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The Tarski Theorems and Elementary Free Groups

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Abstract

Around 1945, Alfred Tarski proposed several questions concerning the elementary theory of non-abelian free groups. These remained open for 60 years until they were proved by O. Kharlampovich and A. Myasnikov and independently by Z. Sela. The proofs, by both sets of authors, were monumental and involved the development of several new areas of infinite group theory. In this paper we explain precisely the Tarski problems and what was actually proved. We then discuss the history of the solution as well the components of the proof and provide the basic startegy for the proof. We finish with a brief discussion of elementary free groups, that is groups that have exactly the same elementary theory as the class of nonabelian free groups

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1 Introduction

Around 1945, Alfred Tarski proposed several questions concerning the elementary theory of non-abelian free groups. These questions then became well-known conjectures but remained open for 60 years. They were proved in the period 1996-2006 independently by O. Kharlampovich and A. Myasnikov [KhM 1-5] and by Z. Sela [Se 1-5]. The proofs, by both sets of authors, were monumental, and involved the development of several new areas of infinite group theory. Because of the tremendous amount of material developed and used in the two different proofs, the details of the solution are largely unknown, even to the general group theory population. The book [FGMRS], presented an introductory guide through the material. In this paper and the talk presented we provide, for a general mathematical audience, an introduction to both the Tarksi theorems and the vast new ideas that went into the proof. These ideas straddle the line between algebra and mathematical logic and hence most group theorists don't know enough logic to fully understand the details while in the other direction most logicians don't understand enough infinite group theory. Details and an explanation of the proof can be found in the book Elementary Theory of Groups by B.Fine, A. Gaglione, A. Myasnikov, G. Rosenberger and D. Spellman.

2 Elementary and Universal Theory

A first-order sentence in group theory has logical symbols $\forall, \exists, \vee, \wedge, \sim$ but no quantification over sets. A first-order theorem in a free group is a theorem that says a first-order sentence is true in all non-abelian free groups. We make this a bit more precise:

We start with a first-order language appropriate for group theory. This language, which we denote by L_0, is the first-order language with equality containing a binary operation symbol $\cdot$ a unary operation symbol $^{-1}$ and a constant symbol 1. A **universal sentence** of L_0 is one of the form $\forall \overline{x}\{\phi(\overline{x})\}$ where $\overline{x}$ is a tuple of distinct variables, $\phi(\overline{x})$ is a formula of L_0 containing no quantifiers and containing at most the variables of $\overline{x}$. Similarly an **existential sentence** is one of the form $\exists \overline{x}\{\phi(\overline{x})\}$ where $\overline{x}$ and $\phi(\overline{x})$ are as above.

If G is a group, then the **universal theory** of G, denoted by $Th_{\forall}(G)$, consists of the set of all universal sentences of L_0 true in G. Since any universal sentence is equivalent to the negation of an existential sentence it follows that two groups have the same universal theory if and only if they have the same **existential theory**. We say that two groups G, H are **universally equivalent** if $Th_{\forall}(G) = Th_{\forall}(H)$.

The set of all sentences of L_0 true in G is called the **first-order theory** or the **elementary theory** of G, denoted by $Th(G)$. Being **first-order** or **elementary** means that in the intended interpretation of any formula or sentence all of the variables (free or bound) are assumed to take on as values only individual group elements - never, for example, subsets of nor functions, on the group in which they are interpreted.

We say that two groups G and H are **elementarily equivalent**, denoted $G \equiv H$ if they have the same first-order theory, that is $Th(G) = Th(H)$.

Group monomorphisms which preserve the truth of first-order formulas are called elementary embeddings. Specifically, if H and G are groups and

$$f : H \to G$$

is a monomorphism then f is an **elementary embedding** provided whenever $\phi(x_0, ..., x_n)$ is a formula of L_0 containing free at most the distinct variables $x_0, ..., x_n$ and $(h_0, ..., h_n) \in H^{n+1}$ then $\phi(h_0, , ..., h_n)$ is true in H if and only if

$$\phi(f(h_0), , ..., f(h_n))$$

is true in G. If H is a subgroup of G and the inclusion map $i : H \to G$ is an elementary embedding then we say that G is an **elementary extension** of H.

Two important concepts in the elementary theory of groups, are **completeness** and **decidability**. Given a non-empty class of groups $\mathcal{X}$ closed under isomorphism then we say its first-order theory is **complete** if given a sentence ϕ of L_0 then either ϕ is true in every group in $\mathcal{X}$ or ϕ is false in every group in $\mathcal{X}$. The first-order theory of $\mathcal{X}$ is **decidable** if there exists a recursive algorithm which, given a sentence ϕ of L_0 decides whether or not ϕ is true in every group in $\mathcal{X}$.

3 The Tarski Problems

Tarski first asked the general question whether all non-abelian free groups share the same elementary theory. Vaught, a student of Tarksi's, proved almost immediately that all free groups of infinite rank do have the same elementary theory, and thus reduced the question to the class of non-abelian free groups of finite rank. After this, Tarski's question was formalized into the following conjectures.

Tarski Conjecture 1 *Any two non-abelian free groups are elementarily equivalent. That is any two non-abelian free groups satisfy exactly the same first-order theory.*

Tarski Conjecture 2 *If the non-abelian free group H is a free factor in the free group G then the inclusion map $i : H \to G$ is an elementary embedding.*

The second conjecture implies the first. Hence the theory of the non-abelian free groups is **complete**, that is, given a sentence ϕ of L_0 then either ϕ is true in every non-abelian free group or ϕ is false in every non-abelian free group.

After a long series of partial results the positive solution to the Tarksi conjectures was given by O. Kharlampovich and A. Myasnikov [KhM 1-5] and independently by Z.Sela [Se 1-5]. The proofs by both sets of authors involved the development of whole new areas of mathematics, in particular an algebraic geometry (Sela calls this diophantine geometry) over free groups. The basic theorems eventually proved were:

Theorem 3.1. *(Tarski 1:) Any two non-abelian free groups are elementarily equivalent. That is any two non-abelian free groups satisfy exactly the same first-order theory.*

Theorem 3.2. *(Tarski 2:) If the non-abelian free group H is a free factor in the free group G then the inclusion map $H \to G$ is an elementary embedding.*

Kharlampovich and Myasnikov in addition to the proofs of the main Tarksi conjectures also proved that the theory is decidable (see [KhM 5])

Theorem 3.3. *Tarski 3: The elementary theory of the non-abelian free groups is decidable.*

Although Tarksi was never explicit on the origin of the basic question, it is motivated by several results, and concepts, in the theory of free groups (see [MKS],[LS], and [FGMRS] for complete discussions of free groups). First is the observation that most free group properties, involving elements, are rank independent, that is, true for all free groups independent of rank. For example all non-abelian free groups are torsion-free and all abelian subgroups of non-abelian free groups are cyclic.

A second possible motivation, which also shows that all non-abelian free groups have the same universal theory, is the following. Let F_2 be a free group of rank 2. It is a straightforward consequence of the Reidemeister-Schreier process (see [MKS]) that the commutator subgroup of F_2 is free of infinite rank. This implies that if we let F_ω denote a free group of countably infinite rank, then $F_\omega \subset F_2$. It follows that for any $m, n \geq 2$ with $m < n$ we have the string of inclusions

$$... \subset F_\omega \subset F_2 \subset F_m \subset F_n \subset ... \subset F_\omega \subset ...$$

This shows that $F_n \subset F_m$ and $F_m \subset F_n$. Its like a snake eating its tail.

If $G \subset H$ then any universal sentence in H must also be true in G, that is $Th_\forall(H) \subset Th_\forall(G)$. This observation combined with the observations above prove that all non-abelian free groups have the same universal theory and hence are universally equivalent.

Theorem 3.4. *All non-abelian free groups are universally equivalent.*

A group with the same universal theory as a non-abelian free group is called a **universally free group**. The above theorem then opens the question as to whether the class of universally free groups extends beyond the class of free groups. One of the initial steps toward the proof of the Tarksi problems was a group theoretical characterization of universally free groups. In the finitely generated case these turn out to be the **fully residually free groups**.

4 The History of the Solution

The final proof of the Tarski theorems was a monumental collection of work by both sets of authors. In addition to dealing with already existing ideas in group theory and logic, the solution involved the development of several new areas of group theory. In particular three areas of group theory had to be fully developed before the proof could be completed. First: the theory of **fully residually free groups**. In Sela's approach these were called **limit groups**. Next, the **Makhanin-Razborov technique** for solving equations within free groups and finally the development of **algebraic geometry over groups**. Sela calls this **diophantine geometry**.

We will discuss each of these in turn. First we look at the initial partial results that were done between the statement of the problem by Tarski (in 1945) and the final proofs (1998-2006).

The first progress was due to Vaught, a student of Tarski, who showed that the Tarski conjectures 1,2 are true if G and H are both free groups of infinite rank. This reduced the problem to free groups of finite rank, that is, in showing that all non-abelian free groups of finite rank share the same elementary theory or even stronger that the embedding of a free group of rank m into a free group of rank n, with $m < n$, is an elementary embedding.

The basic idea in Vaught's proof is to use the following criteria for elementary embeddings; if H_0 is a subgroup of H and that to every finite subset $\{a_1, ..., a_n\}$ of H_0 and every element $b \in H$ there exists an automorphism σ of H fixing $a_1, ..., a_n$ and mapping b into H_0, then the inclusion map from H_0 into H is an elementary embedding. Applying this criterion to free groups of infinite rank, suppose that F is free on an infinite subset S and that G is free on an infinite subset S_0 of S. Then permutations of S will induce enough automorphisms to guarantee that the inclusion map of G into F is an elementary embedding.

The next significant progress was due to Merzljakov [Mer]. A **positive sentence** is a first-order sentence which is logically equivalent to a sentence constructed using (at most) the connectives $\vee, \wedge, \forall, \exists$. The **positive theory** of a group G consists of all the positive sentences true in G.

Merzljakov showed that the non-abelian free groups have the same positive theory.

Merzljakov's proof used what are now called generalized equations and a quantifier elimination process. This was a precursor to the methods used in the eventual solution of the overall Tarksi problems.

Two non-abelian free groups satisfy the same universal theory. Sacerdote [Sa] extended this to **universal-existential sentences**. The set of universal-existential sentences true in a group G is called the Π_2-theory of G. Sacerdote's [Sa] result is then that all non-abelian free groups have the same Π_2-theory.

That all non-abelian free groups have the same universal theory coupled with the fact that universally free is equivalent to existentially free says that Tarski conjecture 1 is true if there is only one quantifier. Sacerdote's extension to Π_2-theory shows that the Tarski conjecture 1 is true if there are two quantifers. Sacerdote's theorem becomes the initial step in the final proof which employs an induction based on the number of quantifiers.

A first step to the initial proofs was to completely characterize those groups that are universally free. This was accomplished within the study of fully residually free groups. A group G is **residually free** if for each non-trivial $g \in G$ there is a homomorphism $\phi : G \to F$ where F is a free group and $\phi(g) \neq 1$. A group G is **fully residually free** if for each finite subset of non-trivial elements $g_1, ..., g_n$ in G there is a homomorphism $\phi : G \to F$ where F is a free group and $\phi(g_i) \neq 1$ for all $i = 1, .., n$.

Fully residually free groups arise in Sela's approach as limiting groups of homomorphisms from a group G into a free group. Sela shows that such groups in the finitely generated case are equivalent to fully residually free groups. Hence. a finitely generated fully residually free group is also called a **limit group**. This has become the more common designation (see [FGMRS] for a proof of the equivalence)

Two concepts are crucial in the study of limit groups. A group G is **commutative transitive** or **CT** if commutativity is transitive on the set of non-trivial elements of G. That is if $[x, y] = 1$ and $[y, z] = 1$ for non-trivial elements $x, y, z \in G$ then $[x, z] = 1$. A group G is CSA or **conjugately separated abelian** if maximal abelian subgroups are malnormal. A subgroup $H \subset G$ is malnormal if $g^{-1}Hg \cap H \neq \{1\}$ implies that $g \in H$. CSA groups are always CT but there exist CT groups that are not CSA. As we will see, in the presence of residual freeness they are equivalent. A classification of CT non-CSA groups was given in [FGRS 3].

In 1967 Benjamin Baumslag [BB] proved the following result who's innocuous beginnings belied its much greater later importance. It was in this paper that the concept of full residual freeness was first explored.

Theorem 4.1. *(B.Baumslag [BB]) Suppose G is residually free. Then the following are equivalent:*
(1) G is fully residually free,
(2) G is commutative transitive,

Gaglione and Spellman [GS] and independently Remeslennikov [Re] extended B.Baumslag's Theorem and this extension became one of the cornerstones of the proof of the Tarksi problems

Theorem 4.2. *([GS],[Re]) Suppose G is residually free. Then the following are equivalent:*
(1) G is fully residually free,
(2) G is commutative transitive,
(3) G is universally free if non-abelian.

Further the result can be extended to include the equivalence with CSA. In addition Remeslennikov and independently Chiswell (see [Ch]) showed that if a group G is finitely generated then being fully residually free is equivalent to being universally free. Therefore the finitely generated universally free groups are precisely the finitely generated fully residually free groups which are non-abelian

Theorem 4.3. *Let G be finitely generated. Then G is a limit group if and only if G is universally free.*

Ciobanu, Fine and Rosenberger [CFR] recently greatly extended the class of groups satisfying both B.Baumslag's original theorem and the theorem of Gaglione, Spellman and Remeslennikov.

The solution of the Tarski conjectures involved analyzing groups which have the same elementary theory as a free group. Clearly this includes the universally free groups and therefore the theory of limit groups became essential to the proof and to analyzing those groups which have the same elementary theory as a free group

It was clear that to deal with the Tarski problems it was necessary to give a precise definition of solution sets of equations and inequations over free groups. In this direction R. Lyndon [L] introduced the concept of an exponential group, that is a group which allows parametric exponents in an associative unitary ring A. In particular he studied the free exponential group $F^{\mathbb{Z}[t]}$ where exponents are allowed from the polynomial ring $\mathbb{Z}[t]$ over the integers $\mathbb{Z}$. Lyndon established that the free exponential group $F^{\mathbb{Z}[t]}$ and hence any finitely generated subgroup of it, is fully residually free and hence, if it is non-abelian, universally free. Kharlampovich and Myasnikov [KhM 7,8] established the converse; therefore a finitely generated group is fully residually free if and only if it is embeddable in $F^{\mathbb{Z}[t]}$.

Advances on solving equations in free group were given by Makanin and Razborov (see [Mak 1,2],[Ra]). Makanin proved that there exists an algorithm to determine, given a finite system of equations over a free group, whether the system possesses at least one solution. Razborov working with the Makanin algorithm determined an algorithm to effectively describe the solution sets of a finite system of equations over a free group.

Kharlampovich and Myasnikov further refined the Makanin-Razborov method. Their technique allows one to transform arbitrary finite systems of equations in free groups to some canonical forms and describe precisely the irreducible components of algebraic sets in free groups.

These canonical forms consist of finitely many quadratic equations in a triangular form. The following result is a corollary of the decidability of the Diophantine problem

Theorem 4.4. *(Makanin) [Mak 1,2] (1) The existential (and hence the universal) theory of a free group is decidable.*
(2) The positive theory of a free group is decidable

The final ingredient that was needed for the proof was the development of an **algebraic geometry over groups**. In analogy with the classical theory of equations over number fields, algebraic geometry over groups was developed by G. Baumslag, A. Myasnikov and V. Remeslennikov [BMR

1,2]. The theory of algebraic geometry over groups translated the basic notions of the classical algebraic geometry: algebraic sets, the Zariski topology, Noetherian domains, irreducible varieties, radicals and coordinate groups to the setting of equations over groups.

This provided the necessary machinery to transcribe important geometric ideas into pure group theory. The proof of the Tarski conjectures depends on the algebraic geometry of free groups. In particular it depends on the description of a fully residually free group as the coordinate group of an irreducible algebraic variety. A full description of the algebraic geometry of free group is given in [FGMRS]

What did not translate immediately was the Noetherian property which is crucial in classicial algebraic geomerty. For the group based algebraic geometry, what had to be introduced was **equationally Noetherian groups** which is the group theoretic counterpart of the Noetherian condition. The Noetherian condition in rings is defined in terms of the ascending chain condition and implies that every ideal is finitely generated. What is important about this condition in algebraic geometry is the Hilbert Basis theorem that asserts that every algebraic set is finitely based. That is if S be a set of polynomials in $k[x_1, ..., x_n]$ then $V(S) = V(S_1)$ for some finite set of polynomials. This is what is recast in terms of group theory. First a G-group H is a group which has a distinguished subgroup isomorphic to G. If S is a set of equations over a group G then $V(S)$ is its set of solutions in G.

Definition 4.1. *A G-group H is said to be G-***equationally Noetherian** *if for every $n > 0$ and every subset S of $G[x_1, ..., x_n]$ there exists a finite subset S_0 of S such that*

$$V(S) = V(S_0).$$

The first major examples of equationally Noetherian groups are linear groups over commutative Noetherian rings. This was proved originally by R. Bryant [Bry] in the one variable case and then extended by V. Guba [Gu] to the case of free groups. The general result is the following.

Theorem 4.5. *Let H be a linear group over a commutative, Noetherian ring with unity and in particular a field. Then H is equationally Noetherian.*

In particular, it follows that a finitely generated non-abelian free group is equationally noetherian.

Extremely important in the application of the algebraic geometry of groups to the proof of the Tarski problems is the description of the coordinate groups of systems of equations. Radicals of a system of equations and coordinate group are defined as in classical algebraic geometry. Examining the relationship between the coordinate groups and groups embeddable by a sequence of extensions of centralizers in the free exponential group $F^{\mathbb{Z}[t]}$, shows that the coordinate groups of irreducible algebraic varieties are precisely the finitely generated fully residually free groups (limit groups).

5 Strategy for the Proof

All these components had to be combined and integrated to provide the final proofs. Here we outline the strategy that was followed. Recall that Vaught proved Tarski Conjecture 2 for all free groups of infinite rank and hence reduced the problem to non-abelian free groups of finite rank. Vaught's main result was that if the infinite rank free group F_1 is a free factor of the infinite rank free group F_2 then F_1 is an elementary subgroup of F_2, that is the identity map embedding F_1 into F_2 is an elementary embedding. Sacerdote went on to prove that all free groups of finite rank have the same Π_2-theory, that is they satisfy exactly the same $\forall\exists$ (and equivalently $\exists\forall$) sentences. It is Sacerdote's result that pinpoints the main strategy in solving the whole problem and provides the first step in an induction.

The main technique Vaught used in proving the Tarksi conjecture for infinite rank and Sacerdote used for the Π_2-theory is the following, that is known as the **Tarski-Vaught Test**.

Tarski-Vaught Test If H is a subgroup of G then H is an elementary subgroup of G if and only if for any formula $\phi(x, \overline{z})$ and for any tuple $(\overline{h})$ of elements from H there exists a $c \in G$ such that $\phi(c, \overline{h})$ is satisfied in G implies that there exists $c \in H$ such that $\phi(c, \overline{h})$ is satisfied in H.

Roughly the Tarski-Vaught Test says that a subgroup H of G is an elementary subgroup if and only if H is **algebraically closed** in G. In analogy with commutative algebra if we consider first order sentences with variables as our equations then any equation with constants from H which has a solution in G already has a solution within H.

If we wish to apply the Tarksi-Vaught Test to the case of a free factor in a free group of finite rank we must then understand the nature of solving equations in free groups and over free groups. The work of Makanin and Razborov became crucial. Their work provided first a method to determine if an equation over a free group was solvable and hence provided a technique for Kharlampovich and Myasnikov to show that the elementary theory of non-abelian free groups was decidable.

Here is where, however, it was the introduction of algebraic geometry over free groups that led to the necessary understanding of groups that have the same elementary theory as a non-abelian free group of finite rank.

The proofs of Kharlampovich-Myasnikov and Sela show that a general system of equations, with a few special cases that must be handled separately, can be shown to be equivalent to what is called a **quasi-trianglular system** of quadratic equations.

The coordinate groups of such systems are called QT-groups and are limit groups. A special subclass of them, called special NTQ-groups, are precisely the groups that can be shown to have the same elementary theory as the non-abelian free groups.

The structure of the algebraic variety of a system of equations can be broken down by the Makanin-Razborov method and is tied to the group theoretic breakdown of the coordinate groups.

Since the coordinate groups are limit groups this breakdown is well-understood as the JSJ decomposition of limit groups. The JSJ decompositon of a finitely generated group was developed originally by Rips and Sela [RiS]. It is graph of groups decomposition with abelian edge groups that encodes all other amalgam decompositions of a group.

It is the JSJ decomposition of the coordinbate groups combined with a type of implicit function theorem that provides for a quantifier elimination process that permits an induction starting with Sacerdote's Π_2-result.

After all these massive preliminaries the proof itself is then an induction on the number of quantifiers, based on a **quantifer elimination process**. In the Kharlampovich-Myasnikov approach the quantifier elimination is handled by an **implicit function** theorem for quadratic systems. A summary of the proof can be found in the book [FGMRS].

6 Elementary Free Groups

The proof of the **Tarski theorems** provided a complete characterizations of those finitely generated groups that have exactly the same first order theory as the non-abelian free groups. Such groups are called **elementary free groups** and extend beyond the class of purely non-abelian free groups. In the Kharlampovich-Myasnikov approach these are the special NTQ-groups and in the Sela approach the hyperbolic ω-residually free towers. The primary examples of non-free elementary free groups are the orientable surface groups S_g of genus $g \geq 2$ and the non-orientable surface groups N_g of genus $g \geq 4$. That these groups are elementary free provides a powerful tool to prove some results concerning surface groups that are otherwise quite difficult. For example J.Howie [H] and independently O. Bogopolski and O,Bogopolski and V. Sviridov [Bo],[BoS] proved that a theorem of Magnus about the normal closures of elements in free groups holds also in surface groups of appropriate genus. Their proofs were non-trivial. However it was proved (see [FGRS 1,2] and [GLS]) that this result is first order and hence automatically true in any elementary free group. In

[FGRS 1] a large collection of such results was given. Such results were called *something for nothing results.* Of course any such first order result true in a non-abelian free group must hold in any elementary free group. However elementary free groups satisfy many other properties beyond first order results. This second idea were explored in the paper [FGRS 2]. In this final section we survey briefly these two ideas.

Magnus proved the following theorem about the normal closures of elements in non-abelian free groups:

Theorem 6.1. *(Magnus) Let F be a non-abelian free group and $R, S \in F$. Then if $N(R) = N(S)$,it follows that R is conjugate to either S or S^{-1}. Here $N(g)$ denotes the normal closure in F of the element g.*

J. Howie [H] and independently O. Bogopolski and Bogopolski and V.Sviridov [BoS] gave a proof of this for surface groups. Howie's proof was for orientable surface groups while Bogopolski and Sviridov also handled the non-orientable case. Their proofs were non-trivial and Howie's proof used the topological properties of surface groups. Howie further developed, as part of his proof of Magnus' theorem for surface groups, a theory of one-relator surface groups. These are surface groups modulo a single additional relator. Bogopolski and Bogopolski-Sviridov proved in addition that Magnus's Theorem holds in even a wider class of groups. In [FGRS 1] (see also [FGRS 2] and [GLS]) it was proved that Magnus' result is actually a first-order theorem on non-abelian free groups and hence from the solution to the Tarski problems it holds automatically in all elementary free groups. In particular Magnus' theorem will hold in surface groups, both orientable and non-orientable of appropriate genus. If G is a group and $g \in G$ then $N(g)$, as in the statement of Magnus's Theorem above, will denote the normal closure in G of the element g.

Theorem 6.2. *Let G be an elementary free group and $R, S \in G$. Then if $N(R) = N(S)$ it follows that R is conjugate to either S or S^{-1}.*

As corollaries we recover the results of Howie [H], Bogopolski [Bo] and Bogopolski-Sviridov [BoS] which extend Magnus's Theorem to surface groups

Corollary 6.1. *([H],[Bo], [BoS]) Let S_g be an orientable surface group of genus $g \geq 2$. Then S_g satisfies Magnus's theorem, that is if $u, v \in S_g$ and $N(u) = N(v)$ it follows that u is conjugate to either v or v^{-1}. Further if N_g is a non-orientable surface group of genus $g \geq 4$, then N_g also satisfies Magnus's theorem. For N_G The genus $g \geq 4$ is essential here.*

In [FGRS 1] a collection of results about elementary free groups and surface groups was presented, their proofs being consequences of the Tarski theorem. We mention one such result that is not obvious in a surface group. The following theorem can be easily proved in free groups.

Theorem 6.3. *Let F be a free group and n, k non-zero integers. For all $x, y \in F$ if $[x^n, y] = [x, y^k]$ then either $n = k = 1$ or x, y commute and both are powers of a single element.*

The first part of the result that either $n = k$ or $[x, y] = 1$ is first-order given by a sequence of elementary sentences, one for each $(n, k) \in \mathbb{Z}^2 \setminus \{(1, 1)\}$ with neither n nor k zero;

$$\forall x, y \in F([x^n, y] = [x, y^k]) \implies [x, y] = 1$$

Therefore this part of the result must hold in any elementary free group. Further if the elementary free group is finitely generated the second part must also hold.

Corollary 6.2. *Let G be an elementary free group. If $x, y \in G$ and if $[x^n, y] = [x, y^k]$ then either $n = k = 1$ or x, y commute. If G is finitely generated then both x and y are powers of a single element $w \in G$.*

Since surface groups are finitely generated we have the following.

Corollary 6.3. *Let G be either an orientable surface group of genus $g \geq 2$ or a non-orientable surface group of genus $g \geq 4$. If $x, y \in G$ and if $[x^n, y] = [x, y^k]$ then either $n = k = 1$ or x,y commute and then both x and y are powers of a single element $w \in G$.*

In another direction in [FGRS 2] properties of all elementary free groups, which may not be first order were explored. A finitely generated elementary free group G must be a limit group and many of its properties follow from from the structure theory of limit groups. Hence such a group must be CSA and any 2-generator subgroup is either free or abelian.

In [FGRS 2] it was proved that a finitely generated elementary free group has cyclic centralizers. This is not a first order statement, however from this we get that if two elements commute in a finitely generated elementary free group then they are both powers of a single element. This is not true in a general elementary free group. An example where it does not hold in the infinitely generated case is given in [FGRS 2]. From the cyclic centralizer property we can obtain that a finitely generated elementary free group must be hyperbolic, stably hyperbolic and a Turner group, that is the test elements, if there are any, in any finitely generated elementary free group are precisely those elements that do not lie in any proper retract. It was also proved in [FGRS 2] that any finitely generated elementary free group is conjugacy separable and hence has a solvable conjugacy problem. in [FKMRR] it was shown the automorphism group of a finitely generated elementary free group is tame.

The next theorem summarizes many of these results. The proofs can be found [FGKRS].

Theorem 6.4. *Let G be a finitely generated elementary free group. Then:*

(1) (Magnus's Theorem) if $N(R) = N(S)$ if $R, S \in G$ it follows that R is conjugate to either S or S^{-1}

(2) G has cyclic centralizers of non-trivial elements. It follows that if $x, y \in G$ and x,y commute then both x and y are powers of a single element $w \in G$.

(3) if $x, y, u, v \in G$ with $[x, y] \neq 1$ and u, v in the subgroup generated by x, y it follows that if $[x, y]$ is conjugate to a power of $[u, v]$ within $\langle x, y \rangle$ that is there exists a k with $[x, y] = g([u, v]^k)g^{-1}$ for some $g \in \langle x, y \rangle$ and $[x, y^m] = [u, v^n]$ it follows that $m = n$. Further if $m = n \geq 2$ then y is conjugate within $\langle x, y \rangle$ to v or v^{-1}.

(4) G is conjugacy separable.

(5) G is hyperbolic and stably hyperbolic.

(6) G is a Turner group, that is the test elements in G are precisely those elements that do not fall in a proper retract

(7) if G is freely indecomposable then the automorphism group of G is tame.

(8) G has a faithful representation in $PSL(2, \mathbb{C})$.

7 References

[BB] B.Baumslag, Residually free groups, **Proc. London Math. Soc.** (3), 17,1967, 635 – 645. 1–4.

[BMR] G. Baumslag, A.Myasnikov and V.Remeslennikov, Algebraic geometry over groups I. Algebraic sets and ideal theory, **J. of Algebra**, 219, 1999, 16-79.

[GB] G.Baumslag, On generalised free products **Math. Z.**, 78 ,1962, 423–438

[Bo] O. Bogopolski, A surface analogue of a theorem of Magnus, **Cont. Math**, vol. 352, 2005, 55-89

[BoS] O. Bogopolski and K. Sviridov, A Magnus theorem for some one-relator groups, in **The Zieschang Gedenkschrift** v.14, 2008, 63-73.

[Ch 2] I. Chiswell, Abstract length functions in groups, **Math. Proc. Cambridge Philos. Society**, 80, 1976, 451–463.

[CFR] L. Ciobanu, B.Fine and G. Rosenberger, Classes of Groups Generalizing a Theorem of Benjamin Baumslag, - to appear - Comm. in Alg.

[FR] B. Fine and G. Rosenberger, **Algebraic Generalizations of Discrete Groups**, Marcel-Dekker, 1999

[FR 3] B. Fine and G. Rosenberger, Faithful Representations of Hyperbolic Limit Groups Journal Groups, Complexity and Cryptology , Vol.3 No.2, 2011, 349-355

[FKMRR] B. Fine, O.Kharlampovich, A.Myasnikov, V.Remeslennikov and G. Rosenberger, Tame Automorphisms of Elementary Free Groups, **Comm. in Alg.**, 1, 2012, 1-15

[FRSS] B.Fine, G.Rosenberger, D. Spellman and M.Stille, Test Elements, Generic Elements and Almost Primitivity in Free groups, - **Pacific J. of Math.** 190 No. 2, 1999 277-297

[FGMRS 1] B.Fine, A. Gaglione, A. Myasnikov, G.Rosenberger and D. Spellman, **The Elementary Theory of Groups**, DeGruyter 2014

[FGMRS 2] B.Fine, A. Gaglione, A. Myasnikov, G.Rosenberger and D. Spellman, A Classification of Fully Residually Free Groups of Rank Three or Less, **J. of Algebra** , 200, 1998, 571–605

[FGRS 1] B.Fine, A. Gaglione, G.Rosenberger and D. Spellman, Something for Nothing:Some Consequences of the Solution of the Tarski Problems, to appear **Proc. of Groups St Andrews 2013**

[FGRS 2] B.Fine, A. Gaglione, G.Rosenberger and D. Spellman, On Elementary Free Groups, to appear Cont. Math . 2015

[FGRS 3] B.Fine, A. Gaglione, G.Rosenberger and D. Spellman, On CT and CSA Groups, to appear Int. J. Grp Theory

[GS] A. Gaglione and D. Spellman, Even More Model Theory of Free Groups, in **Infinite Groups and Group Rings** edited by J.Corson,M.Dixon,M.Evans,F.Rohl, World Scientific Press, 1993, 37-40

[GLS] A. Gaglione, S. Lipschutz and D. Spellman, Almost Locally Free Groups and a Theorem of Magnus, **J. groups,compleity and Cryptology**, 1, 2009, 181-198

[Gu] V. Guba, Equivalence of infinite systems of equations in free groups and semigroups to finite subsystems, **Mat. Zametki**, 40, 1986, 321324.

[H] J.Howie, Some Results on One-Relator Surface Groups, **Boletin de la Sociedad Matematica Mexicana**, 10, 2004, 255-262

[KhM 1] O. Kharlamapovich and A.Myasnikov *Irreducible affine varieties over a free group: I. Irreducibility of quadratic equations and Nullstellensatz* **J. of Algebra**, 200, 1998, 472-516

[KhM 2] O. Kharlamapovich and A.Myasnikov *affine varieties over a free group: II. Systems in triangular quasi-quadratic form and a description of residually free groups* **J. of Algebra**, 200, 1998, 517-569

[KhM 3] O. Kharlamapovich and A.Myasnikov *The Implicit Function Theorem over Free groups* **J. Alg.** , 290, 2005, 1-203

[KhM 4] O. Kharlamapovich and A.Myasnikov *Effective JSJ Decompositions* **Cont. Math.** , 378, 2005, 87-211

[KhM 5] O. Kharlamapovich and A.Myasnikov *Elementary Theory of Free non-abelian Groups* **J. Alg.** , 302, 2006, 451-552

[KhM 6] O.Kharlamapovich and A.Myasnikov *Hyperbolic Groups and Free Constructions* **Trans. Amer. Math. Soc.**, 350, 2, 1998, 571-613

[KhM 7] O. Kharlamapovich and A.Myasnikov **Algebraic Geometry over Free Groups** to appear

[KhM 8] O. Kharlamapovich and A.Myasnikov *Algebraic Geometry over Free Groups: Lifting Solutions Into Generic Poitns* **Cont. Math.** , 378, 2005, 212-313

[KhM 9] O. Kharlampovich and A.G. Myasnikov,*Implicit function theorem and genus problem* **Proc. of Birmanfest, AMS/99, Studies Adv. Math.** 124 (2001), 77 - 83.

[L] R. C. Lyndon, Groups with parametric exponents, **Trans. Amer. Math. Soc.**, 96, 1960, 518533.

[LS] R.C. Lyndon and P.E. Schupp, **Combinatorial Group Theory**, Springer-Verlag 1977

[MKS] W.Magnus, A. Karrass and D.Solitar, **Combinatorial group theory: Presentations of groups in terms of generators and relations** Interscience Publishers - John Wiley and Sons, Inc., New York, London, Sydney. 1966.

[Mak 1] G.S. Makanin, Equations in a free group (Russian), **Izv. Akad. Nauk SSSR, Ser. Mat.** 46, 1982, 1199–1273 transl. in Math. USSR Izv., V. 21, 1983; MR 84m:20040.

[Mak 2] G.S. Makanin, Decidability of the universal and positive theories of a free group, **Math. USSR Izvestiya** 25, 1985, 75–88

[Mer] Y.I. Merzlyakov, Positive Formulas on Free Groups, **Algebra i Logika**, 5, 1966,25-42.

[Ra] A.A. Razborov, On systems of equations in free groups, **Izv.Akad. Nauk SSSR** , 48, 1984, 779-832 Englisg transl: Math, USSR Izv. 25 115-162.

[Re] V.N. Remeslennikov, $\exists$-free groups **Siberian Mat. J.**, 30, 1989, 998–1001.

[Sa] G.S. Sacerdote, Elementary properties of free groups,**Trans. Amer. Math. Soc.** , 178, 1972, 127-138.

[RiS] E.Rips and Z.Sela, Cyclic Splittings of Finitely Presented Groups and the Canonical JSJ Decomposition, **Ann. of Math. (2)** , 146. 1997, 53-109

[Se 1] Z. Sela, The Isomorphism Problem for Hyperbolic Groups I., **Ann. of Math**, 141, 2, 1995, 217-283

[Se 2] Z. Sela, Diophantine Geometry over Groups I: Makanin-Razborov Diagrams, **Publ. Math. de IHES** 93, 2001, 31-105

[Se 3] Z. Sela, Diophantine Geometry over Groups II: Completions, Closures and Fromal Solutions, **Israel Jour. of Math.**, 104, 2003, 173-254

[Se 4] Z. Sela, Diophantine Geometry over Groups III: Rigid and Solid Solutions, **Israel Jour. of Math.**, 147, 2005, 1-73

[Se 5] Z. Sela, Diophantine Geometry over Groups IV: An Itertaive Procedure for Validation of a Sentence, **Israel Jour. of Math.**, 143, 2004, 1-71

[Se 6] Z. Sela, Diophantine Geometry over Groups V: Quantifier Elimination, **Israel Jour. of Math.**, 150, 2005, 1-97