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A sufficient condition for a finite group to be a Borsuk-Ulam group

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1 Introduction

In this paper, we always assume that a group means a finite group. A G-map $f\colon X\to Y$ is said to be a G-isovariant map if $G_x=G_{f(x)}$ for any $x\in X$, where G_x is the isotropy subgroup, that is, $G_x=\{g\in G\mid g\cdot x=x\}$. We call a group G is a BUG (Borsuk-Ulam group) [5] if

$$\dim V-\dim V^G\leq\dim W-\dim W^G$$

for any isovariant G-map $f\colon V\to W$ between G-representation spaces V and W.

Let C_2 be a cyclic group of order 2 and let $f\colon V\to W$ be an isovariant C_2-map between C_2-representation spaces V and W. Fixing a G-invariant inner product, f induces a free C_2-map $S(f|_{V-V^{C_2}})\colon S(V-V^{C_2})\to S(W-W^{C_2})$ between C_2-representation spheres, where $V-V^{C_2}$ is an orthogonal vector subspace of V^{C_2} in V. By Borsuk-Ulam theorem, this map gives $\dim S(V-V^{C_2})\leq\dim S(W-W^{C_2})$. Since $\dim S(V-V^{C_2})=\dim V-\dim V^{C^2}-1$, C_2 is a BUG. For a cyclic group C_p of prime order p, Kobayashi [2] showed that $\dim S(V)\leq\dim S(W)$ for a free C_p-map $S(f')\colon S(V)\to S(W)$ between representation spheres and thus C_p is a BUG.

Let G be a group extension of K by H: $1\to H\to G\to K\to 1$ and $f\colon V\to W$ be an isovariant G-map. Since the equality

$$\begin{aligned}\dim W-\dim W^G\ -&(\dim V-\dim V^G)\\ &=(\dim W-\dim W^H-(\dim V-\dim V^H))\\ &\quad+(\dim W^H-\dim W^G-(\dim V^H-\dim V^G))\end{aligned}$$

holds, if K and H are BUGs then G is a BUG [5]. Therefore any solvable group is a BUG. Then it is natural to ask whether a group is a BUG or not.

Wasserman [5] proposed a prime condition which implies a sufficient condition for a group to be a BUG. A positive integer n satisfies the *prime condition* if

$$p_1^{-1}+p_2^{-1}+\cdots+p_r^{-1}<1,$$

where $p_1,\dots,p_r$ are primes and $e_1,\dots,e_r$ are positive integers such that $n=p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r}$. A group G satisfies the *prime condition* if the order of any cyclic subgroup of G satisfies the prime condition.

Theorem 1 ([5]) *If a group G satisfies the prime condition, then G is a BUG.*

Let $\mathrm{Cycl}(G)$ be the set of all cyclic subgroups of G. Nagasaki and Ushitaki [3] proposed a Möbius condition: A group G satisfies the *Möbius condition* if

$$\sum_{\substack{D\in \mathrm{Cycl}(G)\\ C\leq D}} \mu(\frac{|D|}{|C|}) \geq 0$$

for any cyclic subgroup C of G, where $\mu\colon \mathbb{N} \to \{0, \pm 1\}$ is the Möbius function, that is,

$$\mu(n) = \begin{cases} 1 & n = 1 \\ 0 & \text{if } p^2 | n \text{ for some prime } p \\ (-1)^r & n = p_1p_2\cdots p_r \text{ for distinct primes } p_1, p_2, \dots, p_r. \end{cases}.$$

Theorem 2 ([3]) *If a group G satisfies the Möbius condition, then G is a BUG.*

Since if K and H are BUGs then a group extension of H by K is a BUG, if we obtain that every simple group is a BUG, then any group is a BUG. By the above theorem, Nagasaki and Ushitaki showed that projective linear groups $\mathrm{PSL}(2, q)$ are BUGs. In this paper, we give a sufficient condition for a group to be a BUG and apply projective linear groups $\mathrm{PSL}(3, q)$ and alternating groups A_n.

2 A sufficient condition

Let V and W be G-representation spaces and let $f\colon V \to W$ be an isovariant G-map. For a subgroup H of G, let

$$g_f(H) = (\dim W - \dim W^H) - (\dim V - \dim V^H).$$

Note that If G is a cyclic group then $g_f(G) \geq 0$.

Proposition 3 *Let H_1 and H_2 be a subgroups of G with $H_1 \triangleleft H_2$ and f an isovariant G-map between representation spaces.*

$$g_f(H_2) - g_f(H_1) = g_{f^{H_1}}(H_2/H_1)$$

holds. In particular, if H_2/H_1 is a BUG, *$g_f(H_2) \geq g_f(H_1)$ holds.*

Let $\mathcal{S}(G)$ denote the set of all subgroups of G. It is made into a poset by defining $H \leq K$ in $\mathcal{S}(G)$ if H is a subgroup of K. Let $\mathrm{Cycl}(G)$ be the full subposet of $\mathcal{S}(G)$ which contains all cyclic subgroups of G.

We put

$$\mu(C, D) = \begin{cases} \mu(\frac{|D|}{|C|}), & C \leq D \\ 0, & \text{otherwise.} \end{cases}$$

Nagasaki and Ushitaki [3] showed that $\mathrm{PSL}(2, q)$ satisfies the Möbius condition by using the following equation.

Theorem 4 ([3]) *Let $f\colon V \to W$ be a G-map between representation spaces.*

$$|G|g_f(G) = \sum_{C\in \mathrm{Cycl}(G)} \left(\sum_{D\in \mathrm{Cycl}(G)} \mu(C,D) \right) |C| g_f(C)$$

holds. If G satisfies the Möbius condition then G is a BUG.

Let $\mathrm{RCycl}(G)$ be the set of representatives of conjugacy classes of all cyclic subgroups of G and let $\mathrm{RCycl}_1(G)$ be the set of representatives of conjugacy classes of all nontrivial cyclic subgroups of G. Recall that $g_f(\{e\}) = 0$.

Let

$$\tilde{\mu}(C,D) = \begin{cases} \mu(\frac{|D|}{|C|}), & (C) \le (D) \\ 0, & \text{otherwise.} \end{cases}$$

where (C) denotes the conjugacy class of C.

Lemma 5 *For $C \in \mathrm{RCycl}(G)$, $\sum_{D\in \mathrm{Cycl}(G)} \mu(C,D) = \sum_{D\in \mathrm{RCycl}(G)} \frac{|N_G(C)|}{|N_G(D)|}\tilde{\mu}(C,D)$.*

Proof Let $C < D \in \mathrm{Cycl}(C_G(C))$, $S = \{E \in \mathrm{Cycl}(G) \mid (D) = (E), E \ge C\}$ and let $f\colon N_G(C) \to S$ be a map which sends g to $g^{-1}Dg$. Let $E \in S$. Then $E = g^{-1}Dg$ for some $g \in G$. C and $g^{-1}Cg$ is a subgroup of E with same index. Since for any $k > 0$ dividing $|D|$, a subgroup of order k of the cyclic group D is unique, we see that $C = g^{-1}Cg$ and thus $g \in N_G(C)$ and $f(g) = E$. Therefore, the map f is surjective. Thus, $\#S = \frac{|N_G(C)|}{|N_G(D)|}$. If D_1 and D_2 are conjugate, then $\tilde{\mu}(C,D_1) = \tilde{\mu}(C,D_2)$. Therefore we see that

$$\begin{aligned}
\sum_{D\in \mathrm{Cycl}(G)} \mu(C,D) &= \sum_{D\in \mathrm{RCycl}(G)} \sum_{\substack{E\in \mathrm{Cycl}(G)\\ (E)=(D)}} \mu(C,E) \\
&= \sum_{D\in \mathrm{RCycl}(G)} \sum_{E\in S} \mu(C,D) \\
&= \sum_{D\in \mathrm{RCycl}(G)} \frac{|N_G(C)|}{|N_G(D)|}\tilde{\mu}(C,D).
\end{aligned}$$

∎

Let

$$\beta_G(C,D) = \frac{|C|\tilde{\mu}(C,D)}{|N_G(D)|}$$

and

$$\beta_G(C) = \sum_{D\in \mathrm{RCycl}(G)} \beta_G(C,D).$$

We abbreviate to write $g_f(G)$ as $g(G)$ if f is clear.

Proposition 6

$$g(G) = \sum_{C\in \mathrm{RCycl}(G)} \beta_G(C) g(C). \tag{1}$$

Proof By Theorem 4 and Lemma 5, we see that

$$\begin{aligned} g(G) &= \tfrac{1}{|G|} \sum_{C\in \mathrm{Cycl}(G)} \left(\sum_{D\in \mathrm{Cycl}(G)} \mu(C,D) \right) |C| g(C) \\ &= \tfrac{1}{|G|} \sum_{C\in \mathrm{RCycl}(G)} \sum_{\substack{C'\in \mathrm{Cycl}(G) \\ (C)=(C')}} \left(\sum_{D\in \mathrm{Cycl}(G)} \mu(C',D) \right) |C'| g(C') \\ &= \sum_{C\in \mathrm{RCycl}(G)} \tfrac{|C|}{|N_G(C)|} \left(\sum_{D\in \mathrm{Cycl}(G)} \mu(C,D) \right) g(C) \\ &= \sum_{C\in \mathrm{RCycl}(G)} \tfrac{|C|}{|N_G(C)|} \left(\sum_{D\in \mathrm{RCycl}(G)} \tfrac{|N_G(C)|}{|N_G(D)|} \tilde{\mu}(C,D) \right) g(C) \\ &= \sum_{C\in \mathrm{RCycl}(G)} \left(\sum_{D\in \mathrm{RCycl}(G)} \tfrac{|C|}{|N_G(D)|} \tilde{\mu}(C,D) \right) g(C). \end{aligned}$$

∎

We write $\beta(C) = \beta_G(C)$ for simple. If G is a cyclic group, then

$$\beta(C) = \sum_{D\in \mathrm{RCycl}(G)} \frac{|C|}{|G|} \tilde{\mu}(C,D) = \begin{cases} 0, & C \neq G \\ 1, & C = G. \end{cases} \qquad (2)$$

Lemma 7 $\sum_{C\in \mathrm{RCycl}(G)} \frac{1}{|N_G(C)|} \leq 1.$

Proof By the class equation for G, we see that

$$1 = \sum_{(x)} \frac{1}{|C_G(x)|} \geq \sum_{C\in \mathrm{RCycl}(G)} \frac{1}{|C_G(C)|} \geq \sum_{C\in \mathrm{RCycl}(G)} \frac{1}{|N_G(C)|}.$$

∎

Lemma 8 $|G| = \sum_{C,D\in \mathrm{Cycl}(G)} \mu(C,D)|C|$ *and* $\sum_{C\in \mathrm{RCycl}(G)} \beta(C) = 1$. *If G is nontrivial then* $\sum_{C\in \mathrm{RCycl}_1(G)} \beta(C) > 0$.

Proof Let $u\colon \mathrm{Cycl}(G) \to \mathbb{Q}$ be a map defined as

$$u(C) = \begin{cases} |\mathrm{gen} C| & C \neq \{e\} \\ 1 & C = \{e\} \end{cases}$$

and put $v(G) = \sum_{D\in \mathrm{Cycl}(G)} u(D)$. Then $v(G) = |G|$. By the Möbuis inversion formula, we see

$$u(D) = \sum_{C\leq D\in \mathrm{Cycl}(G)} \mu(|D|/|C|) v(C) = \sum_{C\in \mathrm{Cycl}(G)} \mu(C,D) v(C)$$

and then

$$|G| = v(G) = \sum_{C,D \in \mathrm{Cycl}(G)} \mu(C,D)|C|.$$

Therefore, we see

$$\begin{aligned}
1 &= \tfrac{1}{|G|} \sum_{C \in \mathrm{Cycl}(G)} |C| \left(\sum_{D \in \mathrm{Cycl}(G)} \mu(C,D) \right) \\
&= \tfrac{1}{|G|} \sum_{C \in \mathrm{RCycl}(G)} \sum_{\substack{C' \in \mathrm{Cycl}(G) \\ (C)=(C')}} |C'| \left(\sum_{D \in \mathrm{Cycl}(G)} \mu(C',D) \right) \\
&= \sum_{C \in \mathrm{RCycl}(G)} \tfrac{|C|}{|N_G(C)|} \sum_{D \in \mathrm{Cycl}(G)} \mu(C,D) \\
&= \sum_{C \in \mathrm{RCycl}(G)} \sum_{D \in \mathrm{RCycl}(G)} \tfrac{|C|}{|N_G(C)|} \tfrac{|N_G(C)|}{|N_G(D)|} \tilde{\mu}(C,D) \\
&= \sum_{C \in \mathrm{RCycl}(G)} \sum_{D \in \mathrm{RCycl}(G)} \beta(C,D) \\
&= \sum_{C \in \mathrm{RCycl}(G)} \beta(C).
\end{aligned}$$

If G is nontrivial then, by Lemma 7, we see

$$\beta(\{e\}) = \sum_{D \in \mathrm{RCycl}(G)} \frac{\mu(|D|)}{|N_G(D)|} < \sum_{D \in \mathrm{RCycl}(G)} \frac{1}{|N_G(D)|} \leq 1$$

and thus $\sum_{C \in \mathrm{RCycl}_1(G)} \beta(C) > 0$. ∎

Now we consider $\sum_{C \in \mathrm{RCycl}_1(G)} \beta_G(C)\gamma(C)$ for a map $\gamma\colon \mathrm{RCycl}_1(G) \to \mathbb{Q}_{\geq 0}$. We note that G satisfies the Möbius condition if and only if $\sum_{C \in \mathrm{RCycl}_1(G)} \beta_G(C)\gamma(C) \geq 0$ for an arbitrary map $\gamma\colon \mathrm{RCycl}_1(G) \to \mathbb{Q}_{\geq 0}$. By (2), we see

Proposition 9 *A cyclic group satisfies the Möbius condition.*

We recall Proposition 3. For an isovariant G-map f and subgroups $C \triangleleft D$ of G such that D/C is a BUG, $g_f(C) \leq g_f(D)$. We say G is a CCG (cyclic condition group), if for an arbitrary map $\gamma\colon \mathrm{RCycl}_1(G) \to \mathbb{Q}_{\geq 0}$ such that $\gamma(C) \leq \gamma(D)$ if $(C) \leq (D)$, $\sum_{C \in \mathrm{RCycl}_1(G)} \beta_G(C)\gamma(C) \geq 0$. A CCG is a BUG.

Proposition 10 *A group satisfying the prime condition is a* CCG.

Proof Let $\gamma\colon \mathrm{RCycl}_1(G) \to \mathbb{Q}_{\geq 0}$ be a map such that $\gamma(C) \leq \gamma(D)$ if $(C) \leq (D)$. Since

$$\sum_{C \in \mathrm{RCycl}_1(G)} \beta(C)\gamma(C) = \sum_{D \in \mathrm{RCycl}_1(G)} \sum_{C \in \mathrm{RCycl}_1(G)} \beta(C,D)\gamma(C),$$

we show that $\sum_{C\in \mathrm{RCycl}_1(G)} \beta(C,D)\gamma(C) \geq 0$ for each $D \in \mathrm{RCycl}_1(G)$. For an positive integer n, let $\pi(n)$ be the set of all primes dividing n. Put $r = \#\pi(|D|)$, the number of elements of $\pi(|D|)$, and let D_0 be the subgroup of D with index $\prod_{p\in\pi(|D|)} p$. Since

$$\binom{r}{2s}(r-2s) = \binom{r}{2s+1}(2s+1),$$

we see that

$$\begin{aligned}
&\sum_{C\in \mathrm{RCycl}_1(G)} \beta(C,D)\gamma(C)\\
&= \sum_{\substack{C\in \mathrm{RCycl}_1(G)\\ (D_0)\leq (C)\leq (D)}} \beta(C,D)\gamma(C)\\
&\geq \sum_{s=0}^{\lfloor (r-1)/2\rfloor} \left(\sum_{\substack{E\in \mathrm{RCycl}_1(G)\\ (D_0)<(E)\leq (D)\\ |\pi(|D|/|E|)|=2s}} \beta(E,D)\gamma(E) + \sum_{\substack{F\in \mathrm{RCycl}_1(G)\\ (D_0)\leq (F)<(D)\\ |\pi(|D|/|F|)|=2s+1}} \beta(F,D)\gamma(F) \right)\\
&= \sum_{s=0}^{\lfloor (r-1)/2\rfloor} \sum_{\substack{E\in \mathrm{RCycl}_1(G)\\ (D_0)<(E)\leq (D)\\ |\pi(|D|/|E|)|=2s}} \left(\beta(E,D)\gamma(E) + \frac{1}{2s+1} \sum_{\substack{F\in \mathrm{RCycl}_1(G)\\ (D_0)\leq (F)<(E)\\ |\pi(|D|/|F|)|=2s+1}} \beta(F,D)\gamma(F) \right)\\
&= \sum_{s=0}^{\lfloor (r-1)/2\rfloor} \sum_{\substack{E\in \mathrm{RCycl}_1(G)\\ (D_0)<(E)\leq (D)\\ |\pi(|D|/|E|)|=2s}} \left(\frac{|E|}{|N_G(D)|}\gamma(E) - \frac{1}{2s+1} \sum_{\substack{F\in \mathrm{RCycl}_1(G)\\ (D_0)\leq (F)<(E)\\ |\pi(|D|/|F|)|=2s+1}} \frac{|F|}{|N_G(D)|}\gamma(F) \right)\\
&\geq \sum_{s=0}^{\lfloor (r-1)/2\rfloor} \sum_{\substack{E\in \mathrm{RCycl}_1(G)\\ (D_0)<(E)\leq (D)\\ |\pi(|D|/|E|)|=2s}} \left(\frac{|E|}{|N_G(D)|}\gamma(E) - \frac{1}{2s+1} \sum_{\substack{F\in \mathrm{RCycl}_1(G)\\ (D_0)\leq (F)<(E)\\ |\pi(|D|/|F|)|=2s+1}} \frac{|F|}{|N_G(D)|}\gamma(E) \right)\\
&\geq \sum_{s=0}^{\lfloor (r-1)/2\rfloor} \sum_{\substack{E\in \mathrm{RCycl}_1(G)\\ (D_0)<(E)\leq (D)\\ |\pi(|D|/|E|)|=2s}} \left(1 - \frac{1}{2s+1} \sum_{\substack{F\in \mathrm{RCycl}_1(G)\\ (F)<(E)\\ |\pi(|E|/|F|)|=1}} \frac{|F|}{|E|} \right) \frac{|E|}{|N_G(D)|}\gamma(E)\\
&\geq \sum_{s=0}^{\lfloor (r-1)/2\rfloor} \sum_{\substack{E\in \mathrm{RCycl}_1(G)\\ (D_0)<(E)\leq (D)\\ |\pi(|D|/|E|)|=2s}} \left(1 - \sum_{p\in\pi(|E|)} \frac{1}{p} \right) \frac{|E|}{|N_G(D)|}\gamma(E)
\end{aligned}$$

and thus if G satisfies the prime condition, then it is nonnegative. ∎

3 Through linear programming

Let $\mathrm{RCycl}_1^+(G)$ and $\mathrm{RCycl}_1^-(G)$ be the subsets of $\mathrm{RCycl}_1(G)$ consisting of C with $\beta_G(C) > 0$ and $\beta_G(C) < 0$, respectively. We consider a map

$$\psi\colon \mathrm{RCycl}_1^-(G) \times \mathrm{RCycl}_1^+(G) \to \mathbb{Q}_{\le 0}$$

such that $\beta_G(C) = \sum_{D\in \mathrm{RCycl}_1^+(G)} \psi(C,D)$ for $C \in \mathrm{RCycl}_1^-(G)$ and if C is not subconjugate to D then $\psi(C,D) = 0$. Then

$$\begin{aligned}
g(G) &= \sum_{C\in \mathrm{RCycl}_1(G)} \beta_G(C)g(C) \\
&= \sum_{C\in \mathrm{RCycl}_1^+(G)} \beta_G(C)g(C) + \sum_{C\in \mathrm{RCycl}_1^-(G)} \beta_G(C)g(C) \\
&= \sum_{D\in \mathrm{RCycl}_1^+(G)} \beta_G(D)g(D) + \sum_{C\in \mathrm{RCycl}_1^-(G)} \left(\sum_{D\in \mathrm{RCycl}_1^+(G)} \psi(C,D) \right) g(C) \\
&\ge \sum_{D\in \mathrm{RCycl}_1^+(G)} \left(\beta_G(D) + \sum_{C\in \mathrm{RCycl}_1^-(G)} \psi(C,D) \right) g(D).
\end{aligned}$$

By Lemma 8, $\sum_{C\in \mathrm{RCycl}_1^+(G)} \beta_G(C) + \sum_{C\in \mathrm{RCycl}_1^-(G)} \beta_G(C) > 0$. Thus we may expect the existence of ψ. We determine whether there exist ψ such that $\beta(D) + \sum_{C\in \mathrm{RCycl}_1^-(G)} \psi(C,D) \ge 0$ for $D \in \mathrm{RCycl}_1^+(G)$ by linear programming.

$$\begin{cases}
\psi(C,D) \le 0 \\
\psi(C,D) = 0 \text{ if } (C) \not\le (D) \\
\sum_{D\in \mathrm{RCycl}_1^+(G)} \psi(C,D) \le \beta_G(C) \text{ for } C \in \mathrm{RCycl}_1^-(G) \\
\sum_{C\in \mathrm{RCycl}_1^-(G)} \psi(C,D) \ge -\beta_G(D) \text{ for } D \in \mathrm{RCycl}_1^+(G)
\end{cases}$$

We can check the existence of ψ for the following groups by using the software GAP [1]:

Theorem 11 (1) *Alternating groups A_n, $5 \le n \le 11$ satisfy the prime condition.*

(2) *A_n, $12 \le n \le 21$ are* CCG*s.*

(3) *Symmetric groups S_n, $5 \le n \le 9$ satisfy the prime condition.*

(4) S_n, $10 \leq n \leq 22$ *are* CCG*s.*

(5) *All sporadic groups are* CCG*s.*

(6) *Automorphism groups of all sporadic groups are* CCG*s.*

4 Projective special linear group

If any simple groups are BUGs, then every group is a BUG. It is important to study simple groups. Projective special linear groups $\mathrm{PSL}(3,q)$ are simple groups.

Lemma 12 *Let C be a cyclic subgroup of a group G. Suppose that there is a unique maximal cyclic subgroup D of G with $C < D$. Then $N_G(C) = N_G(D)$, $\beta_G(C) = 0$ and $\beta_G(D) = \frac{|D|}{|N_G(D)|} > 0$.*

Proof Since $C < D$, for $g \in N_G(D)$, $g^{-1}Cg$ is a subgroup of the cyclic group D with index $|D/C|$ and thus $g^{-1}Cg = C$. Therefore, $N_G(D) \leq N_G(C)$. If $g \in N_G(C) \setminus N_G(D)$ exists, then $C < g^{-1}Dg \neq D$. This is contradiction. Therefore the equality $N_G(C) = N_G(D)$ holds.

We see that

$$\begin{aligned}\beta_G(C) &= \sum_{E\in\mathrm{RCycl}(G)} \frac{|C|}{|N_G(E)|}\tilde{\mu}(C,E) = \frac{|D|}{|N_G(D)|}\sum_{E\in\mathrm{RCycl}(D)} \frac{|C|}{|D|}\tilde{\mu}(C,E)\\ &= \frac{|D|}{|N_G(D)|}\beta_D(C) = 0\end{aligned}$$

by (2), and clearly $\beta_G(D) = \frac{|D|}{|N_G(D)|}$. ∎

Let p be a prime and q a power of p. Let C_{q-1}, C_p, and C_{q+1} be cyclic subgroups of $\mathrm{SL}(2,q)$ of order $q-1$, p, and $q+1$ respectively, and let $\pi\colon \mathrm{SL}(2,q) \to G$ be a natural projection. Then $\pi(C_{q\pm1})$ has order $(q\pm1)/\gcd(q-1,2)$.

Proposition 13 $g(\mathrm{PSL}(2,q)) = \frac{1}{2}g(\pi(C_{q-1})) + \frac{1}{2}g(\pi(C_{q+1})) + \frac{p}{|N_{\mathrm{PSL}(2,q)}(C_p)|}g(C_p)$.

Proof We may put

$$\mathrm{RCycl}_1(\mathrm{PSL}(2,q)) = \{H \mid \{1\} < H \leq C_r, r = p, q\pm1\}.$$

The cyclic groups $\pi(C_r)$, $r = p, q\pm1$ are maximal and the orders are $p, (q\pm1)/d$, respectively, which are coprime each other. Therefore, any nontrivial cyclic subgroup of G has a unique maximal cyclic subgroup of G. Thus by Lemma 12, we see

$$g(G) = \sum_{r\in\{q\pm1,p\}} \frac{|C_r|}{|N_{\mathrm{PSL}(2,q)}(C_r)|}g(C_r).$$

∎

Nonsolvable groups $\mathrm{PSL}(2,q)$, $\mathrm{SL}(2,q)$, $\mathrm{PGL}(2,q)$, and $\mathrm{GL}(2,q)$ are all BUGs (see [3]). Furthermore, a group which does not have a simple group except $\mathrm{PSL}(2,q)$ as a subquotient group is a BUG.

Example 14 The simple group $\mathrm{PSL}(3,11)$ does not satisfy the prime condition, since it has an element of order 120. We confuse order (with type) with the cyclic subgroup generated by the corresponding element. For example, $\mathrm{PSL}(3,11)$ has a unique cyclic group of order 110 up to conjugate, and two cyclic subgroups of order 5 up to conjugate, denoted by $5a$ and $5b$, whose are not conjugate. We have

$$\begin{aligned}\mathrm{RCycl}_1^+(\mathrm{PSL}(3,11)) &= \{10b, 10c, 10d, 11a, 110, 120, 133\},\\ \mathrm{RCycl}_1^-(\mathrm{PSL}(3,11)) &= \{2, 5a, 5b, 10a, 11b\}.\end{aligned}$$

Since $\beta(133)+\beta(11b) = 587/1815 > 0$, $\beta(110)+\beta(5b)+\beta(10a) = 0$, and $\beta(10b)+\beta(2)+\beta(5a) = 0$, we see that

$$\begin{aligned}g(\mathrm{PSL}(3,11)) \geq\ & \beta(10c)\gamma(10c)+\beta(10d)\gamma(10d)+\beta(11a)\gamma(11a)\\ & +\beta(120)\gamma(120)+\tfrac{587}{1815}\gamma(133) \geq 0.\end{aligned}$$

The group $\mathrm{PSL}(3,11)$ is a CCG. See the following table corresponds with $\beta_{\mathrm{PSL}(3,11)}(-,-)$. The first columns and first rows are all cyclic subgroups C and D of $\mathrm{RCycl}_1(\mathrm{PSL}(3,11))$ respectively, and the last columns are the values $\beta_{\mathrm{PSL}(3,11)}(-)$.

$C\backslash D$	1	2	3	4	$5a$	$5b$	6	7	8	$10a$
1	1/212427600	−1/13200	−1/240	0	−1/200	−1/13200	1/240	−1/399	0	1/13200
2	0	1/6600	0	−1/120	0	0	−1/120	0	0	−1/6600
3	0	0	1/80	0	0	0	−1/80	0	0	0
4	0	0	0	1/60	0	0	0	0	−1/60	0
$5a$	0	0	0	0	1/40	0	0	0	0	0
$5b$	0	0	0	0	0	1/2640	0	0	0	−1/2640
6	0	0	0	0	0	0	1/40	0	0	0
7	0	0	0	0	0	0	0	1/57	0	0
8	0	0	0	0	0	0	0	0	1/30	0
$10a$	0	0	0	0	0	0	0	0	0	1/1320
$10b$	0	0	0	0	0	0	0	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

$C\backslash D$	$10b$	$10c$	$10d$	$11a$	$11b$	12	15	19	20	22
1	1/100	1/100	1/200	−1/1210	−1/133100	0	1/240	−1/399	0	1/1100
2	−1/50	−1/50	−1/100	0	0	1/120	0	0	1/120	−1/550
3	0	0	0	0	0	0	−1/80	0	0	0
4	0	0	0	0	0	−1/60	0	0	−1/60	0
$5a$	−1/20	0	−1/40	0	0	0	0	0	0	0
$5b$	0	−1/20	0	0	0	0	−1/48	0	0	0
6	0	0	0	0	0	−1/40	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
$10a$	0	0	0	0	0	0	0	0	−1/24	0
$10b$	1/10	0	0	0	0	0	0	0	0	0
$10c$	0	1/10	0	0	0	0	0	0	0	0
$10d$	0	0	1/20	0	0	0	0	0	0	0
$11a$	0	0	0	1/110	0	0	0	0	0	0
$11b$	0	0	0	0	1/12100	0	0	0	0	−1/100
12	0	0	0	0	0	1/20	0	0	0	0
15	0	0	0	0	0	0	1/16	0	0	0
19	0	0	0	0	0	0	0	1/21	0	0
20	0	0	0	0	0	0	0	0	1/12	0
22	0	0	0	0	0	0	0	0	0	1/50
24	0	0	0	0	0	0	0	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

$C\backslash D$	24	30	40	55	60	110	120	133	$\beta(C)$
1	0	$-1/240$	0	$1/1100$	0	$-1/1100$	0	$1/399$	$127/7260$
2	0	$1/120$	0	0	$-1/120$	$1/550$	0	0	$-1/20$
3	0	$1/80$	0	0	0	0	0	0	0
4	$1/60$	0	$1/60$	0	$1/60$	0	$-1/60$	0	0
$5a$	0	0	0	0	0	0	0	0	$-1/20$
$5b$	0	$1/48$	0	$-1/220$	0	$1/220$	0	0	$-1/20$
6	0	$-1/40$	0	0	$1/40$	0	0	0	0
7	0	0	0	0	0	0	0	$-1/57$	0
8	$-1/30$	0	$-1/30$	0	0	0	$1/30$	0	0
$10a$	0	$-1/24$	0	0	$1/24$	$-1/110$	0	0	$-1/20$
$10b$	0	0	0	0	0	0	0	0	$1/10$
$10c$	0	0	0	0	0	0	0	0	$1/10$
$10d$	0	0	0	0	0	0	0	0	$1/20$
$11a$	0	0	0	0	0	0	0	0	$1/110$
$11b$	0	0	0	$-1/100$	0	$1/100$	0	0	$-6/605$
12	$-1/20$	0	0	0	$-1/20$	0	$1/20$	0	0
15	0	$-1/16$	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	$-1/21$	0
20	0	0	$-1/12$	0	$-1/12$	0	$1/12$	0	0
22	0	0	0	0	0	$-1/50$	0	0	0
24	$1/10$	0	0	0	0	0	$-1/10$	0	0
30	0	$1/8$	0	0	$-1/8$	0	0	0	0
40	0	0	$1/6$	0	0	0	$-1/6$	0	0
55	0	0	0	$1/20$	0	$-1/20$	0	0	0
60	0	0	0	0	$1/4$	0	$-1/4$	0	0
110	0	0	0	0	0	$1/10$	0	0	$1/10$
120	0	0	0	0	0	0	$1/2$	0	$1/2$
133	0	0	0	0	0	0	0	$1/3$	$1/3$

Table 1: $\beta_{\mathrm{PSL}(3,11)}(-,-)$

The group $\mathrm{SL}(3,q)$ is of order $q^3(q-1)^2(q+1)(q^2+q+1)$. Put $q=p^u$, $G=\mathrm{SL}(3,q)$, $\delta=1$, $d=\gcd(3,q-\delta)$, $\rho^r=1$, $r=q-\delta$, $r'=r/d$, $s=q+\delta$, $s'=s/\gcd(3,s)$, $t=q^2+\delta q+1$, $t'=t/d$, $\sigma^s=\rho$, $\tau^t=1$, $\omega=\rho^{(q-1)/d}$. A maximal cyclic subgroup of $\mathrm{SL}(3,q)$ is conjugate to one of the followings: $C_{pr}=\langle\begin{pmatrix}\rho & 1 & \\ & \rho & \\ & & \rho^{-2}\end{pmatrix}\rangle$, $C_r^{(a,b)}=\langle\begin{pmatrix}\rho^a & & \\ & \rho^b & \\ & & \rho^{-a-b}\end{pmatrix}\rangle$ ($0\le a<r'$, $a\le b<r$, $(r,a,b)=1$), $C_{rs}=\langle\begin{pmatrix}B & \\ & \rho^{-1}\end{pmatrix}\rangle$, ${C_{dp}}^{(c)}=\langle\omega\begin{pmatrix}1 & \theta^c & \\ & 1 & \theta^c \\ & & 1\end{pmatrix}\rangle$ ($0\le c\le d-1$), C_t, where B is conjugate to $\begin{pmatrix}\sigma^\delta & \\ & \sigma^q\end{pmatrix}$ in $\mathrm{GL}(2,q^2)$ and C_t is generated by an element conjugate to $\begin{pmatrix}\tau & & \\ & \tau^{\delta q} & \\ & & \tau^{q^2}\end{pmatrix}^{q-1}$ in $\mathrm{GL}(3,q^3)$ [4, Table 1a]. Let $\psi\colon \mathrm{SL}(3,q)\to\mathrm{PSL}(3,q)$

be a canonical projection and put $D_{pr'} = \psi(C_{pr})$, $D^{(a,b)}_{r(a,b)} = \psi(C^{(a,b)}_r)$, $D_{r's} = \psi(C_{rs})$, $D^{(c)}_p = \psi(C^{(c)}_{dp})$, $D_{t'} = \psi(C_t)$, where $r(a,b) = r/d$ if $d = 3$ and $ra/d \equiv rb/d \equiv -r(a+b)/d$ modulo r, and $r(a,b) = r$ otherwise. We may assume that $\mathrm{RCycl}(G)$ consists of subgroups of the above cyclic subgroups.

Proposition 15 *If r satisfies the prime condition, then* $\mathrm{PSL}(3,q)$ *is a* CCG.

Proof The order of a maximal cyclic subgroup is r, r', p, $r's$, pr', or t'. We see that $(p, r's) = (pr's, t') = 1$. For $C \in \mathrm{RCycl}_1(G)$, if $D^{(0)}_p \leq C$, $D^{(1,1)}_{r'} < C$ or $C < D_{t'}$, then a maximal cyclic subgroup containing C is unique. We see that

$$\begin{aligned}\sum_{C\in\mathrm{RCycl}_1(G)} \beta_G(C)\gamma(C) &= \sum_{c=0}^{d-1} \beta_G(D^{(c)}_p)\gamma(D^{(c)}_p) + \beta_G(D_{t'})\gamma(D_{t'}) + \beta_G(D_{r's})\gamma(D_{r's}) \\ &\quad + \beta_G(D_{pr'})\gamma(D_{pr'}) + \sum_{(a,b)} \sum_{C\leq D^{(a,b)}_{r(a,b)}} \beta_G(C)\gamma(C).\end{aligned}$$

Note that $\beta_G(D_{pr'}) = 1/r$, $\beta_G(D_{sr'}) = 1/2$, $\beta_G(D_{t'}) = 1/3$ and $\beta_G(D_p) = -s/p^2r$, where D_p is a subgroup of $D_{pr'}$ of order p. Then $\beta_G(D_{pr'}) + \beta_G(D_p) > 0$ and

$$\sum_{C\in\mathrm{RCycl}_1(G)} \beta_G(C)\gamma(C) \geq \sum_{(a,b)} \sum_{C\leq D^{(a,b)}_{r(a,b)}} \beta_G(C)\gamma(C).$$

By the proof of Proposition 10, if r satisfies the prime condition, then for any $D \in \mathrm{RCycl}_1(G)$ of order r, $\sum_C \beta_G(C,D)\gamma(C) \geq 0$ and thus $\sum_{(a,b)} \sum_{C\leq D^{(a,b)}_{r(a,b)}} \beta_G(C)\gamma(C) \geq 0$. ∎

Example 16 The number 30 does not satisfy the prime condition. The following table corresponds with $\beta_{\mathrm{PSL}(3,31)}(C,D)$ such that $\beta_{\mathrm{PSL}(3,31)}(C) \neq 0$.

$C\backslash D$	1	2	3	4	$5b$	6	$10a$	$10b$
3	0	0	1/600	0	0	−1/200	0	0
6	0	0	0	0	0	1/100	0	0
$10b$	0	0	0	0	0	0	0	1/29760
$15a$	0	0	0	0	0	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

$C\backslash D$	$10c$	$10d$	$15a$	$15b$	20	$30a$	$30b$	$30c$	$30d$	$31a$	$31b$
3	0	0	−1/200	−1/200	0	1/200	1/100	1/100	1/200	0	0
6	0	0	0	0	0	−1/100	−1/50	−1/50	−1/100	0	0
$10b$	0	0	0	0	−1/64	0	0	0	−1/60	0	0
$15a$	0	0	1/40	0	0	−1/40	0	−1/20	0	0	0
$15b$	0	0	0	1/40	0	0	−1/20	0	−1/40	0	0
$30a$	0	0	0	0	0	1/20	0	0	0	0	0
$30b$	0	0	0	0	0	0	1/10	0	0	0	0
$30c$	0	0	0	0	0	0	0	1/10	0	0	0
$30d$	0	0	0	0	0	0	0	0	1/20	0	0
$31a$	0	0	0	0	0	0	0	0	0	1/288300	0
$31b$	0	0	0	0	0	0	0	0	0	0	1/930
$31c$	0	0	0	0	0	0	0	0	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

$C\backslash D$	$31c$	$31d$	62	155	310	320	331	$\beta(C)$
3	0	0	0	0	0	0	0	$1/60$
6	0	0	0	0	0	0	0	$-1/20$
$10b$	0	0	0	0	$-1/930$	0	0	$-1/30$
$15a$	0	0	0	0	0	0	0	$-1/20$
$15b$	0	0	0	0	0	0	0	$-1/20$
$30a$	0	0	0	0	0	0	0	$1/20$
$30b$	0	0	0	0	0	0	0	$1/10$
$30c$	0	0	0	0	0	0	0	$1/10$
$30d$	0	0	0	0	0	0	0	$1/20$
$31a$	0	0	$-1/300$	$-1/300$	$1/300$	0	0	$-16/4805$
$31b$	0	0	0	0	0	0	0	$1/930$
$31c$	$1/930$	0	0	0	0	0	0	$1/930$
$31d$	0	$1/930$	0	0	0	0	0	$1/930$
310	0	0	0	0	$1/30$	0	0	$1/30$
320	0	0	0	0	0	$1/2$	0	$1/2$
331	0	0	0	0	0	0	$1/3$	$1/3$

Table 2: $\beta_{\mathrm{PSL}(3,31)}(-,-)$

Since $\beta(6)+\beta(30a)=0$, $\beta(10b)+\beta(30d)>0$, $\beta(15a)+\beta(30c)>0$, $\beta(15b)+\beta(30b)>0$, $\beta(31a)+\beta(310)>0$, we get $g(\mathrm{PSL}(3,31)) \geq \beta(3)\gamma(3)+\beta(31b)\gamma(31b)+\beta(31c)\gamma(31c)+\beta(31d)\gamma(31d)+\beta(320)\gamma(320)+\beta(331)\gamma(331) \geq 0$. Thus $\mathrm{PSL}(3,31)$ is a CCG.

5 Future work

It is not true that for an extension $1 \to H \to G \to K \to 1$, if H and K are CCGs, then G is a CCG.

Proposition 17 *A_{22} is not a* CCG.

Proof Let $x=(1,2,3,4,5,6,7)(8,9,10,11,12)(13,14,15,16)(17,18,19)(20,21) \in A_{22}$ and $S=\{\langle x^3\rangle,\langle x^4\rangle,\langle x^5\rangle,\langle x^7\rangle\}$. Let $\tilde{S}$ be the subset of $\mathrm{RCycl}_1(A_{22})$ consisting of C such that some element of S is subconjugate to C. Then $\tilde{S}=S\cup\{\langle x\rangle,\langle x^2\rangle\}$.

n	1	2	3	4	5	7	sum
$\beta_{A_{22}}(\langle x^n\rangle)$	$\frac{1}{96}$	$-\frac{1}{288}$	$-\frac{1}{384}$	$-\frac{13}{15120}$	$-\frac{7}{3456}$	$-\frac{137}{92160}$	$-\frac{61}{1935360}$

Let $\gamma\colon \mathrm{RCycl}(G) \to \mathbb{Q}_{\geq 0}$ by $\gamma(C)=1$ if C is conjugate to some element of $\tilde{S}$ and $\gamma(C)=0$ otherwise. We see that $\sum_{C\in\mathrm{RCycl}_1(G)} \beta_G(C)\gamma(C) = -61/1935360 < 0$. Therefore A_{22} is not a CCG. ∎

Suppose that $\mathrm{RCycl}(A_{22}) \supset \tilde{S}$. There exists representation A_{22}-spaces V and W such that $\dim V^C = \dim W^C$ for $C \in \mathrm{RCycl}(A_{22}) \smallsetminus \tilde{S}$ including $\dim V = \dim W$, and $\dim W^C - \dim V^C$ is constant positive number for $C \in \tilde{S}$. By these condition, we have $\dim W < \dim V$ (see (1)). Suppose that there is an isovariant map $f\colon V \to W$. Let C be a cyclic subgroup generated by x, and H a solvable subgroup of $N_{A_{22}}(C)$ of order 840 generated by x and $(14,16)(18,19)$. Then H normalizes C and $g_f(C)=1$ and $g_f(H)=\frac{1}{2}$, which is a contradiction. Therefore, there is no isovariant map $V\to W$.

Thus we consider a new condition. For a map $\gamma\colon \mathrm{RCycl}_1(G)\to\mathbb{Q}_{\geq 0}$ and a subgroup H of G, we put

$$\hat{\gamma}(H)=\sum_{C\in\mathrm{RCycl}_1(H)}\beta_H(C)\bar{\gamma}(C),$$

where $\bar{\gamma}\colon \mathrm{Cycl}_1(G)\to\mathbb{Q}$ is a class function which sends a cyclic subgroup C of G to $\gamma(C')$ such that $C'\in\mathrm{RCycl}_1(G)$ is conjugate to C in G. If $H\in\mathrm{RCycl}_1(G)$ then $\hat{\gamma}(H)=\gamma(C)$, and if H_1 and H_2 are conjugate in G then $\hat{\gamma}(H_1)=\hat{\gamma}(H_2)$. Recall that $g_f(H_2)-g_f(H_1)=g_{f^{H_1}}(H_2/H_1)$ and if H_2/H_1 is a BUG then $g_{f^{H_1}}(H_2/H_1)\geq 0$ for an isovariant G-map $f\colon V\to W$. A group G is a SCG (subgroup condition group) if for an arbitrary map $\gamma\colon \mathrm{RCycl}_1(G)\to\mathbb{Q}_{\geq 0}$ such that $\hat{\gamma}(H_1)\leq\hat{\gamma}(H_2)$ for subgroups $H_1\trianglelefteq H_2\leq G$ with H_2/H_1 a CCG, $\hat{\gamma}(G)\geq 0$. A SCG is a BUG.

Question 18 *Is the group A_{22} a* SCG*?*

A sufficient condition to be a BUG is that the minimizing value of the following linear programming is zero.

$$\text{Minimize}\quad \sum_{V\in\mathrm{Irr}_1(G)} x_V\dim V$$

$$\text{subject to}\quad \begin{cases} -1\leq x_V\leq 1, \quad V\in\mathrm{Irr}_1(G)\\ \displaystyle\sum_{V\in\mathrm{Irr}_1(G)} x_V(\dim V^{H_1}-\dim V^{H_2})\geq 0, \quad H_1\triangleleft H_2\leq G, H_2/H_1 \text{ solvable}\end{cases}$$

where $\mathrm{Irr}_1(G)$ is the set of all irreducible nontrivial representation G-spaces.

Since there are many inequalities, we could not check whether the minimizing value is zero or not for A_{22}. We must reduce partial condition to compute.

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