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Diophantine Frobenius problems from semigroup's series and identities for zeta functions

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1 Introduction

The *Frobenius Problem* is to determine the largest positive integer that is NOT representable as a nonnegative integer combination of given positive integers that are coprime (see [11] for general references).

Given positive integers $d_1, \dots, d_m$ with $\gcd(d_1, \dots, d_m) = 1$, it is well-known that all sufficiently large b the equation

$$d_1 x_1 + \cdots + d_m x_m = b \tag{1}$$

has a solution with nonnegative integers $x_1, \dots, x_m$.

The *Frobenius number* $F(d_1, \dots, d_m)$ is the LARGEST integer b such that (1) has no solution in nonnegative integers. For $m = 2$, we have

$$F(d_1, d_2) = (d_1 - 1)(d_2 - 1) - 1$$

(Sylvester (1884) [15]). For $m \ge 3$, exact determination of the Frobenius number is difficult. The Frobenius number cannot be given by closed formulas of a certain type (Curtis 1990 [4]), the problem to determine $F(d_1, \dots, d_m)$ is NP-hard under Turing reduction (see, e.g., Ramírez Alfonsín [11]).

Some formulae for the Frobenius number in three variables can be seen in [17].

Proposition 1. *Let*

$$q := \left\lfloor \frac{a}{a-\ell} \right\rfloor \quad \text{and} \quad r := a - q(a-\ell) = (a-\ell)\left\{ \frac{a}{a-\ell} \right\}.$$

If $\ell > k$ and $br < cq$, then

$$F(a,b,c) = \begin{cases} -a + b\big((\lambda+1)(a-\ell) + r - 1\big) & \text{if } \lambda \ge \frac{c(q-1)-br}{b(a-\ell)+c}; \\ -a + b(a-\ell-1) + c(q-\lambda-1) & \text{if } \lambda \le \frac{c(q-1)-br}{b(a-\ell)+c}, \end{cases}$$

where $\lambda := \left\lfloor \frac{cq-br}{b(a-\ell)+c} \right\rfloor$.

Proposition 2. *Let*

$$\bar{q} := \left\lfloor \frac{a}{\ell} \right\rfloor \quad \text{and} \quad \bar{r} := a - \bar{q}\ell = \ell \left\{ \frac{a}{\ell} \right\} .$$

If $\ell > k$ *and* $b(\ell - \bar{r}) < c(\bar{q} + 1)$, *then*

$$F(a,b,c) = \begin{cases} -a + b(\ell - 1) + c(\bar{q} - 1) & \text{if } 0 \le \bar{r} < \ell - k; \\ -a + b(\bar{r} - 1) + c\bar{q} & \text{if } \ell - k \le \bar{r} < \ell . \end{cases}$$

Consider the number of solutions. Sylvester (1882) gave the number of positive integers with no nonnegative integer representation by d_1 and d_2 by

$$g(d_1, d_2) = \frac{(d_1 - 1)(d_2 - 1)}{2} . \tag{2}$$

The number of solutions of the equation (1) in nonnegative integers $x_1, \dots, x_m$, denoted by $N(d_1, \dots, d_m; b)$. For $m = 2$, there exists an explicit formula for the number of solutions.

Proposition 3. *Tripathi (2000) [16]*

$$N(d_1, d_2; b) = \frac{b + d_1 d_1' + d_2 d_2'}{d_1 d_2} - 1 ,$$

where $d_1' \equiv -b d_1^{-1} \pmod{d_2}$, $d_2' \equiv -b d_2^{-1} \pmod{d_1}$ *with* $1 \le d_1' \le d_2$ *and* $1 \le d_2' \le d_1$.

But, the problem becomes fairly hard if $m \ge 3$.

We give the method for computing the desired number. For the set $\{a_1, \dots, a_n\} \subset \{1, 2, \dots\}$ with $\gcd(a_1, \dots, a_n) = 1$, we have

$$\begin{aligned} \mathcal{N}(x) : &= \sum_{b=0}^{\infty} N(d_1, \dots, d_m; b) x^b = \frac{1}{(1 - x^{d_1}) \cdots (1 - x^{d_m})} \\ &= \frac{c_1}{1 - x} + \cdots + \frac{c_m}{(1 - x)^m} \\ &\quad + \sum_{k=1}^{d_1 - 1} \frac{A_{d_1}(k)}{1 - \zeta_{d_1}^{-k} x} + \cdots + \sum_{k=1}^{d_m - 1} \frac{A_{d_m}(k)}{1 - \zeta_{d_m}^{-k} x} , \end{aligned} \tag{3}$$

where $\zeta_{d_l} = e^{2\pi i/d_l}$ $(l = 1, 2, \ldots, m)$. For the first decomposition into ordinary partial fractions, putting

$$\sum_{t=0}^{\infty} P_A(t)x^t = \frac{c_1}{1-x} + \cdots + \frac{c_m}{(1-x)^m},$$

we know that

$$P_A(t) = \sum_{l=1}^{\infty} c_l \binom{b+l-1}{b},$$

where we take $c_l = 0$ for $l > n$.

Then, we have the following expression ([2]).

Theorem 1.

$$\begin{aligned} P_A(t) &= \frac{1}{d_1 \cdots d_m} \sum_{l=0}^{m-1} \frac{(-1)^l}{(m-l-1)!} \\ &\quad \times \sum_{k_1+\cdots+k_m=l} d_1^{k_1} \cdots d_m^{k_m} \frac{B_{k_1} \cdots B_{k_m}}{k_1! \cdots k_m!} b^{m-l-1} \\ &= \frac{1}{d_1 \cdots d_m} \sum_{l=0}^{m-1} \frac{(-1)^l}{(m-l-1)!} \sum_{k_1+2k_2+\cdots+lk_l=l} \frac{(-1)^{k_2+\cdots+k_l}}{k_1! \cdots k_l!} \\ &\quad \times \left(\frac{B_1 S_1}{1 \cdot 1!}\right)^{k_1} \cdots \left(\frac{B_l S_l}{l \cdot l!}\right)^{k_l} b^{m-l-1}, \end{aligned}$$

where $S_j = d_1^j + \cdots + d_m^j$ and B_m is the m-th Bernoulli number.

If we write

$$P_A(t) = \sum_{l=1}^{\infty} c_l \binom{b+l-1}{b} = \sum_{j=0}^{m-1} d_j b^j,$$

d_j can be expressed as follows ([9]).

Theorem 2. *For $l \ge 0$ we have*

$$d_{m-l-1} = \frac{(-1)^l}{(m-l-1)!l!P} \mathbf{Y}_l \left(B_1 S_1, -\frac{B_2 S_2}{2}, \ldots, (-1)^{l+1} \frac{B_l S_l}{l}\right),$$

where $P = \prod_{j=1}^{m} d_j$, $S_n = \sum_{j=1}^{m} d_j^n$, B_n *is the n-th Bernoulli number, and* $\mathbf{Y}_n(y_1, \dots, y_n)$ *are Bell polynomials defined by*

$$\exp\left(\sum_{k=1}^{\infty} y_k \frac{x^k}{k!}\right) = \sum_{n=0}^{\infty} \mathbf{Y}_n(y_1, \dots, y_n) \frac{x^n}{n!}$$

with $\mathbf{Y}_0 = 1$, *and expressed as*

$$\mathbf{Y}_n(y_1, \dots, y_n) = \sum_{\substack{k_1+2k_2+\cdots+nk_n=n \\ k_1,k_2,\dots,k_n \ge 0}} \prod_{i=1}^{n} \frac{n! y_i^{k_i}}{k_i!(i!)^{k_i}}.$$

For the second decomposition including the periodic sequences in (3), we know that for $l = 1, 2, \dots, m$

$$A_{d_l}(k) = \frac{1}{d_l} \frac{1}{(1 - \zeta_{d_l}^{d_1 k}) \cdots (1 - \zeta_{d_l}^{d_{l-1} k})(1 - \zeta_{d_l}^{d_{l+1} k}) \cdots (1 - \zeta_{d_l}^{d_m k})}.$$

2 Numerical semigroups

A *numerical semigroup* $S(\mathbf{d}^m) = \langle d_1, \dots, d_m \rangle$ is said to be generated by a minimal set of natural numbers $\mathbf{d}^m = \{d_1, \dots, d_m\}$ with $\gcd(d_1, \dots, d_m) = 1$ if neither of its elements is linearly representable by the rest of them. Namely,

$$S(\mathbf{d}^m) = \left\{ s \in \mathbb{N} \cup \{0\} | s = \sum_{i=1}^{m} x_i d_i, \ x_i \in \mathbb{N} \cup \{0\} \right\}.$$

Here, $d_1, \dots, d_m$ are called generators. Put $\pi_m = \prod_{i=1}^{m} d_i$ and $\sigma_m = \sum_{i=1}^{m} d_i$.
$\mu = \min\{d_1, \dots, d_m\}$ is called multiplicity.
$G(\mathbf{d}^m) = \mathbb{N} \backslash S(\mathbf{d}^m)$: set of gaps of semigroup
$F(\mathbf{d}^m) = \max\{G(\mathbf{d}^m)\}$: *Frobenius number*
$g(\mathbf{d}^m) = \#\{G(\mathbf{d}^m)\}$: genus of semigroup
$c(\mathbf{d}^m) = 1 + F(\mathbf{d}^m)$: conductor of semigroup, so that $c(\mathbf{d}^m) \le 2g(\mathbf{d}^m)$
$\rho(\mathbf{d}^m) = 1 - \dfrac{g(\mathbf{d}^m)}{c(\mathbf{d}^m)}$: density of non-gaps
$H(\mathbf{d}^m; z) := \sum_{s \in S(\mathbf{d}^m)} z^s$: Hilbert series
$\Phi(\mathbf{d}^m; z) := \sum_{s \in G(\mathbf{d}^m)} z^s$: Generating function of gaps, so that $H(\mathbf{d}^m; z) + \Phi(\mathbf{d}^m; z) = \dfrac{1}{1-z}$

Several special numerical semigroups $S(\mathbf{d}^m)$ are as follows.

Proposition 4 (Roberts (1956) Arithmetic sequence [12]). *For* $\mathbf{d}^m = \{a, a+d, \dots, a+(m-1)d\}$

$$F(\mathbf{d}^m) = a\left\lfloor \frac{a-2}{m-1} \right\rfloor + d(a-1)$$

Proposition 5 (Selmer (1997) [14]; Rödseth (1994) [13] Almost arithmetic sequence). *For* $\mathbf{d}^m = \{a, ha+d, ha+2d, \dots, ha+(m-1)d\}$

$$F(\mathbf{d}^m) = ha\left\lfloor \frac{a-2}{m-1} \right\rfloor + a(h-1) + d(a-1)$$

Proposition 6 (Selmer (1997) [14]; Rödseth (1994) [13]) Almost arithmetic sequence). *For* $\mathbf{d}^m = \{a, a+1, a+2, a+2^2, \dots, a+2^{m-2}\}$

$$F(\mathbf{d}^m) = \frac{a(a+1)}{2^{m-2}} + \sum_{k=0}^{m-3} 2^k \left\lfloor \frac{a+2^k}{2^{m-2}} \right\rfloor + a(m-4) - 1$$

Proposition 7 (Ong & Ponomarenko (2008) Geometric sequence [10]). *For* $\mathbf{d}^m = \{a^{m-1}, a^{m-2}b, a^{m-3}b^2, \dots, b^{m-1}\}$

$$F(\mathbf{d}^m) = b^{m-2}(ab-a-b) + \frac{(b-1)a^2(a^{m-2}-b^{m-2})}{a-b}$$

A semigroup $S(\mathbf{d}^m)$ is called *symmetric* if for any integer s

$$s \in S(\mathbf{d}^m) \Longrightarrow F(\mathbf{d}^m) - s \notin S(\mathbf{d}^m)\,.$$

In fact, we have

$$c(\mathbf{d}^m) = 2g(\mathbf{d}^m), \quad \rho(\mathbf{d}^m) = \frac{1}{2}$$

Otherwise, $S(\mathbf{d}^m)$ is called nonsymmetric.

Proposition 8 (Watanabe (1973) [18]). *Let* $H_1 = \langle d_1, \dots, d_m \rangle$ *be a semigroup. For positive integers* a *and* b*, satisfying* $a \in H_1 \backslash \{d_1, \dots, d_m\}$ *and* $\gcd(a,b) = 1$*, denote* $H := \langle a, bH_1 \rangle = \langle a, bd_1, \dots, ad_m \rangle$*. Then*

$$H \textit{is symmetric} \Longleftrightarrow H_1 \textit{is symmetric}\,.$$

Proposition 9 (Johnson (1960), [8]).

$$F(H) = bF(H_1) + (b-1)a\,.$$

The semigroup $S(\mathbf{d}^2)$ is always symmetric.

Proposition 10 (Sylvester (1884) [15], Rödseth (1994) [13]).

$$F(\mathbf{d}^2; z) = d_1 d_2 - d_1 - d_2,$$
$$H(\mathbf{d}^2; z) = \frac{1 - z^{d_1 d_2}}{(1 - z^{d_1})(1 - z^{d_2})}.$$

However, the Hilbert series $H(\mathbf{d}^3; z)$ and the power sum $g_n(\mathbf{d}^3; z)$ are not so simple. For given $\mathbf{d}^3 = (d_1, d_2, d_3)$, Johnson's *minimal relations* (1960) [8] are constructed as follows.

$$a_{11}d_1 = a_{12}d_2 + a_{13}d_3,\ a_{22}d_2 = a_{21}d_1 + a_{23}d_3,\ a_{33}d_3 = a_{31}d_1 + a_{32}d_2\,,$$

where

$$a_{11} = \min\{v_{11} | v_{11} \geq 2,\ v_{11}d_1 = v_{12}d_2 + v_{13}d_3,\ v_{12}, v_{13} \in \mathbb{N} \cup \{0\}\}\,,$$
$$a_{22} = \min\{v_{22} | v_{22} \geq 2,\ v_{22}d_2 = v_{21}d_1 + v_{23}d_3,\ v_{21}, v_{23} \in \mathbb{N} \cup \{0\}\}\,,$$
$$a_{33} = \min\{v_{33} | v_{33} \geq 2,\ v_{33}d_3 = v_{31}d_1 + v_{32}d_2,\ v_{31}, v_{32} \in \mathbb{N} \cup \{0\}\}\,.$$

The auxiliary invariants a_{ij} $(i \neq j)$ are uniquely determined by this definition and

$$\gcd(a_{11}, a_{12}, a13) = \gcd(a_{22}, a_{22}, a23) = \gcd(a_{31}, a_{32}, a33) = 1\,.$$

The denominator of the Hilbert series is given by $(1-z^{d_1})(1-z^{d_2})(1-z^{d_3})$.

The numerator of the Hilbert series $Q(\mathbf{d}^3; z)$ for *nonsymmetric* semigroups $S(\mathbf{d}^3)$ is given by the following.

$$Q(\mathbf{d}^3; z) = 1 - (z^{a_{11}d_1} + z^{a_{22}d_2} + z^{a_{33}d_3}) + z^{1/2\left(\langle \mathbf{a}, \mathbf{d} \rangle - J(\mathbf{d}^3)\right)} + z^{1/2\left(\langle \mathbf{a}, \mathbf{d} \rangle + J(\mathbf{d}^3)\right)}\,,$$

where

$$\langle \mathbf{a}, \mathbf{d} \rangle = a_{11}d_1 + a_{22}d_2 + a_{33}d_3$$

and

$$J(\mathbf{d}^3) = \sqrt{\langle \mathbf{a}, \mathbf{d} \rangle - 4 \sum_{i>j} a_{ii}a_{jj}d_i d_j + 4d_1 d_2 d_3}\,.$$

The numerator of the Hilbert series for symmetric semigroup $S(\mathbf{d}^3)$ is given by

$$Q(\mathbf{d}^3; z) = (1 - z^{a_{22}d_2})(1 - z^{a_{33}d_3})\,.$$

3 Semigroup's series for negative degrees of the gaps values

We derive an explicit form for an inverse power series over values of gaps of numerical semigroups generated by two integers.

Let $S_m = \langle d_1, \ldots, d_m \rangle$ be the semigroup generated by a set of integers $\{d_1, \ldots, d_m\}$ such that

$$1 < d_1 < \cdots < d_m, \quad \gcd(d_1, \ldots, d_m) = 1\,.$$

This sum of integer powers of values the gaps in numerical semigroups $S_m = \langle d_1, \ldots, d_m \rangle$ is referred often as semigroup's series

$$g_n(S_m) = \sum_{s \in \mathbb{N} \setminus S_m} s^n \quad (n \in \mathbb{Z})\,,$$

and $g_0(S_m)$ is known as a genus of S_m.

For $n \geq 0$, an explicit expression of $g_n(S_2)$ was given.

Proposition 11. *Rödseth (1994) [13]) For $n \geq 0$,*

$$g_n(S_2) = \frac{1}{(n+1)(n+2)} \sum_{k=0}^{n+1} \sum_{l=0}^{n+1-k} \binom{n+2}{k} \binom{n+2-k}{l} B_k B_l d_1^{m+1-k} d_2^{n+1-l} - \frac{B_{n+1}}{n+1}\,,$$

where B_n is n-th Bernoulli number.

Remark. For $n = 0$, it is reduced to Sylvester's expression [15]:

$$g_0(S_2) = \frac{(d_1 - 1)(d_2 - 1)}{2}\,.$$

For $n = 1$, the result was given by Brown and Shiue in 1993 [3].

$$g_1(S_2) = \frac{g_0(S_2)}{6}(2d_1 d_2 - d_1 - d_2 - 1)\,.$$

An implicit expression of $g_n(S_3)$ was given by Fel and Rubinstein in 2007 [6].

We derive a formula for semigroup series

$$g_{-n}(S_2) = \sum_{s\in\mathbb{N}\setminus S_2} s^{-n} \quad (n \ge 1)\,.$$

Consider the numerical semigroup $S_2 = \langle d_1, d_2\rangle$, where $d_1, d_2 \ge 2$. We introduce the Hilbert series $H(z; S_2)$ and the gaps generating function $\Phi(z; S_2)$ are given by

$$H(z; S_2) = \sum_{s\in S_2} z^s \quad \text{and} \quad \Phi(z; S_2) = \sum_{s\in\mathbb{N}\setminus S_2} z^s\,,$$

so that

$$H(z; S_2) + \Phi(z; S_2) = \frac{1}{1-z} \quad (z < 1)\,. \tag{4}$$

Here, $\min\{\mathbb{N}\setminus S_2\} = 1$. $\max\{\mathbb{N}\setminus S_2\} = d_1 d_2 - d_1 - d_2$ is exactly the same as Frobenius number.

The rational representation of $H(z; S_2)$ is given by

$$H(z; S_2) = \frac{1 - z^{d_1 d_2}}{(1 - z^{d_1})(1 - z^{d_2})}\,. \tag{5}$$

Introduce a new generating function $\Psi_1(z; S_2)$ by

$$\Psi_1(z; S_2) = \int_0^z \frac{\Phi(t; S_2)}{t} dt = \sum_{s\in\mathbb{N}\setminus S_2} \frac{z^s}{s}\,.$$

Hence,

$$\Psi_1(1; S_2) = \sum_{s\in\mathbb{N}\setminus S_2} \frac{1}{s} = g_{-1}(S_2)\,. \tag{6}$$

Substituting (4) into (6), we obtain

$$\Psi_1(z; S_2) = \int_0^z \left(\frac{1}{1-t} - H(t; S_2)\right)\frac{dt}{t}\,. \tag{7}$$

Present an integral in (7) as follows.

$$\Psi_1(z; S_2) = \int_0^z \left(\sum_{k=0}^{\infty} t^{k-1} - \frac{H(t; S_2)}{t}\right) dt\,,$$

$$\frac{H(t;S_2)}{t} = \sum_{j=0}^{2} h_j(t;S_2)\,, \tag{8}$$

$$h_0(t;S_2) = \frac{1}{t},\ h_1(t;S_2) = \sum_{k_1=1}^{d_2-1} t^{k_1 d_1 - 1},$$

$$h_2(t;S_2) = \sum_{k_1=0}^{d_2-1}\sum_{k_2=1}^{\infty} t^{k_1 d_1 + k_2 d_2 - 1}\,. \tag{9}$$

Perform integration in (8) as

$$\Psi_1(z;S_2) = \sum_{k=1}^{\infty}\frac{z^k}{k} - \frac{1}{d_1}\sum_{k_1=1}^{d_2-1}\frac{z^{k_1 d_1}}{k_1} - \sum_{k_1=0}^{d_2-1}\sum_{k_2=1}^{\infty}\frac{z^{k_1 d_1 + k_2 d_2}}{k_1 d_1 + k_2 d_2}\,,$$

so by (6) we obtain

$$g_{-1}(S_2) = \sum_{k=1}^{\infty}\frac{1}{k} - \sum_{k_1=0}^{d_2-1}\sum_{k_2=1}^{\infty}\frac{1}{k_1 d_1 + k_2 d_2} - \frac{1}{d_1}\sum_{k_1=1}^{d_2-1}\frac{1}{k_1}\,.$$

4 A sum of the negative degrees of the gaps values $g_{-n}(S_2)$

We can have a general formula as $g_{-1}(S_2)$ by introducing of a new generating function $\Psi_n(z;S_2)$ $(n \geq 2)$ by

$$\Psi_n(z;S_2) = \int_0^z \frac{dt_1}{t_1}\int_0^{t_1}\frac{dt_2}{t_2}\cdots\int_0^{t_{n-1}}\frac{dt_n}{t_n}\Phi(t_n;S_2)$$
$$= \sum_{s\in\mathbb{N}\setminus S_2}\frac{z^s}{s^n},\quad \text{so,}\quad \Psi_n(1;S_2) = g_{-n}(S_2), \tag{10}$$

satisfying the recursive relation:

$$\Psi_{k+1}(t_{n-k-1};S_2) = \int_0^{t_{n-k-1}}\frac{dt_{n-k}}{t_{n-k}}\Psi_k(t_{n-k};S_2)\quad (k \geq 0)$$

with $\Psi_0(t_n;S_2) = \Phi(t_{n-1};S_2)$ and $t_0 = z$.

Hence,

$$\Psi_1(t_{n-1}; S_2) = \int_0^{t_{n-1}} \frac{dt_n}{t_n} \Psi_0(t_n; S_2)\,,$$
$$\Psi_2(t_{n-2}; S_2) = \int_0^{t_{n-2}} \frac{dt_{n-1}}{t_{n-1}} \Psi_1(t_{n-1}; S_2)\,.$$

Performing integration in (10), we obtain

$$\Psi_n(z; S_2) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} - \frac{1}{d_1^n} \sum_{k_1=1}^{d_2-1} \frac{z^{k_1 d_1}}{k_1^n} - \sum_{k_1,k_2 \in \mathbb{K}_2} \frac{z^{k_1 d_1 + k_2 d_2}}{(k_1 d_1 + k_2 d_2)^n}\,.$$

Thus, setting $z = 1$, we have for $n \geq 2$

$$g_{-n}(S_2) = \sum_{k=1}^{\infty} \frac{1}{k^n} - \sum_{k_1=0}^{d_2-1} \sum_{k_2=1}^{\infty} \frac{1}{(k_1 d_1 + k_2 d_2)^n} - \frac{1}{d_1^n} \sum_{k_1=1}^{d_2-1} \frac{1}{k_1^n}\,.$$

Define a ratio $\delta = d_1/d_2$ and represent the last expression as follows.

$$g_{-n}(S_2) = \sum_{k=1}^{\infty} \frac{1}{k^n} - \frac{1}{d_2^n} \sum_{k_2=1}^{\infty} \frac{1}{k_2^n} - \frac{1}{d_2^n} \sum_{k_1=1}^{d_2-1} \sum_{k_2=1}^{\infty} \frac{1}{(k_1 \delta + k_2)^n} - \frac{1}{d_1^n} \sum_{k_1=1}^{d_2-1} \frac{1}{k_1^n}\,.$$

Making use of the Hurwitz $\zeta(n,q) = \sum_{k=0}^{\infty} (k+q)^{-n}$ and Riemann zeta functions $\zeta(n) = \zeta(n,1)$, we obtain

$$g_{-n}(S_2) = \left(1 - \frac{1}{d_2^n}\right) \zeta(n) - \frac{1}{d_2^n} \sum_{k_1=1}^{d_2-1} \zeta(n, k_1\delta) \quad (n \geq 2)\,. \tag{11}$$

Interchanging d_1 and d_2 in (11), we get an alternative expression for $g_{-n}(S_2)$:

$$g_{-n}(S_2) = \left(1 - \frac{1}{d_1^n}\right) \zeta(n) - \frac{1}{d_1^n} \sum_{k_2=1}^{d_1-1} \zeta\left(n, \frac{k_2}{\delta}\right)\,. \tag{12}$$

5 Identities for Hurwitz zeta functions

Combining formulas (11) and (12), we get the identity

$$\delta^n \sum_{k=1}^{d_2-1} \zeta(n, k\delta) = (1 - \delta^n)\,\zeta(n) + \sum_{k=1}^{d_1-1} \zeta\left(n, \frac{k}{\delta}\right)\,.$$

Another spinoff of formulas (11) and (12) is a set of identities for Hurwitz zeta functions.

For example, consider the numerical semigroup $\langle 3,4\rangle$ with three gaps $\mathbb{N}\setminus\langle 3,4\rangle=\{1,2,5\}$. Substituting it into (11) and (12), we have

$$\zeta\left(n,\frac{3}{4}\right)+\zeta\left(n,\frac{6}{4}\right)+\zeta\left(n,\frac{9}{4}\right)=(4^n-1)\zeta(n)-\left(4^n+2^n+\left(\frac{4}{5}\right)^n\right)$$

and

$$\zeta\left(n,\frac{4}{3}\right)+\zeta\left(n,\frac{8}{3}\right)=(3^n-1)\zeta(n)-\left(3^n+\left(\frac{3}{2}\right)^n+\left(\frac{3}{5}\right)^n\right),$$

respectively.

We[1] shall show the identity (11) can be reduced to the multiplication theorem in Hurwitz zeta functions (see, e.g., [1, p.249],[5, (16),p.71]). It is similar for (12).

Since $\gcd(d_1,d_2)=1$, if $k_1d_1\equiv k_2d_1 \pmod{d_2}$ then $k_1\equiv k_2 \pmod{d_2}$. Therefore,

$$\begin{aligned}&\zeta\left(n,\left\{\frac{d_1}{d_2}\right\}\right)+\zeta\left(n,\left\{\frac{2d_1}{d_2}\right\}\right)+\cdots+\zeta\left(n,\left\{\frac{(d_2-1)d_1}{d_2}\right\}\right)\\&=\zeta\left(n,\frac{1}{d_2}\right)+\zeta\left(n,\frac{2}{d_2}\right)+\cdots+\zeta\left(n,\frac{d_2-1}{d_2}\right),\end{aligned}\tag{13}$$

where $\{x\}$ denotes the fractional part of a real number x. There exists a nonnegative integer a such that

$$\frac{ad_1}{d_2}<1<\frac{(a+1)d_1}{d_2}.$$

Then for any integer k' with $a<k'\le d_2-1$ there exists a positive integer l' such that $1\le k'd_1-l'd_2<d_2$, and

$$\begin{aligned}\zeta\left(n,\frac{k'd_1}{d_2}\right)&=\zeta\left(n,\frac{k'd_1-l'd_2}{d_2}\right)-\left(\frac{d_2}{k'd_1-l'd_2}\right)^n\\&\quad-\left(\frac{d_2}{k'd_1-(l'-1)d_2}\right)^n-\cdots-\left(\frac{d_2}{k'd_1-d_2}\right)^n,\end{aligned}\tag{14}$$

[1] This part was suggested by Dr. Ade Irma Suriajaya (RIKEN) in February 2018.

where

$$\frac{k'd_1 - l'd_2}{d_2} = \left\{\frac{k'd_1}{d_2}\right\}.$$

For any positive integer r, there exist integers x and y such that $r = xd_1 + yd_2$. If $0 \le x < d_2$, then r can be expressed uniquely. Thus, if $y \ge 0$, then $r \in S_2$. If $y < 0$, then $r \notin S_2$. The largest integer is given by $(d_2 - 1)d_1 - d_2$, that is exactly the same as Frobenius number $F(d_1, d_2)$. Thus, $k'd_1 - l''d_2 \notin S_2$ for all l'' with $1 \le l'' \le l'$ in (14). In addition, if $k_1d_1 - l_1d_2 = k_2d_1 - l_2d_2$, then by $\gcd(d_1, d_2) = 1$ we have $d_1 | (k_1 - k_2)$ and $d_2 | (l_1 - l_2)$. As $0 < k_1, k_2 < d_2$ and $0 < l_1, l_2 < d_1$, we get $k_1 = k_2$ and $l_1 = l_2$. Thus, all such numbers of the form $kd_1 - ld_2 \notin S_2$ are different.

In [7, (3.32)] for a real ξ and $d = \gcd(d_1, d_2)$

$$\sum_{k=0}^{d_2-1} \left\lfloor \frac{kd_1 + \xi}{d_2} \right\rfloor = d \left\lfloor \frac{\xi}{d} \right\rfloor + \frac{(d_1-1)(d_2-1)}{2} + \frac{d-1}{2}. \tag{15}$$

Hence, by (15) with $d = 1$ and $\xi = 0$, the total number of non-representable positive integers of the form $kd_1 - ld_2$ ($a < k < d_2$, $l = 1, 2, \dots, \lfloor kd_1/d_2 \rfloor - 1$) is

$$\sum_{k=1}^{d_2-1} \left\lfloor \frac{kd_1}{d_2} \right\rfloor = \frac{(d_1-1)(d_2-1)}{2},$$

that is exactly the same as the number of integers without non-negative integer representations by d_1 and d_2 in (2). Therefore, the right-hand side of (11) is

$$\begin{aligned} &\left(1 - \frac{1}{d_2^n}\right)\zeta(n) - \frac{1}{d_2^n}\sum_{k_1=1}^{d_2-1}\zeta\left(n, \frac{k_1d_1}{d_2}\right) \\ &= \left(1 - \frac{1}{d_2^n}\right)\zeta(n) - \frac{1}{d_2^n}\left(\sum_{k_1=1}^{d_2-1}\zeta\left(n, \left\{\frac{k_1d_1}{d_2}\right\}\right) - d_2^n \sum_{s \in \mathbb{N}\setminus S_2} s^{-n}\right) \\ &= \left(1 - \frac{1}{d_2^n}\right)\zeta(n) - \frac{1}{d_2^n}\sum_{k_1=1}^{d_2-1}\zeta\left(n, \frac{k}{d_2}\right) + \sum_{s \in \mathbb{N}\setminus S_2} s^{-n}. \end{aligned}$$

On the other hand, the left-hand side of (11) is

$$g_{-n}(S_2) = \sum_{s \in \mathbb{N}\setminus S_2} s^{-n}.$$

Therefore, we obtain that

$$\sum_{k=1}^{d_2} \zeta\left(n, \frac{k}{d_2}\right) = d_2^n \zeta(n)\,,$$

that is the multiplication theorem in Hurwitz zeta functions.

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References

[1] T. M. Apostol, *Introduction to Analytics Number Theory*, Springer, New York, 1976.

[2] M. Beck, I. M. Gessel and T. Komatsu, *The polynomial part of a restricted partition function related to the Frobenius problem*, Electron. J. Combin. **8** (No.1) (2001), #N7.

[3] T. C. Brown and P. J. Shiue, *A remark related to the Frobenius problem*, Fibonacci Quart. **31** (1993), 32–36.

[4] F. Curtis, *On formulas for the Frobenius number of a numerical semigroup*, Math. Scand. **67** (1990), 190–192.

[5] H. Davenport, *Multiplicative Number Theory*, Second Edition, Revised by H. L. Montgomety, Springer, New York, 1980.

[6] L. G. Fel and B. Y. Rubinstein, *Power sums related to semigroups* $S(d_1, d_2, d_3)$, Semigroup Forum **74** (2007), 93–98.

[7] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, Reading MA, 1988.

[8] S. M. Johnson, *A Linear Diophantine problem*, Canad. J. Math. **12** (1960), 390–398.

[9] T. Komatsu, *On the number of solutions of the Diophantine equation of Frobenius–General case*, Math. Communications **8** (2003), 195–206.

[10] D. C. Ong and V. Ponomarenko, *The Frobenius number of geometric sequences*, Integers **8** (2008), Article A33, 3 p.

[11] J. L. Ramìrez Alfonsìn, *The Diophantine Frobenius Problem*, Oxford University Press, Oxford, 2005.

[12] J. B. Roberts, *Notes on linear forms*, Proc. Amer. Math. Soc. **7** (1956), 465–469.

[13] Ö. J. Rödseth, *A note on Brown and Shiues paper on a remark related to the Frobenius problem*, Fibonacci Quart. **32** (1994), 407–408.

[14] E. S. Selmer, *On the linear diophantine problem of Frobenius*, J. Reine Angew. Math. **293/294** (1997), 1–17.

[15] J. J. Sylvester, *Mathematical questions with their solutions*, Educational Times **41** (1884), 21.

[16] A. Tripathi, *The number of solutions to $ax + by = n$*, Fibonacci Quart. **38** (2000), 290–293.

[17] A. Tripathi, *Formulae for the Frobenius number in three variables*, J. Number Theory **170** (2017), 368–389.

[18] K. Watanabe, *Some examples of 1-dim Gorenstein domains*, Nagoya Math. J. **49** (1973), 101–109.

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