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ON p-ADIC FAMILIES OF THE D-TH SAITO-KUROKAWA LIFTS

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1. Introduction

Let p be an odd prime, N an odd positive integer not divisible by p and D the discriminant of an imaginary quadratic field satisfying $p \mid D$. We will establish p-adic interpolation of Fourier coefficients of the D-th Saito-Kurokawa lifts of primitive forms of level N varying in a Coleman family. This generalizes known results for Hida families of tame level $N = 1$ to the case of Coleman families of tame level N. The main theorem is Theorem 6.7.

Notation and terminology. Throughout the paper, we fix an odd prime p, a positive integer N satisfying $(N, 2p) = 1$ and a non-negative rational number α. We assume that $Np \geq 4$ to ensure that $\Gamma_1(Np)$ is torsion-free. We denote by $\bar{\mathbb{Q}}$ and $\bar{\mathbb{Q}}_p$ an algebraic closure of the rational number field $\mathbb{Q}$, and the p-adic number field $\mathbb{Q}_p$, respectively. Let $\mathbb{C}$ be the complex number field and $\mathbb{C}_p$ the p-adic completion of $\bar{\mathbb{Q}}_p$. We fix two embeddings $i_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $i_p : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, and an isomorphism $\mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$ which commutes with i_∞ and i_p. Let val_p be the normalized p-adic additive valuation on $\mathbb{C}_p$ so that $\mathrm{val}_p(p) = 1$. For $z \in \mathbb{C}$, we define $\sqrt{z} = z^{1/2}$ so that $\pi/2 < \arg(z^{1/2}) \leq \pi/2$ and put $z^{k/2} := (\sqrt{z})^k$ for each integer k. We put $e(z) := \exp(2\pi\sqrt{-1}z)$ and $e^m(z) := e(mz)$. For a Dirichlet character χ, we denote by c_χ the conductor of χ, χ_0 the primitive character attached to χ and $G(\chi) := \sum_{i=0}^{c_\chi - 1} \chi_0(i) e(i/c_\chi)$. For a non-zero integer a, we let χ_a denote the *Kronecker symbol* $\chi_a(b) := \left(\frac{a}{b}\right)$ defined by [MFM, (3.1.9)]. We call D a *fundamental discriminant* if D is either 1 or the discriminant of a quadratic field. We denote by $\mathbb{1}_M$ the trivial Dirichlet character modulo M, i.e., for any integer n, $\mathbb{1}_M(n) := 1$ if $(n, M) = 1$ and $\mathbb{1}_M(n) := 0$ otherwise. By $d \parallel n$, we mean $d \mid n$ and $(d, n/d) = 1$.

2. Siegel cusp forms

Let g be a positive integer. Note that our concern is only for $g = 1, 2$.

2.1. Definition of Siegel cusp forms

Put $1_g := \mathrm{diag}(1, \ldots, 1), 0_g := \mathrm{diag}(0, \ldots, 0) \in M_g(\mathbb{Z})$ and

(2-1-1)
$$J_g := \begin{pmatrix} 0_g & -1_g \\ 1_g & 0_g \end{pmatrix} \in \mathrm{GL}_{2g}(\mathbb{Z})$$

For a commutative ring R and an integer M,

(2-1-2)
$$\mathrm{GSp}_g(R) := \{\gamma \in \mathrm{GL}_{2g}(R) \mid {}^t\gamma J_g \gamma = \nu(\gamma) J_g \text{ for some } \nu(g) \in R^\times\},$$

(2-1-3)
$$\mathrm{Sp}_g(R) := \{\gamma \in \mathrm{GL}_{2g}(R) \mid {}^t\gamma J_g \gamma = J_g\} = \mathrm{Ker}(\nu),$$

(2-1-4)
$$\Gamma_0^g(M) := \{\gamma \in \mathrm{Sp}_g(\mathbb{Z}) \mid c_\gamma \equiv 0_g \pmod{M}\},$$

where we denote by c_γ the left lower $g \times g$ matrix of γ and from now on we use the notation

(2-1-5)
$$\begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} := \gamma \in \mathrm{GL}_{2g}(R).$$

For a subring R of the real number field $\mathbb{R}$, we put

(2-1-6) $$\mathrm{GSp}_g^+(R) := \{\gamma \in \mathrm{GL}_{2g}(R) \mid {}^t\gamma J_g \gamma = \nu(\gamma) J_g \text{ for some } \nu(g) > 0\}.$$

We denote by $\mathfrak{H}_g := \{Z \in \mathrm{Sym}_g(\mathbb{C}) \mid \mathrm{Im}(Z) > 0\}$ the Siegel upper-half plane of genus g, where $\mathrm{Sym}_g(\mathbb{C})$ is the set of symmetric $g \times g$ matrices whose entries in $\mathbb{C}$. Let $\gamma \in \mathrm{GSp}_g^+(\mathbb{R})$ act on $Z \in \mathfrak{H}_g$ by

(2-1-7) $$\gamma Z := (a_\gamma Z + b_\gamma)(c_\gamma Z + d_\gamma)^{-1}.$$

Definition 2.1. Let χ be a Dirichlet character modulo M. A *Siegel modular form* F of genus g, weight k, level M and character χ is a holomorphic function on $\mathfrak{H}_g$ satisfying

(2-1-8) $$F|_k\gamma(Z) := \det(c_\gamma Z + d_\gamma)^{-k} F(\gamma Z) = \chi^{-1}(\det(a_\gamma))F(Z)$$

for any $\gamma \in \Gamma_0^g(M)$ and additional conditions of holomorphy at the cusps when $g = 1$. We denote the space of all such functions F by $M_k^g(M, \chi)$. We call $F \in M_k^g(M, \chi)$ a *Siegel cusp* form of genus g, weight k, level M and character χ if $F|_k\gamma|\Phi = 0$ for any $\gamma \in \mathrm{Sp}_g(\mathbb{Z})$, where Φ is the Siegel operator (see [QFHO, Section 2.3.4]). We denote the space of such all functions F by $S_k^g(M, \chi)$. Note that a Siegel cusp form of genus g, weight k, level M and character χ is a cusp form of weight k and character χ for $\Gamma_0^g(M)$ in terms of [QFHO, p.78] and $S_k^g(M, \chi)$ is written as $\mathfrak{N}_k(\Gamma_0^g(M), \chi)$ in [QFHO, p.82].

For any $F \in S_k^g(M, \chi)$ and $Z \in \mathfrak{H}_g$, we write the Fourier expansion of F as

(2-1-9) $$F(Z) = \sum_{T \in \mathcal{L}_{>0}} a_T(F) e(\mathrm{tr}\, TZ),$$

where $\mathcal{L}_{>0}$ is the set of positive definite half-integral symmetric $g \times g$ matrices (see [QFHO, Theorem 2.3.12] for the Fourier expansions).

2.2. Hecke algebras

For a group G, a subgroup Γ of G and a commutative ring R, we put

(2-2-1) $$\mathcal{H}_R(G, \Gamma) := R[\Gamma \backslash G / \Gamma].$$

Put $\mathbb{Z}_{(M)} := \bigcap_{\ell | M} \mathbb{Z}_{(\ell)}$ (the intersection of the localizations $\mathbb{Z}_{(\ell)}$ of $\mathbb{Z}$ at $\ell\mathbb{Z}$ for all primes $\ell \mid M$) and

(2-2-2) $$\Delta_0^g(M) := \mathrm{GSp}_g^+(\mathbb{Q}) \cap \mathrm{GL}_{2g}(\mathbb{Z}_{(M)}) \cap M_{2g}(\mathbb{Z})$$

For a prime $\ell \nmid M$, we denote the integral Hecke algebra at ℓ over $\mathbb{Z}$ by

(2-2-3) $$\mathcal{H}_\ell^g(M) := \mathcal{H}_\mathbb{Z}(\Delta_0^g(M) \cap \mathrm{GL}_{2g}(\mathbb{Z}[\ell^{-1}]), \Gamma_0^g(M))).$$

Then $\mathcal{H}_\ell^g(M)$ is generated over $\mathbb{Z}$ by the following elements:

(2-2-4) $$T^g(\ell) := \Gamma_0^g(M) \mathrm{diag}(1_g, \ell 1_g) \Gamma_0^g(M),$$

(2-2-5) $$T_i^g(\ell^2) := \Gamma_0^g(M) \mathrm{diag}(1_{g-i}, \ell 1_i, \ell^2 1_{g-i}, \ell 1_i) \Gamma_0^g(M)$$

for $i = 1, 2, \ldots, g$ (see [QFHO, Theorem 3.3.23] and note $\underline{L}_p^n(q) = \mathcal{H}_p^n(q) \otimes_\mathbb{Z} \mathbb{Q}$). We define

(2-2-6) $$\mathcal{H}^g(M) := \otimes'_\ell \mathcal{H}_\ell^g(M),$$

where $\otimes'_\ell$ is the restricted tensor product running over all primes ℓ, i.e., $\mathcal{H}^g(M)$ is defined to be the $\mathbb{Z}$-algebra generated by $\mathcal{H}_\ell^g(M)$ for all primes ℓ over $\mathbb{Z}$. Note that $\mathcal{H}^g(M) = \mathcal{H}_\mathbb{Z}(\Delta_0^g(M), \Gamma_0^g(M))$ and that $\mathcal{H}^g(M)$ is commutative ([QFHO, Theorem 3.3.7 and 3.3.12]). We let $T := \Gamma_0^g(M)\gamma\Gamma_0^g(M) = \bigcup_i \Gamma_0^g(M)\gamma_i \in \mathcal{H}^g(M)$ act on $F \in S_k^g(M, \chi)$ by

(2-2-7) $$T(F)(Z) := \nu(\gamma)^{g(2k-g-1)/2} \sum_i \chi(\det(a_\gamma)) F|_k \gamma_i(Z)$$

Since $\nu(\ell 1_{2g}) = \ell^g$ by the definition (2-1-2), the definition (2-2-7) requires that

(2-2-8) $$T_g^g(\ell) \text{ acts as } \chi(\ell^g)\ell^{g^2(k-g-1)/2} \text{ on } S_k^g(M, \chi) \text{ if } \ell \nmid M.$$

We refer to $F \in S_k^g(M, \chi)$ as a *Hecke eigenform* if F is an eigenvector for any $T \in \mathcal{H}^g(M)$.

2.3. Hecke fields

For a Hecke eigenform $F \in S_k^g(M, \chi)$, we define $\lambda_F \in \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{T}^\Sigma(S_k^g(M, \chi)), \mathbb{C})$ by $\lambda_F(T)$ to be the eigenvalue of T at F and the *Hecke field* $\mathbb{Q}(F)$ of F by

(2-3-1) $$\mathbb{Q}(F) := \mathbb{Q}(\{\lambda_F(T) \mid T \in \mathcal{H}^g(M)\}).$$

Moreover, we denote by $\mathbb{Z}(F)$ the ring of integers of $\mathbb{Q}(F)$, $\mathbb{Z}_{(p)}(F)$ the localization of $\mathbb{Z}(F)$ at the prime above $p\mathbb{Z}$, $\mathbb{Z}_p(F)$ the p-adic completion of $\mathbb{Z}(F)$, and $\mathbb{Q}_p(F) := \mathrm{Frac}(\mathbb{Z}_p(F))$ the field of fractions. For a Dirichlet character ψ, we denote by $\mathbb{Q}(F, \psi)$ the field obtained by adjoining the values of ψ to $\mathbb{Q}(F)$, $\mathbb{Z}(F, \psi)$ the ring of integers of $\mathbb{Q}(F, \psi)$, $\mathbb{Z}_{(p)}(F, \psi)$ the localization of $\mathbb{Z}(F, \psi)$ at the prime above $p\mathbb{Z}$, $\mathbb{Z}_p(F, \psi)$ the p-adic completion of $\mathbb{Z}(F, \psi)$, and $\mathbb{Q}_p(F, \psi) := \mathrm{Frac}(\mathbb{Z}_p(F, \psi))$ the field of fractions.

2.4. Petersson inner products

For $F, G \in S_k^g(M, \chi)$, the normalized *Peterson inner product* of F and G is defined by

(2-4-1) $$\langle F, G \rangle := [\mathrm{Sp}_g(\mathbb{Z}) : \Gamma_0^g(M)]^{-1} \int_{\Gamma_0^g(M)\backslash\mathfrak{H}_g} F(Z)\overline{G(Z)} \det(Y)^{k-(g+1)} dZ,$$

where $Z = X + \sqrt{-1}Y = (x_{\alpha\beta}) + \sqrt{-1}(y_{\alpha\beta})$,

(2-4-2) $$dZ := \prod_{1 \le \alpha \le \beta \le g} dx_{\alpha\beta} y_{\alpha\beta},$$

and $(\det Y)^{-(g+1)} dZ$ is a $\mathrm{Sp}_g(\mathbb{R})$-invariant measure ([QFHO, Proposition 1.2.9]). By [QFHO, Theorem 2.5.3], $\langle F, G \rangle$ is absolutely convergent, independent of choice of the subgroup $\Gamma_0^g(N)$ with $F, G \in S_k^g(N, \chi)$ and a positive definite Hermitian form.

2.5. Notation and terminology for genus 1

We often omit g from any notation when $g = 1$ for simplicity. We denote by $S_k^{\mathrm{new}}(M, \varepsilon)$ the orthogonal complement of the subspace of old forms of level M in $S_k(M, \varepsilon)$ with respect to the Petersson inner product. We refer to a Hecke eigenform in $S_k^{\mathrm{new}}(M, \varepsilon)$ as a *primitive form* of level M if $T^1(n)f = a_n(f)f$ for all positive integers n, where $a_n(f)$ is the n-th Fourier coefficient of f. We denote by $S_k(M, \varepsilon)_\alpha$ the subspace of $S_k(M, \varepsilon)$ spanned by the generalized eigenspaces for eigenvalues λ of T_p with $\mathrm{val}_p(\lambda) = \alpha$. Let $\mathbb{Z}[\varepsilon]$ be the ring generated by the values of ε over $\mathbb{Z}$. For a $\mathbb{Z}[\varepsilon]$-algebra R and $q := \exp(2\pi\sqrt{-1}z)$, we put

(2-5-1) $$S_k(M, \varepsilon; R)_\alpha := (S_k(M, \varepsilon)_\alpha \cap \mathbb{Z}[\varepsilon][[q]]) \otimes_{\mathbb{Z}[\varepsilon]} R,$$

(2-5-2) $$S_k^{\mathrm{new}}(M, \varepsilon; R)_\alpha := (S_k^{\mathrm{new}}(M, \varepsilon) \cap S_k(M, \varepsilon; \mathbb{Z}[\varepsilon])_\alpha) \otimes_{\mathbb{Z}[\varepsilon]} R.$$

For $f \in S_k(M,\varepsilon)$ and a Dirichlet character ψ, we denote by $f \otimes \psi \in S_k(L, \varepsilon\psi^2)$ the *ψ-twist* of f defined by $a_n(f \otimes \psi) := \psi_0(n) a_n(f)$ for all $n \geq 1$, where L is the least common multiple of M, c_ψ^2, and $c_\psi c_\varepsilon$ ([MFM, Lemma 4.3.10.(2)]). We put $L(s,f) := \sum_{n=1}^{\infty} a_n(f) n^{-s}$.

2.6. Notation for genus 2

Let $F \in S_k^2(M,\chi)$ be a Hecke eigenform such that $T^2(\ell)F = \lambda_F(\ell)F$ for any prime ℓ and $T_1^2(\ell^2)F = \lambda_F(\ell^2)F$ for each prime $\ell \nmid M$, where $T^2(\ell)$ is defined as (2-2-4) even when $\ell \mid M$. Recall that we have always $T_2^2(\ell)F = \chi(\ell^2)\ell^{2(k-3)}F$ by (2-2-8). The spinor L-function $L(s,F,\mathrm{spin})$ attached to F is defined by

(2-6-1)
$$L(s,F,\mathrm{spin}) := \prod_{\ell \mid M} \left(1 - \lambda_F(\ell)\ell^{-s}\right)^{-1} \prod_{\ell \nmid M} Q_\ell(\ell^{-s})^{-1}$$

with

(2-6-2)
$$\begin{aligned} Q_\ell(X) :=& 1 - \lambda_F(\ell)X + \left(\ell\lambda_F(\ell^2) + \chi(\ell^2)(\ell^2+1)\ell^{2k-5}\right)X^2 \\ &- \chi(\ell^2)\lambda_F(\ell)\ell^{2k-3}X^3 + \chi(\ell^4)\ell^{4k-6}X^4. \end{aligned}$$

3. D-th Saito-Kurokawa lifts

Let $k \geq 2$ be an integer, M an odd positive integer and χ a Dirichlet character modulo M.

3.1. Kohnen plus spaces

Put $\tilde{\chi} := \chi_\epsilon \chi$ with $\epsilon := \chi(-1)$. We denote the *Kohnen plus space* by

(3-1-1)
$$S_{k-1/2}^{+}(M,\chi) := \left\{ g \in S_{2k+1}^{\mathrm{Sh}}(4M,\tilde{\chi}) \mid a_n(g) = 0 \text{ if } \chi(-1)(-1)^{k-1}n \equiv 2,3 \pmod 4 \right\},$$

where $S_{2k+1}^{\mathrm{Sh}}(4M,\tilde{\chi})$ is the space of cusp forms of half-integral weight $k-1/2$ with level $4M$ and a character $\tilde{\chi}$ modulo $4M$ in the sense of Shimura [Shi73, p. 447]. For $g \in S_{k-1/2}^{+}(M,\chi)$ and each prime ℓ, the Hecke operator $T^+(\ell)$ is defined by

(3-1-2)
$$a_n(T^+(\ell)g) = a_{\ell^2 n}(g) + \tilde{\chi}\chi_{(-1)^{k-1}n}(\ell)\ell^{k-2}a_n(g) + \chi(\ell^2)\ell^{2k-3}a_{n/\ell^2}(g)$$

for any positive integer n with $\chi(-1)(-1)^{k-1}n \equiv 0,1 \pmod 4$. For $g, h \in S_{k-1/2}^{+}(M,\chi)$, we define the Petersson inner product by

(3-1-3)
$$\langle g,h\rangle_{4M} := \int_{\Gamma_0(4M)\backslash\mathfrak{H}} g(z)\overline{h(z)}y^{k-5/2}dxdy \ (z = x+\sqrt{-1}y),$$

(3-1-4)
$$\langle g,h\rangle := [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4M)]^{-1}\langle g,h\rangle_{4M}.$$

3.2. D-th Shintani lifts θ_D

Let D be a fundamental discriminant with $\chi(-1)(-1)^{k-1}D > 0$ and $(D, c_\chi) = 1$. We define the D-th *Shimura lift* Sh_D by

(3-2-1)
$$\mathrm{Sh}_D(g) := \sum_{n\geq 1}\left(\sum_{d\mid n}\chi_D\chi(d)d^{k-2}a_{n^2|D|/d^2}(g)\right)q^n$$

(see [KT04, (3-1)]).

Remark 3.1. Although [KT04] assumes $(D,M) = 1$, we see that the results below do not require this assumption except for Theorem 3.3. Note that $\mathrm{Sh}_D = 0$ if $(D, c_\chi) \neq 1$ by [KT04, (3-2)].

Assume that

(3-2-2) either $k \geq 3$, M is square-free, or M is cubic-free and $\chi = \mathbb{1}$.

Then the image of the D-th Shimura lift Sh_D is contained in the space of cusp forms ([Koh85, p.241, l.4-9]). Now we define the D-th *Shintani lift* $\theta_D : S_{2k-2}(M, \chi^2) \to S^+_{k-1/2}(M, \chi)$ as the adjoint map of Sh_D with respect to the Petersson inner products, i.e., the map satisfying

$$\langle g, \theta_D(f) \rangle = \langle \mathrm{Sh}_D(g), f \rangle \tag{3-2-3}$$

for every $g \in S^+_{k-1/2}(M, \chi)$ and $f \in S_{2k-2}(M, \chi^2)$. Since the D-th Shimura lift Sh_D is Hecke equivariant in the sense that $T^1(\ell) \circ \mathrm{Sh}_D = \mathrm{Sh}_D \circ T^+(\ell)$ for all primes ℓ by [KT04, Theorem 3.1] and the Hecke operators are Hermitian operators with respect to the Petersson inner products, we have the following:

Theorem 3.2. *Let $k \geq 2$ be an integer, M an odd positive integer, χ a Dirichlet character modulo M and D a fundamental discriminant with $\chi(-1)(-1)^{k-1}D > 0$ and $(D, c_\chi) = 1$. Assume (3-2-2). Then the D-th Shintani lift θ_D is a $\mathbb{C}$-homomorphism from $S_{2k-2}(M, \chi^2)$ into $S^+_{k-1/2}(M, \chi)$ and Hecke equivariant in the sense that $T^+(\ell) \circ \theta_D = \theta_D \circ T^1(\ell)$ for all primes ℓ.*

Suppose that $c_\chi \parallel M$. Let ℓ be a prime factor of M/c_χ, We put $v_\ell := \mathrm{val}_\ell(M/c_\chi) = \mathrm{val}_\ell(M)$. Let γ_ℓ be an element in $\mathrm{SL}_2(\mathbb{Z})$ such that

$$\gamma_\ell \equiv \begin{cases} J_1 & \pmod{\ell^{2v_\ell}}, \\ 1_2 & \pmod{(M/\ell^{v_\ell})^2}. \end{cases} \tag{3-2-4}$$

We put $\eta_\ell := \gamma_\ell \cdot \mathrm{diag}(\ell^{v_\ell}, 1)$ (see [MFM, (4.6.21)]). We define the eigenvalue of f for the Atkin-Lehner involution η_ℓ by

$$w_\ell(f) := \chi_0^2(\ell^{v_\ell}) a_1(f|_{2k-2}\eta_\ell). \tag{3-2-5}$$

If $v_\ell = 1$, then we have $a_1(f|_{2k-2}\eta_\ell) = -\chi_0^{-2}(\ell)\ell^{-k+2}a_\ell(f)$ by [MFM, Corollary 4,6,18.(2)] and hence

$$w_\ell(f) = -\ell^{-k+2}a_\ell(f) \in \{\pm 1\} \tag{3-2-6}$$

by [MFM, Theorem 4.6.17.(2)].

Theorem 3.3 ([KT04, (4-19, 20, 21, and 22)]). *Let $f \in S^{\mathrm{new}}_{2k-2}(M, \chi^2)$ be a primitive form. Suppose that $c_\chi \parallel M$ and $(D, M) = 1$. We put*

$$R_D(f) := \prod_\ell \left(1 + \chi_D\chi_0(\ell^{v_\ell})w_\ell(f) \left(\frac{1 - \chi_D\chi_0^{-1}(\ell)\ell^{-k+1}a_\ell(f)}{1 - \chi_D\chi_0(\ell)\ell^{-k+1}a_\ell(f)^c} \right) \right), \tag{3-2-7}$$

where $\prod_\ell$ is taken over all prime factors ℓ of M/c_χ and $a_\ell(f)^c$ is the complex conjugate of $a_\ell(f)$. Then

$$a_{|D|}(\theta_D(f)) = R_D(f)|D|^{k-3/2}c_\chi^{2k-3}\pi^{-(k-1)}\Gamma(k-1)L\left(k-1, f \otimes \chi_D\chi^{-1}\right). \tag{3-2-8}$$

Remark 3.4. Let the notation and the assumption be the same as the theorem above. If $\chi^2 = \mathbb{1}$ and M/c_χ is square-free, then $R_D(f) \in \{0, 2^{\nu(M/c_\chi)}\}$ by [Shi72, Proposition 1.3] and (3-2-6), where $\nu(M/c_\chi)$ is the number of distinct prime factors of M/c_χ. In particular, if $\chi = \mathbb{1}$, then the following conditions are equivalent:

(1) $R_D(f) \neq 0$.
(2) $R_D(f) = 2^{\nu(M)}$.
(3) $\chi_D(\ell) = w_\ell(f)$ for any prime divisor ℓ of M.

In this case, the formula (3-2-8) is nothing but the result of Kohnen in [Koh85] and the sign of the functional equation of $L(s, f \otimes \chi_D)$ is $(-1)^{k-1}\chi_D(-1)$, i.e., if $(-1)^{k-1}\chi_D(-1) = -1$, then $L(k-1, f \otimes \chi_D) = 0$.

3.3. *D*-th Saito-Kurokawa lifts SK_D

Let $k \geq 2$ be an even integer, $M \geq 1$ an odd integer, χ a Dirichlet character modulo M with $\chi(-1) = 1$ and D a fundamental discriminant with $D < 0$ and $(D, c_\chi) = 1$. Assume (3-2-2). We then define the D-th *Saito-Kurokawa lift* SK_D by composing θ_D with the Eichler-Zagier map EZ and the Maass lift L (see [Mak, Section 4 and 5] for definition of EZ and L, respectively):

$$\mathrm{SK}_D : S_{2k-2}(M, \chi^2) \xrightarrow{\theta_D} S^+_{k-1/2}(M, \chi) \xrightarrow{\mathrm{EZ}} J^{\mathrm{cusp}}_{k,1}(M, \chi) \overset{L}{\hookrightarrow} S^2_k(M, \chi). \tag{3-3-1}$$

Let $f \in S_{2k-2}(M, \chi^2)$ be any element. Since EZ is an isomorphism ([Mak, Theorem 4.2]) and L is an injective homomorphism ([Mak, Theorem 5.1]), we see that

$$\mathrm{SK}_D(f) \neq 0 \text{ if and only if } \theta_D(f) \neq 0 \tag{3-3-2}$$

(see [Mak, Theorem 3.6] for the non-vanishing criterion for $\theta_D(f)$). For $T = [a, b, c] \in \mathcal{L}_{>0}$, the T-th Fourier coefficient of $\mathrm{SK}_D(f)$ is given by

$$a_T(\mathrm{SK}_D(f)) = \sum_{\substack{0<d|(a,b,c)\\(d,M)=1}} \chi(d) d^{k-1} a_{\det(T)/d^2}(\theta_D(f)). \tag{3-3-3}$$

For any prime ℓ,

$$T^2(\ell) \circ \mathrm{SK}_D = \mathrm{SK}_D \circ (T^1(\ell) + \chi(\ell)(\ell^{k-2} + \ell^{k-1})), \tag{3-3-4}$$

$$\left(\ell T^2_1(\ell^2) + \chi(\ell)^2(\ell^2+1)\ell^{2k-5}\right) \circ \mathrm{SK}_D = \mathrm{SK}_D \circ \left(\chi(\ell)(\ell^{k-1} + \ell^{k-2})T^1(\ell) + 2\chi(\ell)^2\ell^{2k-3}\right). \tag{3-3-5}$$

Namely, we have the following:

Theorem 3.5. *Let $k \geq 2$ be an even integer, $M \geq 1$ an odd integer, χ a Dirichlet character modulo M with $\chi(-1) = 1$ and D a fundamental discriminant with $D < 0$ and $(D, c_\chi) = 1$. Assume (3-2-2). Let $f \in S^{\mathrm{new}}_{2k-2}(M, \chi^2)$ be a primitive form. If $\theta_D(f) \neq 0$, then $\mathrm{SK}_D(f) \in S^2_k(M, \chi)$ is a Hecke eigenform satisfying*

$$L(s, \mathrm{SK}_D(f), \mathrm{spin}) = L(s-k+1, \chi)L(s-k+2, \chi)L(s, f). \tag{3-3-6}$$

Remark 3.6. It is known that the image of SK_D is characterized by the generalized Maass relation (see [Hei17]).

4. *p*-adic families

Let K be a complete discretely valued subfield of $\mathbb{C}_p$. The *weight space* $\mathcal{W}$ attached to $\mathcal{O}_K[\![\mathbb{Z}_p^\times]\!]$ is the rigid analytic variety whose $\mathbb{C}_p$-valued points are given by

$$\mathrm{Hom}^{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}_K\text{-alg}}(\mathcal{O}_K[\![\mathbb{Z}_p^\times]\!], \mathbb{C}_p). \tag{4-0-1}$$

For a K-Banach algebra R and an R-valued point $k \in \mathcal{W}(R)$, we will use a notation t^k instead of $k(t)$ for $t \in \mathbb{Z}_p^\times$. For a K-rigid analytic variety X, we denote by $A(X)$ the ring of rigid analytic functions on X and $A^\circ(X)$ the subring consisting of elements that are power bounded with respect to the supremum semi-norm $|\ |$ (see [BGR, Definition 6.2.1/2]). By [BGR, Proposition 6.2.3/1], we have $A^\circ(X) = \{f \in A(X) \mid |f| \leq 1\}$. We denote by $B_K[a, r]$ the affinoid closed disk over K of radius $r \in |K|$ about $a \in \mathcal{O}_K$ whose $\mathbb{C}_p$-valued points are given by

$$B_K[a, r](\mathbb{C}_p) = \{x \in \mathcal{O}_K \mid |x - a|_p < r\}. \tag{4-0-2}$$

4.1. Coleman families

Let $f \in S_w^{\mathrm{new}}(N,\varepsilon)_\alpha$ be a primitive form. Assume that the characteristic polynomial

(4-1-1) $$X^2 - a_p(f)X + \varepsilon(p)p^{w-1} \in \mathbb{Z}(f)[X]$$

of $T^1(p)$ on the subspace spanned by f and $f|V_p$ has no double roots, i.e., $a_p(f)^2 \neq \varepsilon(p)p^{w-1}$. Let $\alpha_p(f)$ be the root of this polynomial satisfying $\mathrm{val}_p(\alpha_p(f)) = \alpha$. We refer to

(4-1-2) $$f^*(z) := f(z) - \varepsilon(p)p^{w-1}\alpha_p(f)^{-1}f(pz)$$

as the *p-stabilization* of f. The p-stabilization f^* is the Hecke eigenform of level Np with the same eigenvalues as f outside p and $T^1(p)$-eigenvalue $a_p(f^*) = \alpha_p(f)$.

Theorem 4.1 ([Col97]). *Let $f \in S_{w_0}^{\mathrm{new}}(N,\varepsilon)_\alpha$ be a primitive form with $w_0 > \alpha+1$ and K a complete discretely valued subfield of $\mathbb{C}_p$ containing the p-adic completion of the Hecke field $\mathbb{Q}(f^*)$. Assume $a_p(f)^2 \neq \varepsilon(p)p^{w_0-1}$. Then there exists a positive integer M and a formal power series*

(4-1-3) $$\mathbf{f} = \sum_{n=1}^{\infty} a_n(\mathbf{f})q^n \in A^\circ(B_K[w_0, p^{-M}])[[q]]$$

such that for any w in

(4-1-4) $$W(M) := \{w \in \mathbb{Z} \mid w \equiv w_0 \pmod{(p-1)p^M}, w > \alpha + 1\}$$

except for at most one, the specialization $\mathbf{f}(w)$ at w given by

(4-1-5) $$\mathbf{f}(w) := \sum_{n=1}^{\infty} a_n(\mathbf{f})(w)q^n$$

is the p-stabilization of some primitive form in $S_w^{\mathrm{new}}(N,\varepsilon;\mathcal{O}_K)_\alpha$ and $\mathbf{f}(w_0) = f^*$. More precisely, there exists a primitive form $f_w \in S_w^{\mathrm{new}}(N,\varepsilon;\mathcal{O}_K)_\alpha$ satisfying the following conditions:

1. $\mathbf{f}(w) = f_w^*$.
2. $f_{w_0} = f$.
3. $\mathbf{f}(w_1) \in S_{w_1}^{\mathrm{new}}(Np,\varepsilon)_\alpha$ *is primitive if there exists an exceptional weight* $w_1 \in W(M)$.

In particular, for any positive integer m and $w \in W(M)$, if $w \equiv w_0 \pmod{(p-1)p^{M+m}}$, then

(4-1-6) $$f_w^* \equiv f^* \pmod{p^m\mathcal{O}_K}$$

We refer to the family $\{f_w\}_{w\in W(M)}$ of primitive forms obtained in the theorem above as a *Coleman family* passing through f over K. By the theorem above, for any positive integer m and $w \in W(M)$, if $w \equiv w_0 \pmod{(p-1)p^{M+m}}$, then for any positive integer n with $p \nmid n$,

(4-1-7) $$a_n(f_w) \equiv a_n(f) \pmod{p^m\mathcal{O}_K}.$$

4.2. p-adic families of the D-th Saito-Kurokawa lifts

By Theorem 4.1 and Theorem 3.5, we immediately see that the family $\{F_w := \mathrm{SK}_D(f_w)\}_w$ forms a p-adic family in the sense that the T-eigenvalue $\lambda_{F_w}(T)$ gives a p-adic analytic function $w \mapsto \lambda_{F_w}(T)$ from $W(M)$ into K as follows:

Corollary 4.2. *Let $k_0 \geq 2$ be an even integer, χ a Dirichlet character modulo N with $\chi(-1) = 1$ and D a fundamental discriminant with $D < 0$ and $(D, c_\chi) = 1$. Assume (3-2-2) for $k = k_0$ and $M = Np$. Let $f \in S_{w_0}^{\mathrm{new}}(N,\chi^2)_\alpha$ be a primitive form with $w_0 := 2k_0 - 2 > \alpha + 1$ and K a complete discretely valued subfield of $\mathbb{C}_p$ containing the p-adic completion of the Hecke field $\mathbb{Q}(f^*)$. Assume*

$a_p(f)^2 \neq \chi^2(p)p^{w_0-1}$. Let $\{f_w\}_{w\in W(M)}$ be a Coleman family passing through f over K. Then for any w in

$$W^{\mathrm{SK}}(M) := \{2k-2 \in W(M) \mid k \in 2\mathbb{Z}\}, \tag{4-2-1}$$

if $w \equiv w_0 \pmod{(p-1)p^{M+m}}$, then for any $T \in \mathcal{H}^2(Np)$,

$$\lambda_{F_w^*}(T) \equiv \lambda_{F_{w_0}^*}(T) \pmod{p^m\mathcal{O}_K}, \tag{4-2-2}$$

where $F_w^* := \mathrm{SK}_D(f_w^*)$. In particular, for any positive integer n with $p \nmid n$, we have

$$\lambda_{F_w}(T^2(n)) \equiv \lambda_{F_{w_0}}(T^2(n)) \pmod{p^m\mathcal{O}_K}. \tag{4-2-3}$$

Remark 4.3. It is possible that $\{\mathrm{SK}_D(f_w^*)\}_{w\in W^{\mathrm{SK}}(M)}$ and $\{\mathrm{SK}_D(f_w)\}_{w\in W^{\mathrm{SK}}(M)}$ vanishes identically. However, it does follow from $\theta_D(f) \neq 0$ that $\mathrm{SK}_D(f_w^*) \neq 0$ and $\mathrm{SK}_D(f_w) \neq 0$ for any $w \in W^{\mathrm{SK}}(M)$ with sufficiently large M by p-adic interpolation of Fourier coefficients.

5. Cohomological interpretation

5.1. Modular symbols and elliptic cusp forms

Let Δ_0 be a subsemigroup of $\mathrm{M}_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$ containing $\Gamma_0(M)$. Let $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ be the group of divisors of degree 0 supported on the rational cusps $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{i\infty\}$ of the complex upper half plane $\mathfrak{H}$. We let Δ_0 act on $\mathfrak{H}$ by fractional linear transformations, i.e.,

$$\gamma z := \begin{cases} (az+b)(cz+d)^{-1} & \text{if } \det(\gamma) > 0, \\ (a\bar{z}+b)(c\bar{z}+d)^{-1} & \text{if } \det(\gamma) < 0, \end{cases} \quad \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \in \mathfrak{H}\right). \tag{5-1-1}$$

This induces a natural action of Δ_0 on $\mathfrak{H}^* := \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$ and $\mathbb{P}^1(\mathbb{Q})$. Then Δ_0 acts on $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ by linear fractional transformations. Let R be a commutative ring and E a left $R[\Delta_0]$-module. We let $\gamma \in \Delta_0$ acts on $\Phi \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), E)$ by

$$(\Phi|\gamma)(D) := \gamma\Phi(\gamma D). \tag{5-1-2}$$

Then the abstract Hecke algebra $\mathcal{H}_R(\Delta_0, \Gamma_0(M))$ acts on the group of E-valued *modular symbols* over $\Gamma_0(M)$:

$$\mathrm{Symb}_{\Gamma_0(M)}(E) := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), E)^{\Gamma_0(M)}. \tag{5-1-3}$$

Let $\tilde{E}$ be the locally constant sheaf on the open modular curve $Y := \Gamma_0(M)\backslash\mathfrak{H}$ attached to E. Assume that

(5-1-4) the orders of the torsion elements of $\Gamma_0(M)$ act invertibly on E.

Then by [AS86, Proposition 4.2], there exists a Hecke equivariant canonical isomorphism

$$H^1_c(Y, \tilde{E}) \xrightarrow{\sim} \mathrm{Symb}_{\Gamma_0(M)}(E). \tag{5-1-5}$$

Throughout the paper, we will identify the group of compactly supported cohomology classes with the group of modular symbols under the assumption that (5-1-4). Note that (5-1-4) holds if either E is a vector space over a field of characteristic 0, E is a $\mathbb{Z}_p$-module with $p \geq 5$, or $\Gamma_0(M)$ is torsion-free. The matrix $\iota := \mathrm{diag}(1, -1)$ induces the natural involution on $\mathrm{Symb}_{\Gamma_0(M)}(E)$ and the decomposition

$$\mathrm{Symb}_{\Gamma_0(M)}(E) = \mathrm{Symb}^+_{\Gamma_0(M)}(E) \oplus \mathrm{Symb}^-_{\Gamma_0(M)}(E) \tag{5-1-6}$$

if 2 acts invertibly on E. Indeed, each element Φ decomposes as $\Phi = \Phi^+ + \Phi^-$, where

$$\Phi^{\pm} := 2^{-1}(\Phi \pm \Phi|\iota). \tag{5-1-7}$$

For a non-negative integer n, let $L(n;R)$ be the R-module of homogeneous polynomials in (X,Y) of degree n with coefficients in R. For an R-valued Dirichlet character ε modulo M, we denote by $L(n,\varepsilon;R)$ the $R[\Gamma_0(M)]$-module $L(n;R)$ endowed with the following action; for $\gamma \in \Gamma_0(M)$ and $P \in L(n,\varepsilon;R)$,

$$(\gamma P)(X,Y) = \varepsilon(d_\gamma)P((X,Y)^t\gamma). \tag{5-1-8}$$

For each cusp form $f \in S_{w+2}(M,\varepsilon)$, we define the $L(w,\varepsilon;\mathbb{C})$-valued differential form on $\mathfrak{H}$ by

$$\omega_f := f(z)\left(\sum_{i=0}^{w}(-z)^i X^{w-i}Y^i\right)dz. \tag{5-1-9}$$

The additive map

$$\Phi_f : \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})) \to L(w,\varepsilon;\mathbb{C}) \ ; \ \{c_2\}-\{c_1\} \mapsto \int_{c_1}^{c_2}\omega_f \tag{5-1-10}$$

defines a modular symbol in $\mathrm{Symb}_{\Gamma_0(M)}(L(w,\varepsilon;\mathbb{C}))$ and the map

$$\Phi : S_{w+2}(M,\varepsilon) \to \mathrm{Symb}^-_{\Gamma_0(M)}(L(w,\varepsilon;\mathbb{C})) \ ; \ f \mapsto \Phi_f^- \tag{5-1-11}$$

is an injective $\mathbb{C}[\mathcal{H}^1(M)]$-homomorphism.

5.2. Cohomological interpretation of θ_D

Proposition 5.1 ([Mak17, Propostion 3.2]). *Let $k \geq 2$ be an integer, M an odd positive integer, χ a Dirichlet character modulo M and D a fundamental discriminant with $\chi(-1)(-1)^{k-1}D > 0$ and $(D,c_\chi) = 1$. Assume (3-2-2). Then we can define a $\mathbb{C}$-homomorphism Θ_D satisfying the commutative diagram*

$$\begin{array}{ccc} \mathrm{Symb}^-_{\Gamma_0(M)}(L(2k-4,\chi^2;\mathbb{C})) & \xrightarrow{\ \Theta_D\ } & \mathbb{C}[[q]] \\ \Phi\uparrow & & \uparrow \text{Fourier expansion} \\ S_{2k-2}(M,\chi^2) & \xrightarrow{\ \theta_D\ } & S^+_{k-1/2}(M,\chi), \end{array} \tag{5-2-1}$$

5.3. Algebraic litts θ_D^{alg} and $\mathrm{SK}_D^{\mathrm{alg}}$

Let $f \in S^{\mathrm{new}}_{w+2}(N,\varepsilon)$ be a primitive form. Let V_f and V_{f^*} be the the representation spaces of Galois representations ρ_f and ρ_{f^*} attached to f and f^* over $\mathbb{Q}_p(f)$ and $\mathbb{Q}_p(f^*)$, respectively. By the Comparison Theorem between étale and Betti induced by a fixed isomorphism $\bar{\mathbb{Q}}_p \cong \mathbb{C}$, we have

$$V_f \otimes_{\mathbb{Q}_p(f)} \bar{\mathbb{Q}}_p \cong \mathrm{Symb}(\mathbb{C})[f] := \bigcap_\ell \mathrm{Ker}\left((T(\ell)-a_\ell(f)) \,|\, \mathrm{Symb}^-_{\Gamma_0(N)}(L(w,\varepsilon;\mathbb{C}))\right), \tag{5-3-1}$$

$$V_{f^*} \otimes_{\mathbb{Q}_p(f^*)} \bar{\mathbb{Q}}_p \cong \mathrm{Symb}(\mathbb{C})[f^*] := \bigcap_\ell \mathrm{Ker}\left((T(\ell)-a_\ell(f^*)) \,|\, \mathrm{Symb}^-_{\Gamma_0(Np)}(L(w,\varepsilon;\mathbb{C}))\right). \tag{5-3-2}$$

Since the left-hand sides of both (5-3-1) and (5-3-2) are isomorphic by the Brauer-Nesbitt Theorem and the Chebotarev Density Theorem, we have

$$\mathrm{Symb}(\mathbb{C})[f] \cong \mathrm{Symb}(\mathbb{C})[f^*]. \tag{5-3-3}$$

By [Kit94, Proposition 3.3], these are free of rank one over $\mathbb{C}$, and hence we may assume that $\Phi_f \mapsto \Phi_{f^*}$ gives the isomorphism (5-3-3). By [Kit94, Proposition 3.3], the eigenmodules

$$\mathrm{Symb}(\mathbb{Z}_{(p)}(f))[f] := \mathrm{Symb}(\mathbb{C})[f] \cap \mathrm{Symb}^{-}_{\Gamma_0(N)}(L(w,\varepsilon;\mathbb{Z}_{(p)}(f)), \tag{5-3-4}$$

$$\mathrm{Symb}(\mathbb{Z}_{(p)}(f^*))[f^*] := \mathrm{Symb}(\mathbb{C})[f^*] \cap \mathrm{Symb}^{-}_{\Gamma_0(Np)}(L(w,\varepsilon;\mathbb{Z}_{(p)}(f^*)) \tag{5-3-5}$$

are free of rank one over $\mathbb{Z}_{(p)}(f)$ and $\mathbb{Z}_{(p)}(f^*)$, respectively. Let Φ°_f be a generator of $\mathrm{Symb}(\mathbb{Z}_{(p)}(f))[f]$, which is contained in

$$\mathrm{Symb}(\mathbb{Z}_{(p)}(f^*))[f] := \mathrm{Symb}(\mathbb{C})[f] \cap \mathrm{Symb}^{-}_{\Gamma_0(N)}(L(w,\varepsilon;\mathbb{Z}_{(p)}(f^*)). \tag{5-3-6}$$

Then there exists $\Omega(f) \in \mathbb{C}^\times$ such that

$$\Phi^\circ_f = \Omega(f)^{-1} \cdot \Phi^-_f \in \mathrm{Symb}(\mathbb{Z}_{(p)}(f))[f]. \tag{5-3-7}$$

Then the isomorphism (5-3-3) implies

$$\Omega(f)^{-1} \cdot \Phi^-_{f^*} \in \mathrm{Symb}(\mathbb{Z}_{(p)}(f^*))[f^*]. \tag{5-3-8}$$

Theorem 5.2. *Let $k \geq 2$ be an integer, χ a Dirichlet character modulo N and D a fundamental discriminant with $\chi(-1)(-1)^{k-1}D > 0$ and $(D, c_\chi p) = p$. Assume (3-2-2). Let $f \in S^{\mathrm{new}}_{2k-2}(N,\chi^2)$ be a primitive form. Then*

$$(c_D(k,\chi)\Omega(f))^{-1}\,\theta_D(f) \in S^+_{k-1/2}\left(N,\chi;\mathbb{Z}_{(p)}(f,\chi)\right), \tag{5-3-9}$$

$$(c_D(k,\chi)\Omega(f))^{-1}\,\theta_D(f^*) \in S^+_{k-1/2}\left(Np,\chi;\mathbb{Z}_{(p)}(f^*,\chi)\right). \tag{5-3-10}$$

Remark 5.3. Note that the values of χ^2 are contained in $\mathbb{Q}(f)$ for a primitive form $f \in S^{\mathrm{new}}_{2k-2}(N,\chi^2)$ but the values of χ are not necessarily.

We fix, once and for all, the complex period $\Omega(f)$ as (5-3-7) and define

$$\theta^{\mathrm{alg}}_D(f) := \Omega(f)^{-1}\theta_D(f), \tag{5-3-11}$$

$$\theta^{\mathrm{alg}}_D(f^*) := \Omega(f)^{-1}\theta_D(f^*), \tag{5-3-12}$$

$$\mathrm{SK}^{\mathrm{alg}}_D(f) := L\left(\mathrm{EZ}\left(\theta^{\mathrm{alg}}_D(f)\right)\right) = \Omega(f)^{-1}\,\mathrm{SK}_D(f), \tag{5-3-13}$$

$$\mathrm{SK}^{\mathrm{alg}}_D(f^*) := L\left(\mathrm{EZ}\left(\theta^{\mathrm{alg}}_D(f^*)\right)\right) = \Omega(f)^{-1}\,\mathrm{SK}_D(f^*). \tag{5-3-14}$$

Note that $c_D(k,\chi)^{-1}\,\mathrm{SK}^{\mathrm{alg}}_D(f) \in \mathbb{Z}_{(p)}(f,\chi)[[q]]$ and $c_D(k,\chi)^{-1}\,\mathrm{SK}^{\mathrm{alg}}_D(f^*) \in \mathbb{Z}_{(p)}(f^*,\chi)[[q]]$ by (3-3-3). For a Dirichlet character ψ and $j \in [1, k-1] \cap \mathbb{Z}$ with $\psi(-1)(-1)^{j-1} = -1$, we have

$$L^{\mathrm{alg}}(j, f\otimes\psi) := \frac{G(\psi^{-1})\Gamma(j)L(j, f\otimes\psi)}{(-2\pi\sqrt{-1})^j\Omega(f)} \in \mathbb{Z}_{(p)}(f,\psi) \tag{5-3-15}$$

by [Kit94, Lemma 4.1]. If $c_\chi \parallel N$ and $(D, N) = 1$, then

$$a_{|D|}(\theta^{\mathrm{alg}}_D(f)) = (-1)^{k-1}c_D(k,\chi)(Dc_\chi)^{k-2}R_D(f)L^{\mathrm{alg}}\left(k-1, f\otimes\chi_D\chi^{-1}\right). \tag{5-3-16}$$

By inner product formula obtained in [Mak], we have the following:

Corollary 5.4. *Let $k \geq 2$ be an even integer, $M \geq 1$ a square-free odd integer, χ a Dirichlet character modulo M with $\chi^2 = \mathbb{1}$ and $\chi(-1) = 1$ and D a fundamental discriminant with $D < 0$ and $(D, M) = 1$. Let $f \in S^{\mathrm{new}}_{2k-2}(M,\mathbb{1})$ be a primitive form. Then*

$$\frac{\|\,\mathrm{SK}^{\mathrm{alg}}_D(f)\|^2}{\|f\|^2} = C_D(k,M,\chi)L^{\mathrm{alg}}(k,f)\,L^{\mathrm{alg}}\left(k-1, f\otimes\chi_D\chi^{-1}\right) \tag{5-3-17}$$

where $\| \mathrm{SK}_D^{\mathrm{alg}}(f)\|^2 := \langle \mathrm{SK}_D^{\mathrm{alg}}(f), \mathrm{SK}_D^{\mathrm{alg}}(f)\rangle$, $\|f\|^2 := \langle f, f\rangle$ *and*

(5-3-18) $$C_D(k,M,\chi) := \frac{(-2\sqrt{-1})^{2k-1}R_D(f)|D|^{k-3/2}M^2c_\chi^{2k-3}\operatorname{res}_{s=1}L(s,\chi)}{2^3 3G(\chi_D\chi)}$$

In particular, if $\chi = \mathbb{1}$*, then*

(5-3-19) $$\frac{\| \mathrm{SK}_D^{\mathrm{alg}}(f)\|^2}{\|f\|^2} \in \mathbb{Q}(f).$$

Remark 5.5. Since $\operatorname{res}_{s=1}L(s,\chi) \in \bar{\mathbb{Q}}$ if and only if $\chi = \mathbb{1}$ by [MFM, Theorem 3.3.4], we see that

(5-3-20) $$\frac{\| \mathrm{SK}_D^{\mathrm{alg}}(f)\|^2}{\|f\|^2} \in \bar{\mathbb{Q}} \text{ if and only if } \chi = \mathbb{1}.$$

6. p-adic interpolation of Fourier coefficients

In this section, we first present p-adic interpolation of $\{a_n(\theta_D(f_w^*))\}_{w\in W^{\mathrm{SK}}(M)}$. Using this result, we establish p-adic interpolation of $\{a_T(\mathrm{SK}_D(f_w))\}_{w\in W^{\mathrm{SK}}(M)}$ for $T \in \mathcal{L}_{>0}$ following Guerzhoy's method in [Gue00] which is discussed in [Kaw] as well.

6.1. p-adic interpolation of $\theta_D^{\mathrm{alg}}(f^*)$

Theorem 6.1 ([Mak17, Theorem 5.7]). *Let $k_0 \geq 2$ be an integer, χ a Dirichlet character with $c_\chi \mid N$ and D a fundamental discriminant with $\chi(-1)(-1)^{k_0-1}D > 0$ and $(D, c_\chi p) = p$. Assume (3-2-2) for $k = k_0$ and $M = Np$. Let $f \in S_{w_0}^{\mathrm{new}}(N,\chi^2)_\alpha$ be a primitive form with $w_0 := 2k_0 - 2 > \alpha + 1$ and K a complete discretely valued subfield of $\mathbb{C}_p$ containing the p-adic completion of the field obtained by adjoining $c_D(k_0,\chi)$ to $\mathbb{Q}(f^*,\chi)$. Assume $a_p(f)^2 \neq \chi^2(p)p^{w_0-1}$. Let $\{f_w\}_{w\in W(M)}$ be a Coleman family passing through f over K. Then for sufficiently large M, we can define $\theta_D(\mathbf{f}) \in A(B_K[w_0, p^{-M}])[[q]]$ such that for any $w \in W^{\mathrm{SK}}(M)$, there exists $e_w \in K^\times$ independent of D satisfying*

(6-1-1) $$\theta_D(\mathbf{f})(w) = e_w\theta_D^{\mathrm{alg}}(f_w^*)$$

and $e_{w_0} = 1$.

6.2. p-adic interpolation of $\theta_D^{\mathrm{alg}}(f)$

Theorem 6.2. *Let $k_0 \geq 2$ be an integer, χ a Dirichlet character with $c_\chi \mid N$ and D_0 a fundamental discriminant with $\chi(-1)(-1)^{k_0-1}D_0 > 0$ and $(D_0, c_\chi p) = p$. Assume (3-2-2) for $k = k_0$ and $M = Np$. Let $f \in S_{w_0}^{\mathrm{new}}(N,\chi^2)_\alpha$ be a primitive form with $w_0 := 2k_0 - 2 > \alpha + 1$ and K a complete discretely valued subfield of $\mathbb{C}_p$ containing the p-adic completion of the field obtained by adjoining $c_{D_0}(k_0,\chi)$ to $\mathbb{Q}(f^*,\chi)$. Assume $a_p(f)^2 \neq \chi^2(p)p^{w_0-1}$. Let $\{f_w\}_{w\in W(M)}$ be a Coleman family passing through f over K. Let D be a fundamental discriminant with $\chi(-1)(-1)^{k-1}D > 0$ and $(D, c_\chi) = 1$. When both $p \mid D$ and $\chi_{D_0D/p^2}(p) = -1$ holds, we further assume $\chi^2 = \mathbb{1}$, $a_{|D_0|}(\theta_{D_0}(f)) \neq 0$, and the following condition:*

(6-2-1) $$\bigcap_{\ell\nmid N} \operatorname{Ker}\left(\left(T^+(\ell) - a_\ell(f)\right)|S_{k-1/2}^+(N,\chi)\right) \cong \mathbb{C},$$

where $\ell \nmid N$ runs over all primes $\ell \nmid N$. Then for sufficiently large M, we can define $a_{|D|}(\theta_{D_0}(\mathbf{f}^\circ)) \in A(B_K[w_0, p^{-M}])$ such that for any $w = 2k - 2 \in W^{\mathrm{SK}}(M)$, there exists $e_w \in K^\times$ satisfying

(6-2-2) $$a_{|D|}\left(\theta_{D_0}(\mathbf{f}^\circ)\right)(w) = e_w\left(1 - \chi_D\chi_0^{-1}(p)p^{k-2}a_p(f_w^*)^{-1}\right)a_{|D|}\left(\theta_{D_0}^{\mathrm{alg}}(f_w)\right)$$

and $e_{w_0} = 1$. *In particular, for any positive integer* m *and* $w \in W(M)$, *if* $w \equiv w_0 \pmod{(p-1)p^{M+m}}$ *and* $m > \mathrm{val}_p\left(\left(1 - \chi_D\chi_0^{-1}(p)p^{k_0-2}a_p(f^*)^{-1}\right) a_{|D|}\left(\theta_{D_0}^{\mathrm{alg}}(f)\right)\right)$, *then*

$$\begin{aligned} \mathrm{val}_p\left(e_w\left(1 - \chi_D\chi_0^{-1}(p)p^{k-2}a_p(f_w^*)^{-1}\right) a_{|D|}\left(\theta_{D_0}^{\mathrm{alg}}(f_w)\right)\right) \\ = \mathrm{val}_p\left(\left(1 - \chi_D\chi_0^{-1}(p)p^{k_0-2}a_p(f^*)^{-1}\right) a_{|D|}\left(\theta_{D_0}^{\mathrm{alg}}(f)\right)\right). \end{aligned} \tag{6-2-3}$$

Remark 6.3. Let the notation be the same as above.

(1) By (6-2-3), we see that $\theta_D(f_w) \neq 0$ if $\theta_D(f) \neq 0$.

(2) If N is square-free, then the condition (6-2-1) holds by [Koh82, Theorem 2.ii)].

Combining the theorem above with Theorem 3.3 gives the following:

Corollary 6.4. *Let* $k_0 \geq 2$ *be an integer and* χ *a Dirichlet character with* $c_\chi \parallel N$. *Assume (3-2-2) for* $k = k_0$ *and* $M = Np$. *Let* $f \in S_{w_0}^{\mathrm{new}}(N, \chi^2)_\alpha$ *be a primitive form with* $w_0 := 2k_0 - 2 > \alpha + 1$ *and* K *a complete discretely valued subfield of* $\mathbb{C}_p$ *containing the p-adic completion of the field obtained by adjoining* $c_{D_0}(k_0, \chi)$ *to* $\mathbb{Q}(f^*, \chi)$. *Assume* $a_p(f)^2 \neq \chi^2(p)p^{w_0-1}$. *Let* $\{f_w\}_{w \in W(M)}$ *be a Coleman family passing through* f *over* K. *Let* D_0 *and* D *be a fundamental discriminant with* $\chi_{D_0}\chi(-1)(-1)^{k_0-2} = \chi_D\chi(-1)(-1)^{k_0-2} = -1$ *and* $(D_0, Np) = (D, Np) = p$. *Assume* $a_{|D_0|}(\theta_{D_0}(f)) \neq 0$ *and* $a_{|D|}(\theta_D(f)) \neq 0$. *When* $\chi_{D_0D/p^2}(p) = -1$ *holds, we further assume* $\chi^2 = \mathbb{1}$ *and (6-2-1). Then for any* $w = 2k - 2 \in W^{\mathrm{SK}}(M)$ *with sufficiently large* M,

$$\mathrm{val}_p\left(\frac{L^{\mathrm{alg}}\left(k-1, f_w \otimes \chi_{D_0}\chi^{-1}\right)}{L^{\mathrm{alg}}\left(k-1, f_w \otimes \chi_D\chi^{-1}\right)}\right) = \mathrm{val}_p\left(\frac{L^{\mathrm{alg}}\left(k_0-1, f \otimes \chi_{D_0}\chi^{-1}\right)}{L^{\mathrm{alg}}\left(k_0-1, f \otimes \chi_D\chi^{-1}\right)}\right). \tag{6-2-4}$$

Remark 6.5.

$$\frac{G(\chi_{D_0}\chi)L\left(k-1, f_w \otimes \chi_{D_0}\chi^{-1}\right)}{G(\chi_D\chi)L\left(k-1, f_w \otimes \chi_D\chi^{-1}\right)} = \frac{L^{\mathrm{alg}}\left(k-1, f_w \otimes \chi_{D_0}\chi^{-1}\right)}{L^{\mathrm{alg}}\left(k-1, f_w \otimes \chi_D\chi^{-1}\right)} \in \mathbb{Q}(f_w, \chi) \tag{6-2-5}$$

6.3. p-adic interpolation of $\mathrm{SK}_D^{\mathrm{alg}}(f^*)$

Theorem 6.6. *Let* $k_0 \geq 2$ *be an even integer,* χ *a Dirichlet character with* $c_\chi \mid N$ *and* $\chi(-1) = 1$, *and* $D < 0$ *a fundamental discriminant with* $(D, c_\chi p) = p$. *Assume (3-2-2) for* $k = k_0$ *and* $M = Np$. *Let* $f \in S_{w_0}^{\mathrm{new}}(N, \chi^2)_\alpha$ *be a primitive form with* $w_0 := 2k_0 - 2 > \alpha + 1$ *and* K *a complete discretely valued subfield of* $\mathbb{C}_p$ *containing the p-adic completion of the field obtained by adjoining* $c_D(k_0, \chi)$ *to* $\mathbb{Q}(f^*, \chi)$. *Assume* $a_p(f)^2 \neq \chi^2(p)p^{w_0-1}$. *Let* $\{f_w\}_{w \in W(M)}$ *be a Coleman family passing through* f *over* K. *Then for sufficiently large* M, *we can define* $\mathrm{SK}_D^{\mathrm{alg}}(\mathbf{f}) \in A(B_K[w_0, p^{-M}])[[q]]_2$ *such that for any* $w \in W^{\mathrm{SK}}(M)$, *there exists* $e_w \in K^\times$ *satisfying*

$$\mathrm{SK}_D^{\mathrm{alg}}(\mathbf{f})(w) = e_w\,\mathrm{SK}_D^{\mathrm{alg}}(f_w^*) \tag{6-3-1}$$

and $e_{w_0} = 1$.

6.4. p-adic interpolation of $\mathrm{SK}_D^{\mathrm{alg}}(f)$

Theorem 6.7. *Let* $k_0 \geq 2$ *be an even integer,* χ *a Dirichlet character modulo* N *and* $D_0 < 0$ *a fundamental discriminant with* $(D_0, c_\chi p) = p$. *Assume (3-2-2) for* $k = k_0$ *and* $M = Np$. *Let* $f \in S_{w_0}^{\mathrm{new}}(N, \chi^2)_\alpha$ *be a primitive form with* $w_0 := 2k_0 - 2 > \alpha + 1$ *and* K *a complete discretely valued subfield of* $\mathbb{C}_p$ *containing the p-adic completion of the field obtained by adjoining* $c_{D_0}(k_0, \chi)$ *to* $\mathbb{Q}(f^*, \chi)$. *Assume* $a_p(f)^2 \neq \chi^2(p)p^{w_0-1}$. *Let* $\{f_w\}_{w \in W(M)}$ *be a Coleman family passing through* f *over* K. *Let* $T \in \mathcal{L}_{>0}$ *such that* $D := -\det(2T)$ *is a fundamental discriminant with* $(D, c_\chi) = 1$ *and* $D \equiv 1 \pmod 4$. *When both* $p \mid D$ *and* $\chi_{D_0D/p^2}(p) = -1$ *holds, we further assume*

$\chi^2 = \mathbb{1}$, $a_{|D_0|}(\theta_{D_0}(f)) \neq 0$, and (6-2-1). Then for sufficiently large M, we can define an element $a_T(\mathrm{SK}_{D_0}(\mathbf{f}^\circ)) \in A(B_K[w_0, p^{-M}])$ such that for any $w = 2k-2 \in W^{\mathrm{SK}}(M)$, there exists $e_w \in K^\times$ satisfying

$$a_T\left(\mathrm{SK}_{D_0}(\mathbf{f}^\circ)\right)(w) = e_w\left(1 - \chi_D\chi_0^{-1}(p)p^{k-2}a_p(f_w^*)^{-1}\right) a_T\left(\mathrm{SK}_{D_0}^{\mathrm{alg}}(f_w)\right) \tag{6-4-1}$$

and $e_{w_0} = 1$.

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