)$, since the center of them are always trivial, we can say nothing about primitivity of group rings of groups with nontrivial center. In this note, we consider a more general condition than the above one and give a primitivity result for group rings of groups with non-trivial center.

1 Introduction

Let G be a group and M a subset of G. We denote by $\widetilde{M}$ the symmetric closure of M; $\widetilde{M} = M \cup \{x^{-1} \mid x \in M\}$. For non-empty subsets $M_1, M_2 \ldots, M_n$ of G consisting of elements not equal to 1, we say that $M_1, M_2, \ldots, M_n$ are mutually reduced in G if, for each finite number of elements $g_1, g_2, \ldots, g_m \in \bigcup_{i\ 1}^{n} \widetilde{M}_i$, whenever $g_1g_2\cdots g_m = 1$, there exists $i \in [m]$ and $j \in [n]$ so that $g_i, g_{i\ 1} \in \widetilde{M}_j$, where $[n] := \{1, 2, \ldots, n\}$ for any $n \in \mathbb{N}$. If $M_i = \{x_i\}$ for $i \in [n]$ and $M_1, M_2, \ldots, M_n$ are mutually reduced, then we say that $x_1, x_2, \ldots, x_n$ are mutually reduced. For a subset M of G and $x \in G$, we denote by M^x the set $\{x^{-1}fx \mid f \in M\}$.

In [2], we considered the following condition:

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(∗) For each subset M of G consisting of finite number of elements not equal to 1, there exist distinct $x_1, x_2, x \ \in G$ such that M^{x_i} $(i \in [3])$ are mutually reduced.

We have showed the following theorem:

Theorem 1.1. ([2, Theorem 1.1]) *Let G be a group which has a non-abelian free subgroup whose cardinality is the same as that of G, and suppose that G satisfies (∗). Then, if R is a domain with $|R| \leq |G|$, the group ring RG of G over R is primitive. In particular, the group algebra KG is primitive for any field K.*

By making use of Theorem 1.1, we can get many results for primitivity of group rings (see [1], [2] and [3]). For amalgamated free products, we have showed the following result:

Theorem 1.2. ([2, Corollary 4.5]) *Let R be a domain and suppose that $G = A *_H B$ satisfies $B \neq H$ and there exist elements a and a_* in $A \setminus H$ such that $aa_* \neq 1$ and $a^{-1}Ha \cap H = 1$. If $|R| \leq |G|$, then the group ring RG is primitive. In particular, KG is primitive for any field K.*

For $g \in G$, we denote the centralizer of g in G by $C_G(g)$, and let $\ (G) := \{g \in G \mid [G : C_G(g)] < \infty\}$. Clearly, $\ (G)$ includes the center of G. In Theorem 1.1 (and so in Theorem 1.2), $\ (G)$ is always trivial and so is the center of G, which is needed for KG to be primitive for any field K.

It is easy to see that $\ (G)$ is trivial provided G satisfies (∗). In fact, for a nonidentity element g in G, we can see that there exist infinitely many conjugate elements of g. If this is not the case, then the set M of conjugate elements of g in G is a finite set. Since G satisfies (∗), for M, there exists $x_1, x_2 \in G$ such that M^{x_1} and M^{x_2} are mutually reduced. Since g is in M, $(x_1^{-1}gx_1)(x_2^{-1}fx_2)^{-1} \neq 1$ for any $f \in M$, and thus $x_1^{-1}gx_1 \neq x_2^{-1}fx_2$. Hence $(x_1x_2^{-1})^{-1}g(x_1x_2^{-1}) \neq f$ for any $f \in M$, which implies $(x_1x_2^{-1})^{-1}g(x_1x_2^{-1}) \notin M$, a contradiction.

This means that for group rings of groups with nontrivial center, Theorem 1.1 does not say anything. On the other hand, in her Ph D thesis (1975), C. R. Jordan [4] had given the following result:

Theorem 1.3. ([4, Theorem 2.5.2, 2.5.4]) *Let $G = A *_H B$ be the free product of A and B with H amalgamated.*

(1) Let R be a ring. Suppose that $R(H)$ is an uncountable domain and that $|G : H|$ is greater than or equal to the cardinality of $R(H)$. Then $R(G)$ is primitive.

(2) Let $|A : H| \neq 2$ or $|B : H| \neq 2$. Suppose that $R(H)$ is a domain and has countable cardinality. Then $R(G)$ is primitive.

Jordan's results hold even if H is the center of G. So we would like to extend our results to one for groups with nontrivial center.

2 A generalization

Let G be a group and N a proper subgroup of G. Let G be a group and M a subset of G. For non-empty subsets $M_1, M_2, \cdots, M_n$ of G, consisting of elements of $G - N$, we say that $M_1, M_2, \cdots, M_n$ are mutually N-reduced in G, if for each finite number of elements $g_1, g_2, \ldots, g_m \in \bigcup_{i\ \ 1}^{n} \widetilde{M_i}$, whenever $g_1g_2\cdots g_m \in N$, there exists $i \in [m]$ and $j \in [n]$ so that $g_i, g_{i\ \ 1} \in \widetilde{M_j}$. We consider the following condition:

$(* - N)$ For each nonempty subset M of G consisting of finite number of elements of $G - N$, there exist distinct $x_1, x_2, x \ \ \in G$ such that M^{x_i} ($i \in [3]$) are mutually N-reduced.

If $N = 1$, then $(* - N)$ simply means the condition $(*)$. We get the following result:

Theorem 2.1. *Let G be a group which has a non-abelian free subgroup whose cardinality is the same as that of G, and suppose that G satisfies $(* - N)$ for some proper subgroup of G. Then, if R is a domain with $|R| \leq |G|$, the group ring RG of G over R is primitive. In particular, if $\ \ (G) = 1$, then the group algebra KG is primitive for any field K.*

The proof of the above theorem is similar to the one of Theorem 1.1. By the above theorem, even if G has the center $C \neq 1$, if G satisfies $(*-C)$ and $|R| \leq |G|$, then the group ring RG is primitivity. In fact, we can easily show the following corollary:

Corollary 2.2. *Let R be a domain and $G = A *_H B$ the free product of A and B with H amalgamated. Suppose that H is a normal subgroup and $|A| \geq |B| > 1$. If $|R| \leq |G|$, $|A : H| \geq \aleph\ \ $ and $|A : H| \geq |H|$, then RG is primitive.*

We can easily show that G in the above corollary satisfies the conditions needed in Theorem 2.1. That is, G has a nonabelian free subgroup whose cardinality is the same as that of G and also satisfies $(* - H)$.

In fact, since $|A : H| = |G|$, there exist a set I with $|I| = |G|$ and $a_i \in A$ ($i \in I$) such that $\{a_i \mid i \in I\}$ is a complete set of representatives of G/H. We can see then that the subgroup of G generated by $(a_i b)^2$ ($i \in I$) for some $b \in B$ with $b \notin H$ is freely generated by them.

Moreover, for any finite number of elements U_i $(i \in [m])$ in $G-H$, let $U_i = u_{i1} \cdots u_{im_i}$, where either $u_{ij} \in A$ and $u_{i(j\ \ 1)} \in B$ or $u_{ij} \in B$ and $u_{i(j\ \ 1)} \in A$. We set here that $S_i = \{u_{ij} \mid j \in [m_i],\ u_{ij} \in A\}$ and $S = \bigcup_{i\ \ 1}^{m} S_i$. Since $|A : H| \geq \aleph\ $, there are elements a_i in A $(i = 1, 2, 3)$ such that $a_i \notin H \cup (\bigcup_{\alpha \in S} \alpha H)$ and $a_i a_j^{-1} \notin H$ if $i \neq j$. For $b \in B$ with $b \notin H$, let $x_i = a_i^{-1} b a_i$ and $M = \{U_i \mid i \in [m]\}$. We have then that $M^{x_1}, M^{x_2}, M^{x_3}$ are mutually H-reduced.

References

[1] C. R. Abbott and F. Dahmani, *Property P_{naive} for acylindrically hyperbolic groups*, arXive:1610.04143v2

[2] J. Alexander and T. Nishinaka, *Non-noetherian groups and primitivity of their group algebras*, J. Algebra **473** (2017), 221-246

[3] B. B. Solie, *Primitivity of group rings of non-elementary torsion-free hyperbolic groups*, J. Algebra **493** (2018), 438-443.

[4] C. R. Jordan, *Group Rings of Generalised Free Products*, Ph D thesis (1975), 79-81