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Experimental observation on k-summability of divergent solutions of the heat equation with $k > 1$

By

Kunio ICHINOBE* and Masatake MIYAKE**

Abstract

The Borel summability (or 1-summability) problem of divergent solution of the heat equation with locally holomorphic Cauchy data is completely characterized in the paper by Lutz, Miyake and Schäfke [7]. In this paper, we shall discuss the k-summability problem of $k > 1$ when the Cauchy data is an entire function with the exponential growth estimate of order $d > 2$. An experimental observation will be shown after reviewing some related results on this problem.

§ 1. Introduction

We consider the following Cauchy problem for the complex heat equation

$$\text{(H)} \qquad \begin{cases} \partial_t u(t,x) = \partial_x^2 u(t,x), \\ u(0,x) = \varphi(x) \in \mathcal{O}_x, \end{cases}$$

where $t, x \in \mathbb{C}$ and $\mathcal{O}_x$ denotes the set of holomorphic functions in a neighborhood of $x = 0$. This Cauchy problem has a unique formal power series solution of the form

$$\hat{u}(t,x) = \sum_{n\geq 0} \frac{\varphi^{(2n)}(x)}{n!} t^n. \tag{1.1}$$

This solution $\hat{u}(t,x)$ is divergent in general. Exactly, we say that $\hat{u}(t,x)$ is the formal power series of Gevrey order 1 and we denote $\hat{u}(t,x) \in \mathcal{O}_x[[t]]_1$, which means that for any n, we have

$$\max_{|x|\leq r} \left|\varphi^{(2n)}(x)/n!\right| \leq CK^n n!, \tag{1.2}$$

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*Aichi University of Education, 1 Hirosawa, Igaya, Kariya City, Aichi Pref., 448-8542, Japan.
**Nagoya University, Furo-cho, Chikusa-ku, Nagoya 464-8602, Japan.

with some positive constants C, K and r. For the divergent solution, the problem of k-summability with $k = 1$ was proved by Lutz, Miyake and Schäfke [7], where the definition of k-summability will be given in next section.

Theorem 1.1 ([7]). *Let $S(0,\pi;\varepsilon) = S(0;\varepsilon) \cup S(\pi;\varepsilon)$, where $S(\theta;\varepsilon) := \{x \in \mathbb{C}; \arg x - \theta| < \varepsilon/2\}$ and $\varepsilon > 0$. Then the formal solution $\hat{u}(t,x)$ of the Cauchy problem* (H) *is 1-summable in 0 direction if and only if the Cauchy data $\varphi(x) \in \mathcal{O}_x$ satisfies the following conditions.*

(i) *The Cauchy data $\varphi(x)$ can be analytically continued on a region $S(0,\pi;\varepsilon)$.*

(ii) *The Cauchy data has the exponential growth estimate of order at most 2 there, that is, $|\varphi(x)| \le Ce^{\delta|x|^2}$ for $x \in S(0,\pi;\varepsilon)$ and some positive constants C and δ.*

In this case, 1-sum of $\hat{u}(t,x)$ in 0 direction is obtained by

$$u^0(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} \varphi(x+y) e^{-y^2/4t} dy \tag{1.3}$$

with $|t - c_0| < c_0$ and $|x| \le r$ for some positive c_0 and r.

We remark that 1-sum in a sector $S(0,\alpha,\rho)$ for some $\alpha > \pi$ and $\rho > 0$ is obtained by making analytic continuation in t-variable by rotating the integral path $\mathbb{R}$ to $e^{i\theta}\mathbb{R}$ with $|\theta| < \varepsilon/2$.

In the following, we write the conditions (i) and (ii) by

$$\varphi(x) \in \mathrm{Exp}^2(S(0,\pi;\varepsilon))$$

and we call the condition "1-summability condition" or **1-S-C** for short.

As mentioned above we have a complete characterization of the 1-summability for the formal solution of the Cauchy problem (H) when the Cauchy data $\varphi(x)$ is holomorphic in a neighborhood of the origin. Therefore we consider the case when the Cauchy data $\varphi(x)$ is an entire function with the exponential growth estimate of finite order since we can regard the locally holomorphic functions as the entire functions with the exponential growth estimate of infinite order.

It is known that the formal solution $\hat{u}(t,x)$ of the Cauchy problem (H) is convergent in both variables if and only if $\varphi(x)$ is an entire function with the exponential growth estimate of order at most 2, that is, $\varphi(x) \in \mathrm{Exp}^2(\mathbb{C})$. Therefore we consider the case when the Cauchy data $\varphi(x) \in \mathrm{Exp}^d(\mathbb{C})$ with $d > 2$. In this case, the formal solution $\hat{u}(t,x)$ is divergent. Exactly, let $k = 1/(1-2/d) = d/(d-2)(>1)$. Then we have

$$\hat{u}(t,x) \in \mathcal{O}_x[[t]]_{1/k},$$

which means that for any n, we have

$$\max_{|x|\le r} \left|\varphi^{(2n)}(x)/n!\right| \le CK^n\Gamma(1+n/k)$$

with some positive constants C, K and r which are independent of n. In fact, we get the above estimates since we have the following estimates

$$\max_{|x|\le r} |\varphi^{(n)}(x)| \le C_r K_r^n n!^{1-1/d}$$

for positive constants C_r and K_r depending on $r > 0$ by the condition $\varphi(x) \in \mathrm{Exp}^d(\mathbb{C})$. Here when $d = \infty$ for the above inequality, we can get the estimates for the locally holomorphic functions.

In this paper, we ask a condition for the k-summability of the formal solution $\hat{u}(t,x)$ of the Cauchy problem (H) with the Cauchy data $\varphi(x) \in \mathrm{Exp}^d(\mathbb{C})(d > 2)$.

After we give the definition of k-summability in section 2 and related results in section 3, we will give our result in section 4. In section 5, we will give a proof of Lemma 4.2 which is needed for proving our result.

§ 2. Definition of k-summability

In this section, we give some notation and definitions in the way of Ramis or Balser (cf. W. Balser [1] for the details).

For $d \in \mathbb{R}, \beta > 0$ and $\rho (0 < \rho \le \infty)$, we define a sector $S = S(d, \beta, \rho)$ by

$$S(d,\beta,\rho) := \{t \in \mathbb{C}; |d - \arg t| < \beta/2, 0 < |t| < \rho\}, \tag{2.1}$$

where d, β and ρ are called the direction, the opening angle and the radius of S, respectively. We write $S(d, \beta, \infty) = S(d;\beta)$ for short.

For $k > 0$, we define that $\hat{v}(t,x) = \sum_{n=0}^{\infty} v_n(x)t^n \in \mathcal{O}_x[[t]]_{1/k}$ (we say $\hat{v}(t,x)$ is a formal power series of Gevrey order $1/k$) if $v_n(x)$ are holomorphic on a common closed disk $B(\sigma) = \{x \in \mathbb{C}; |x| \le \sigma\}$ for some $\sigma > 0$ and there exist some positive constants C and K such that for any n,

$$\max_{|x|\le\sigma} |v_n(x)| \le CK^n \Gamma\left(1 + \frac{n}{k}\right). \tag{2.2}$$

Here when $v_n(x) \equiv v_n$ (constants) for all n, we use the notation $\mathbb{C}[[t]]_{1/k}$ instead of $\mathcal{O}_x[[t]]_{1/k}$. In the following, we use the similar notation.

Let $k > 0$, $\hat{v}(t,x) = \sum_{n=0}^{\infty} v_n(x)t^n \in \mathcal{O}_x[[t]]_{1/k}$ and $v(t,x)$ be an analytic function on $S(d,\beta,\rho) \times B(\sigma)$. Then we define that

$$v(t,x) \cong_k \hat{v}(t,x) \quad \text{in } S = S(d,\beta,\rho), \tag{2.3}$$

if for any closed subsector S' of S, there exist some positive constants C and K such that for any $N \ge 1$, we have

$$\max_{|x|\le\sigma} \left| v(t,x) - \sum_{n=0}^{N-1} v_n(x)t^n \right| \le CK^N |t|^N \Gamma\left(1 + \frac{N}{k}\right), \quad t \in S'. \tag{2.4}$$

For $k > 0$, $d \in \mathbb{R}$ and $\hat{v}(t,x) \in \mathcal{O}_x[[t]]_{1/k}$, we say that $\hat{v}(t,x)$ is *k-summable* in d direction, and denote it by $\hat{v}(t,x) \in \mathcal{O}_x\{t\}_{k,d}$, if there exist a sector $S = S(d,\beta,\rho)$ with $\beta > \pi/k$ and an analytic function $v(t,x)$ on $S \times B(\sigma)$ such that $v(t,x) \cong_k \hat{v}(t,x)$ in S.

In the paper, we consider the direction as 0 direction only for simplicity. Therefore we use the notation $\mathcal{O}_x\{t\}_k$.

We remark that the function $v(t,x)$ above for a *k-summable* $\hat{v}(t,x)$ is unique if it exists. Therefore such a function $v(t,x)$ is called the *k-sum* of $\hat{v}(t,x)$ in 0 direction and we write it $v^0(t,x)$.

§ 3. Related results

In this section, we give related results.

§ 3.1. Related result by W. Balser [2]

W. Balser studied the same problem in the paper [2], that is, he studied the k-summability of the formal solution $\hat{u}(t,x)$ of the Cauchy problem (H) for the heat equation with the Cauchy data $\varphi(x) \in \mathrm{Exp}^d(\mathbb{C})$ $(d > 2)$ and gave the necessary and sufficient condition for the k-summability of $\hat{u}(t,x)$

Proposition 3.1. *Let* $k = 1/(1-2/d)$ *and* $\hat{u}(t,x) = \sum_{n\geq 0} \varphi^{(2n)}(x)t^n/n! \in \mathcal{O}_x[[t]]_{1/k}$ *be the formal solution of the Cauchy problem* (H) *with the Cauchy data* $\varphi(x) \in \mathrm{Exp}^d(\mathbb{C})$ $(d > 2)$. *Let*

$$\hat{\psi}_j(t) := \partial_x^j \hat{u}(t,0) = \sum_{n\geq 0} \varphi^{(2n+j)}(0)t^n/n! \in \mathbb{C}[[t]]_{1/k}. \tag{3.1}$$

Then $\hat{u}(t,x) \in \mathcal{O}_x\{t\}_k$ *if and only if* $\hat{\psi}_j(t) \in \mathbb{C}\{t\}_k$ *for* $j = 0, 1$.

This result has been extended by many mathematicians (cf. [3], [4], [9], [10] and [11]). We remark that in the above proposition, the conditions for $\hat{\psi}_j(t)$ are not explicitly understood as conditions for the Cauchy data $\varphi(x)$. We want to know the conditions for the Cauchy data $\varphi(x)$ in an explicit form as **1-S-C** in Theorem 1.1 by [7].

§ 3.2. Related results by Miyake-Ichinobe

We consider the following Cauchy problem

$$\text{(CP)} \qquad \begin{cases} \partial_t^p u(t,x) = \partial_x^q u(t,x), \\ u(0,x) = \varphi(x) \in \mathcal{O}_x, \\ \partial_t^j u(0,x) = 0 \ (1 \leq j \leq p-1), \end{cases}$$

where $p, q \in \mathbb{N}$ with $p < q$. This Cauchy problem has a unique formal power series solution of the form

$$\hat{u}(t,x) = \sum_{n \geq 0} \frac{\varphi^{(qn)}(x)}{(pn)!} t^{pn}, \tag{3.2}$$

which belongs to $\mathcal{O}_x[[t]]_{(q-p)/p}$.

We put $k(0) := p/(q-p)$. Then we obtained the following results of $k(0)$-summability and $k(0)$-sum for the formal solution $\hat{u}(t,x)$.

Theorem 3.2 ([5], [8]). *Let $\hat{u}(t,x)$ be the formal solution of* (CP). *Then $\hat{u}(t,x) \in \mathcal{O}_x\{t\}_{k(0)}$ if and only if the Cauchy data satisfies the following condition*

$$\varphi(x) \in \mathrm{Exp}^{\frac{q}{q-p}} \left(\bigcup_{m=0}^{q-1} S\left(\frac{2\pi m}{q}; \varepsilon\right) \right). \tag{$k(0)$-S-C}$$

In this case, $k(0)$-sum is obtained by

$$u^0(t,x) = \int_0^\infty \sum_{m=0}^{q-1} \varphi(x + ye^{2\pi mi/q}) \times E(t,y;p,q)dy, \tag{3.3}$$

where the kernel function of the integral is given in terms of Meijer G-function

$$E(t,y;p,q) = C \times \frac{1}{y} \times G^{q0}_{pq}\left(\frac{p^p}{q^q}\frac{y^q}{t^p}; \begin{matrix} 1/p, 2/p, \cdots, p/p \\ 1/q, 2/q, \cdots, q/q \end{matrix} \right), \tag{3.4}$$

where the constant $C = \prod_{j=1}^p \Gamma(j/p) / \prod_{j=1}^q \Gamma(j/q)$. Here Meijer G-function is given by the following integral

$$G^{q0}_{pq}\left(z; \begin{matrix} 1/p, 2/p, \cdots, p/p \\ 1/q, 2/q, \cdots, q/q \end{matrix} \right) = \frac{1}{2\pi i} \int_I \frac{\prod_{j=1}^q \Gamma(j/q + \tau)}{\prod_{j=1}^p \Gamma(j/p + \tau)} z^{-\tau} d\tau, \tag{3.5}$$

where the path I runs from $-i\infty$ to $+i\infty$ (see [6]).

As mentioned above we have a complete characterization of the $k(0)$-summability of the formal solution for the Cauchy problem (CP) when the Cauchy data $\varphi(x)$ is holomorphic in a neighborhood of the origin. Therefore we consider the case when the Cauchy data $\varphi(x)$ is an entire function with the exponential growth estimate of finite order.

In the paper [8], he gave the result that the formal solution $\hat{u}(t,x)$ of the Cauchy problem (CP) is convergent in both variables if and only if $\varphi(x)$ is an entire function with the exponential growth estimate of order at most $q/(q-p)$. Therefore we consider the case when the Cauchy data $\varphi(x) \in \mathrm{Exp}^d(\mathbb{C})$ with $d > q/(q-p)$. In this case, the formal solution $\hat{u}(t,x)$ is divergent. Exactly, let $k = p/(q-p-q/d)(> k(0) = p/(q-p))$. Then we have

$$\hat{u}(t,x) \in \mathcal{O}_x[[t]]_{1/k}.$$

When d satisfies $d \le q/\ell$ for $\ell \in \mathbb{N}$ and $1 \le \ell \le q-p-1$, we put

$$k(\ell) := \frac{p}{q-p-\ell}. \tag{3.6}$$

Then we can regard that the formal solution $\hat{u}(t,x)$ belongs to $\mathcal{O}_x[[t]]_{1/k(\ell)}(\supset \mathcal{O}_x[[t]]_{1/k})$ instead of $\mathcal{O}_x[[t]]_{1/k}$. In this case, we have the following result.

Theorem 3.3 ([9]). *We assume $q-p>1$. Let $\hat{u}(t,x)$ be the formal solution of the Cauchy problem* (CP) *with the Cauchy data $\varphi(x) \in \mathrm{Exp}^d(\mathbb{C})$, where $q/(q-p) < d \le q/\ell$ for $1 \le \ell \le q-p-1$ and $\ell \in \mathbb{N}$. Then $\hat{u}(t,x) \in \mathcal{O}_x\{t\}_{k(\ell)}$ if the Cauchy data $\varphi(x)$ satisfies the same condition* ($k(0)$-S-C) *as in Theorem 3.2.*

In this case, $k(\ell)$-sum is just same as $k(0)$-sum.

We remark that from the assumption $q-p>1$, this theorem does not contain the case of (H).

§ 4. Result

In this section, we shall give our result. We recall our problem.

The Cauchy problem of the heat equation

$$\text{(H)} \qquad \begin{cases} \partial_t u(t,x) = \partial_x^2 u(t,x), \\ u(0,x) = \varphi(x) \in \mathrm{Exp}^d(\mathbb{C}) \end{cases}$$

with $d>2$ has a unique formal solution of the form

$$\hat{u}(t,x) = \sum_{n\ge 0} \frac{\varphi^{(2n)}(x)}{n!} t^n, \tag{4.1}$$

which belongs to $\mathcal{O}_x[[t]]_{1/k}$, where $k = 1/(1-2/d) = d/(d-2) > 1$. Our problem is to ask a condition for the k-summability of the formal solution $\hat{u}(t,x)$. Our result is stated as follows.

Proposition 4.1. *We assume the following estimates for any n*

$$\text{(A)} \qquad |\varphi^{(n)}(x)| \le CK^n n!^{1-1/d} e^{\delta|x|^2}, \quad x \in S(0,\pi;\varepsilon)$$

with some positive constants C, K and δ which are independent of n. Then the formal solution $\hat{u}(t,x) \in \mathcal{O}_x\{t\}_k$.

In this case, k-sum is given by 1-sum.

$$u^0(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} \varphi(x+y) e^{-y^2/4t} dy. \tag{4.2}$$

Proof. The proof is done by substituting in the expression of 1-sum the following Taylor formula for $\varphi(x)$.

$$\varphi(x+y)=\sum_{j=0}^{n-1}\frac{\varphi^{(j)}(x)}{j!}y^j+\int_0^y\frac{(y-s)^{n-1}}{(n-1)!}\varphi^{(n)}(x+s)ds. \tag{4.3}$$

We put

$$\begin{aligned}u^0(t,x)&=\frac{1}{\sqrt{4\pi t}}\sum_{j=0}^{n-1}\frac{\varphi^{(j)}(x)}{j!}\int_{-\infty}^{+\infty}e^{-\frac{y^2}{4t}}y^j dy\\&\quad+\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{+\infty}e^{-\frac{y^2}{4t}}\int_0^y\frac{(y-s)^{n-1}}{(n-1)!}\varphi^{(n)}(x+s)dsdy\\&=:I_n(t,x)+R_n(t,x).\end{aligned}$$

Then we can easily get

$$I_n(t,x)=\sum_{j=0}^{[(n-1)/2]}\frac{\varphi^{(2j)}(x)}{j!}t^j.$$

Moreover, it is easily shown that under the assumption (A) we can prove the desired asymptotic estimates

$$\max_{|x|\le\sigma}|R_n(t,x)|\le C_1K_1^{\frac{n}{2}}|t|^{\frac{n}{2}}\Gamma\left(1+\frac{n/2}{k}\right)$$

with some positive constants σ, C_1 and K_1 for $t\in S(0,\alpha,\rho)$ with $\pi/k<\alpha<\pi$, where $k=1/(1-2/d)>1$ and ρ is sufficiently small. □

At the end of this section, we give a remark on the class of functions which satisfy the condition (A)

$$|\varphi^{(n)}(x)|\le CK^n n!^{1-1/d}e^{\delta|x|^2},\quad x\in S(0,\pi;\varepsilon). \tag{A}$$

For any entire function $\varphi(x)\in \mathrm{Exp}^d(\mathbb{C})$ the following estimates hold.

$$|\varphi^{(n)}(x_0)|\le C_0K_0^n n!^{1-1/d},\quad n\in\mathbb{N}$$

by some positive constants C_0 and K_0 depending on x_0. However we have to notice that in the condition (A), the constants C and K are independent of the position of x.

Let $d=2m$ $(m\ge 1)$ and $p(x)=\sum_{j=0}^{2m}a_jx^{2m-j}$ with $\operatorname{Re} a_0<0$. Then we put

$$\begin{aligned}&\varphi_1(x):=\exp(p(x)),\\&\varphi_2(x):=q(x)\exp(p(x)),\quad q(x)\in\mathrm{Exp}^2(\mathbb{C}).\end{aligned}$$

In this case, we can prove that these functions satisfy the condition (A). The following formula is crucial to get the estimates (A), whose proof is done by induction and will be given in the next section.

Lemma 4.2. *Let $f(x) = \exp(g(x))$ with a polynomial $g(x)$ of the degree $d > 1$. Then we have*

$$f^{(n)}(x) = f(x) \sum_{k_1,\ldots,k_d} B^n_{k_1,\ldots,k_d} (g')^{k_1} (g'')^{k_2} \cdots (g^{(d)})^{k_d}, \tag{4.4}$$

where the sum is taken over all $k_1, \ldots, k_d \in \mathbb{N}$ such that

$$k_1 + 2k_2 + 3k_3 + \cdots + dk_d = n, \tag{4.5}$$

and

$$B^n_{k_1,\ldots,k_d} = \frac{n!}{k_1!(2!^{k_2} k_2!) \cdots (d!^{k_d} k_d!)}. \tag{4.6}$$

We first show that $\varphi_1(x)$ satisfies (A). Let $\varphi_1(x) = \exp(p(x))$ and $p(x) = -x^{2m}, m \geq 1$ for simplicity. By using Lemma 4.2, we have

$$\varphi_1^{(n)}(x) = \varphi_1(x) \sum_{k_1,\ldots,k_{2m}} B^n_{k_1,\ldots,k_{2m}} P^n_{k_1,\ldots,k_{2m}}(x),$$

where $k_1 + 2k_2 + 3k_3 + \cdots + 2mk_{2m} = n$ and $P^n_{k_1,\ldots,k_{2m}}(x) = (p')^{k_1}(p'')^{k_2} \cdots (p^{(2m)})^{k_{2m}}$. We put $k_1 + k_2 + k_3 + \cdots + k_{2m} = \ell (< n)$. Then since

$$P^n_{k_1,\ldots,k_{2m}}(x) = (-1)^\ell (2m)^{k_1} (2m(2m-1))^{k_2} \cdots ((2m)!)^{k_{2m-1}} ((2m)!)^{k_{2m}} x^{2m\ell - n},$$

we have

$$|P^n_{k_1,\ldots,k_{2m}}(x)| \leq (2m)^n |x|^{2m\ell - n}.$$

Let a be a parameter. Then for $c > 0$, we have

$$\max_{r>0} r^a e^{-cr^{2m}} \leq C_1 K_1^{a/2m} (a/2m)!$$

for some positive constants C_1 and K_1, where the notation $b!$ for $b \notin \mathbb{N}$ means the Gamma function $\Gamma(1+b)$. Therefore, we obtain for $x \in S(0, \pi; \varepsilon)$

$$|\varphi_1(x) P^n_{k_1,\ldots,k_{2m}}(x)| \leq C_2 K_2^n (\ell - n/2m)! \leq C_3 K_3^n \frac{n!^{1-1/2m}}{(n-\ell)!},$$

where C_i and K_i are some positive constants for $i = 2, 3$. Moreover, we have

$$\left| \sum_{k_1,\ldots,k_{2m}} \frac{B^n_{k_1,\ldots,k_{2m}}}{(n-\ell)!} \right| \leq \sum_{\ell=1}^{n} \sum_{\sum_{j=1}^{2m} k_j = \ell} \frac{n!}{\prod_{j=1}^{2m} k_j!(n-\ell)!} = n(2m+1)^n.$$

By summing up the above inequalities, we obtain for $x \in S(0, \pi; \varepsilon)$

$$|\varphi^{(n)}(x)| \leq C_4 K_4^n n!^{1-1/2m}$$

for some positive constants C_4 and K_4.

We next show that $\varphi_2(x)$ satisfies (A). For $q(x) \in \mathrm{Exp}^2(\mathbb{C})$, we put $\varphi_2(x) = q(x)\varphi_1(x)$, where $\varphi_1(x)$ is the function given above. Then we have

$$|q^{(n)}(x)| \leq C_5 K_5^n e^{\delta|x|^2} n!^{1/2}$$

for some positive constants C_5 and K_5. Therefore we get

$$\begin{aligned}
|\varphi_2^{(n)}(x)| &= \left| \sum_{i=0}^{n} \binom{n}{i} q^{(n-i)}(x)\varphi_1^{(i)}(x) \right| \\
&\leq \sum_{i=0}^{n} \binom{n}{i} C_5 K_5^{n-i} e^{\delta|x|^2} (n-i)!^{1/2} C_4 K_4^i i!^{1-1/2m} \\
&\leq C_6 K_6^n n!^{1-1/2m} e^{\delta|x|^2}
\end{aligned}$$

for some positive constants C_6 and K_6. Here we use the inequality $(n-i)!i! \leq n!$ and the equality $\sum_i \binom{n}{i} = 2^n$.

§ 5. Proof of Lemma 4.2

We give a proof of Lemma 4.2 by induction.

When $n = 1$, it is trivial.

We assume that the expression (4.4) holds for $n-1$, that is, we assume the following equation.

$$f^{(n-1)}(x) = f(x) \sum_{k_1,\ldots,k_d} B^{n-1}_{k_1,\ldots,k_d} (g')^{k_1} (g'')^{k_2} \cdots (g^{(d)})^{k_d} =: f(x) \times G(x), \tag{5.1}$$

where the sum is taken over $k_1, \ldots, k_d$ such that $k_1 + 2k_2 + \cdots + dk_d = n-1$. By multiplying $f^{(n-1)}(x)$ by the both side after the logarithmic derivative for the both side, we have

$$f^{(n)}(x) = f^{(n-1)}(x) \times \left\{ \frac{f'(x)}{f(x)} + \frac{G'(x)}{G(x)} \right\} = f(x) \left\{ G(x) \times g'(x) + G'(x) \right\},$$

where we use $f'/f = g'$. Therefore we have

$$f^{(n)}/f = \sum B^{n-1}_{k_1,\ldots,k_d} (g')^{k_1+1} (g'')^{k_2} \cdots (g^{(d)})^{k_d} \tag{5.2}$$

$$+ \sum \sum_{i=1}^{d-1} k_i B^{n-1}_{k_1,\ldots,k_d} (g')^{k_1} (g'')^{k_2} \cdots (g^{(i)})^{k_i - 1} (g^{(i+1)})^{k_{i+1}+1} \cdots (g^{(d)})^{k_d}. \tag{5.3}$$

We put $k_1+1=\ell_1$ and $k_j=\ell_j$ for $j\geq 2$ in the right side of (5.2). For each term of (5.3) we put $k_i-1=\ell_i, k_{i+1}+1=\ell_{i+1}$ and $k_j=\ell_j$ for $j\neq i, i+1$. Then we get

$$f^{(n)}/f=\sum_{\ell_1,\dots,\ell_d}\left\{B^{n-1}_{\ell_1-1,\ell_2,\dots,\ell_d}+\sum_{i=1}^{d-1}k_iB^{n-1}_{\ell_1,\cdots,\ell_i+1,\ell_{i+1}-1,\cdots,\ell_d}\right\}(g')^{\ell_1}\cdots(g^{(d)})^{\ell_d},$$

where the sum is taken over $\ell_1,\dots,\ell_d$ such that $\ell_1+2\ell_2+\cdots+d\ell_d=n$. Finally we obtain the following desired expression.

$$B^{n-1}_{\ell_1-1,\ell_2,\dots,\ell_d}+\sum_{i=1}^{d-1}k_iB^{n-1}_{\ell_1,\cdots,\ell_i+1,\ell_{i+1}-1,\cdots,\ell_d}=B^n_{\ell_1,\dots,\ell_d}.$$

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