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A formal solvability of a coupling equation for PDEs of Briot-Bouquet type

By

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Abstract

We study couplings for a pair of a partial differential equation of Briot-Bouquet type in the t variable and its model equation, without assuming the analytic dependency in t. In this report, we concentrate on the formal solvability —the existence of a formal solution of a special form— of a coupling equation on one side indicated as (Ψ). The precise statement concerning the convergence, together with a similar question on the reversed equation, that is, the coupling equation on the other side indicated as (Φ), will be published elsewhere.

§ 1. Introduction

The notion of coupling equations was introduced by the third author [2], for a theory of a class of transformations between nonlinear partial differential equations of normal form in complex domains. It was extended in [3] and [4] for partial differential equations of Briot-Bouquet type.

In the original coupling theory, the analytic dependency in the independent variables of the original equations plays an important role, and the solutions to a coupling equation were treated as formal power series of a special form in infinitely many variables.

Recently, using a functional analytic approach with the notion of infinite dimensional holomorphy, we studied the coupling equations for partial differential equations of normal form in the t variable, without the requirement of the analytic dependency in

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t. (See [1]). As for couplings for partial differential equations of Briot-Bouquet type in the t variable without analytic dependency in t, we have not succeeded to introduce a similar functional analytic approach for the solvability result. However, as for coupling equations for an equation and its model equation, we succeed to solve them under a weaker assumption of the dependency in t.

In this report, we focus to illustrate the formal solvability of a coupling equation (Ψ).

§ 2. Coupling equations for PDEs of Briot-Bouquet type

Let us briefly recall the solvability results in [3], of coupling equations for PDEs of Briot-Bouquet type in a complex domain.

A partial differential equation of an unknown $u(t,x)$

$$t\frac{\partial u}{\partial t} = F(t,x,u,\frac{\partial u}{\partial x}) \tag{F}$$

with a given differentiable function $F(t,x,z_0,z_1)$ of four variables in a neighborhood of the origin is said to be *of Briot-Bouquet type* in the t variable, if F satisfies the so-called *Briot-Bouquet condition*

$$F(0,x,0,0) = 0, \quad \frac{\partial F}{\partial z_1}(0,x,0,0) = 0. \tag{BB}$$

In this case, the *characteristic exponent* of (F) is defined by

$$\lambda(x) := \frac{\partial F}{\partial z_0}(0,x,0,0), \tag{CE}$$

and F is written as

$$F(t,x,z_0,z_1) = \sum_{k\geq 1} F_k(t,x,z_0,z_1) = a(x)t + \lambda(x)z_0 + F_{\geq 2}(t,x,z_0,z_1). \tag{2.1}$$

Here F_k denotes the homogeneous part of degree k in the Taylor expansion of F in (t,z_0,z_1) variables, and $F_{\geq 2} = \sum_{k\geq 2} F_k$.

Among such equations sharing the same characteristic exponent $\lambda(x)$, a simple example is

$$t\frac{\partial v}{\partial t} = \lambda(x)v, \tag{M}$$

which is actually a linear ordinary differential equation in t with a parameter x. We call (M) a model equation of (F).

In [3], third author considered the case that F is a holomorphic function in (t,x,z_0,z_1) in a neighborhood of the origin in $\mathbb{C}^4$, and studied couplings between (F) and (M).

Actually, he considered the correspondences

$$\Phi : u \mapsto v, \quad v(t,x) = \Phi[u](t,x) := \phi(t,x,((\frac{\partial}{\partial x})^i u(t,x))_{i\in\mathbb{N}}),$$
$$\Psi : v \mapsto u, \quad u(t,x) = \Psi[v](t,x) := \psi(t,x,((\frac{\partial}{\partial x})^i v(t,x))_{i\in\mathbb{N}}),$$

defined via $\phi(t,x,z)$ and $\psi(t,x,z)$ with $z = (z_i)_{i\in\mathbb{N}} = (z_0, z_1, \dots)$, which are regarded as "holomorphic functions of infinitely many variables", and studied the condition for Φ to transform solutions of (F) into those to (M), and that for Ψ to transform solutions vice versa. Such conditions were described as *coupling equations*: $\phi(t,x,z)$ and $\psi(t,x,z)$ should formally satisfy

$$(\Phi) \qquad t\frac{\partial \phi}{\partial t} + \sum_{m\in\mathbb{N}} D^m F(t,x,z_0,\dots,z_{m+1}) \cdot \frac{\partial \phi}{\partial z_m} = \lambda(x)\phi,$$

$$(\Psi) \qquad t\frac{\partial \psi}{\partial t} + \sum_{m\in\mathbb{N}} D^m (\lambda(x)z_0) \cdot \frac{\partial \psi}{\partial z_m} = F(t,x,\psi,D\psi),$$

where D denotes a formal vector field of infinitely many variables, defined by

$$D = \frac{\partial}{\partial x} + \sum_{i\in\mathbb{N}} z_{i+1}\frac{\partial}{\partial z_i}.$$

A notion of "holomorphic functions of infinitely many variables" for ϕ and ψ in this situation was interpreted as a formal power series involving infinitely many variables (t,z) of form

$$(2.2) \qquad \phi, \psi \in \sum_{k\geq 1} \mathcal{O}_x(\mathbb{D}_R)[t, z_0, \dots, z_{k-1}]_k,$$

where $\mathcal{O}_x(\mathbb{D}_R)$ denotes the space of holomorphic functions on $\mathbb{D}_R := \{x \in \mathbb{C} \mid |x| < R\}$, and $\mathcal{O}_x(\mathbb{D}_R)[t, z_0, \dots, z_{k-1}]_k$ denotes the space of homogeneous polynomial of degree k in the $(t, z_0, \dots, z_{k-1})$ variables with coefficients in $\mathcal{O}_x(\mathbb{D}_R)$. In other words, ψ and ϕ admit decompositions into homogeneous parts and into monomials in (t,z) of form

$$(2.3) \qquad \psi(t,x,z) = \sum_{k\geq 1} \psi_k(t,x,z_0,\dots,z_{k-1}) = \sum_{k\geq 1} \sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N}^k, \\ i+|j|=k}} \psi_{i,j}(x) t^i z_0^{j_0} \cdots z_{k-1}^{j_{k-1}},$$

For example, the homogeneous part of degree 1 of ψ reads

$$\psi_1(t,x,z_0) = \psi_{1,0}(x)t + \psi_{0,1}(x)z_0.$$

By substituting these decompositions, we can reduce the coupling equations (Φ) and (Ψ) into recursive relations in $k \in \mathbb{N}$. In fact, for example, the coupling equation (Ψ)

for F of form (2.1) reads

$$\Big\{t\frac{\partial}{\partial t}+\lambda(x)\Big(\sum_{m\in\mathbb{N}} z_m\frac{\partial}{\partial z_m}-1\Big)+\sum_{m\in\mathbb{N}}\sum_{p=0}^{m-1}\lambda_{m,p}(x)z_p\frac{\partial}{\partial z_m}\Big\}\psi$$
$$= a(x)t+F_{\geq 2}(t,x,\psi,D\psi),$$

where $\lambda_{m,p}(x) := \binom{m}{p}(\frac{d}{dx})^{m-p}\lambda(x)$, and the corresponding recursive relation for $\psi_k = \sum_{i+|j|=k}\psi_{i,j}(x)t^iz^j$ is

$$\sum_{i+|j|=k}\Big\{i+\lambda(x)\big(|j|-1\big)\Big\}\psi_{i,j}(x)t^iz^j$$
$$+\sum_{i+|j|=k}\sum_{m\in\mathbb{N}}\sum_{p=0}^{m-1} j_m\lambda_{m,p}(x)\psi_{i,j}(x)t^i(z_p/z_m)z^j$$
$$= \text{terms determined by } F \text{ and } \{\psi_\ell\}_{1\leq\ell<k}.$$

We can interpret the left hand side as a linear operator with diagonal 1st term and off-diagonal 2nd term, applied to a family of unknowns $\psi_{i,j}$ for $(i,j)\in\mathbb{N}\times\mathbb{N}^k$, $i+|j|=k$. Therefore, if $i+\lambda(x)(j-1)$ never vanishes for any $(i,j)\in\mathbb{N}\times\mathbb{N}\setminus\{(0,0),(0,1)\}$ and for any x under consideration, the coupling equation (Ψ) admits a formal power series solution of form (2.3) satisfying $\psi_{0,1}(x)=\beta(x)$ for any holomorphic function $\beta(x)$.

In [3], the coupling equation (Φ) was also studied and similar formal power series solutions were constructed. Moreover, under so-called the *Poincaré condition* on the characteristic exponent $\lambda(x)$:

(P) $$\exists\sigma>0,\ \exists R>0,\ \forall x\in\overline{\mathbb{D}}_R,\ \forall(i,j)\in\mathbb{N}\times\mathbb{N}\setminus\{(0,0),(0,1)\},$$
$$|i+\lambda(x)(j-1)|\geq\sigma(i+j),$$

the convergence results were proved for such formal solutions ϕ and ψ, and some applications were obtained.

§ 3. A case with non-analytic dependency in t

We study the case that F is not necessarily analytic in t. Let $F(t,x,z_0,z_1)$ be a continuous function defined in a neighborhood of the origin in $\mathbb{R}_t\times\mathbb{C}^3_{(x,z_0,z_1)}$, which is holomorphic in x, z_0 and z_1. If moreover F satisfies the Briot-Bouquet condition (BB), the equation (F) is similarly said to be a PDE of Briot-Bouquet type in the t variable with the characteristic exponent $\lambda(x)$ defined by (CE). Note that we shall further pose a differentiability assumption on F in the t variable in order to study the solvability of coupling equations. However, for introducing the notion of couplings, it suffices to assume the continuity in t.

Between the equation (F) and the model equation (M) sharing the same characteristic exponent $\lambda(x)$, the notion of the coupling can be introduced, and we get the same coupling equations (Φ) and (Ψ), completely in the same manner as in the case of complex analytic dependency in Section 2.

On the other hand, we can not expand F into the Taylor series in the (t, z_0, z_1) variables like (2.1). Moreover, we can neither expect our functions ϕ and ψ to be a formal power series in the (t, z) variables like (2.2) and (2.3).

Now we pose the assumption that $F \in C_t^{m+1}\mathcal{O}_{(x,z_0,z_1)}$ for a positive integer m, and that

$$0 < \operatorname{Re}\lambda(0) < m, \quad \lambda(0) \notin \mathbb{Z}.$$

Note that the differentiability assumption for F can be relaxed to $F \in C_t^m\mathcal{O}_{(x,z_0,z_1)}$ by introducing the notion of "continuous solution" to the equation (Ψ), while we skip it here. Moreover, in this report, we restrict ourselves to the case $m = 1$, for the sake of simplicity. That is, we assume that F is C^2 in t, and that the characteristic exponent $\lambda(x)$ satisfies $0 < \operatorname{Re}\lambda(0) < 1$. In this case, F can be written as

$$F(t, x, z_0, z_1) = a(t, x)t + \lambda(x)z_0 + \sum_{k\geq 2} F_k(t, x, z_0, z_1), \tag{3.1}$$

$$F_k(t, x, z_0, z_1) = \sum_{i+j_0+j_1=k} F_{i,j_0,j_1}(t, x)t^i z_0^{j_0} z_1^{j_1}, \tag{3.2}$$

instead of (2.1), with $a(t, x)$, $F_{i,j_0,j_1}(t, x) \in C_t^1\mathcal{O}_x$. Note that it is possible to take the sum only for $i = 0, 1$ in (3.2), and that the expansion (3.1) is not unique. As for an example of the non-uniqueness, a function tz_0^2 can belong to F_3 as a monomial $t^1 z_0^2 z_1^0$ with a coefficient 1, or alternatively to F_2 as a monomial $t^0 z_0^2 z_1^0$ with a coefficient t.

Note also that the hypothesis $0 < \operatorname{Re}\lambda(0) < 1$ implies

$$\begin{aligned} &\exists\sigma > 0,\ \exists R > 0,\ \forall x \in \overline{\mathbb{D}}_R,\ \forall(i, j) \in \mathbb{N}\times\mathbb{N}\setminus\{(0, 0), (0, 1)\}, \\ &\qquad \operatorname{Re}\{i + \lambda(x)(j - 1)\} \geq \sigma(i + j), \end{aligned} \tag{RP$_1$}$$

which is a stronger condition than (P).

In this situation, we want to find a solution ψ to (Ψ), of form

$$\psi(t, x, z) = \sum_{k\geq 1} \sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N}^k,\\ i+|j|=k}} \psi_{i,j}(t, x)t^i z_0^{j_0} \cdots z_{k-1}^{j_{k-1}}, \tag{3.3}$$

with $\psi_{i,j}(t, x) \in C_t^1\mathcal{O}_x$ instead of (2.3).

Actually, we study the following equation:

$$t_{\mathrm{d}}\frac{\partial\hat{\psi}}{\partial t_{\mathrm{d}}} + t_{\mathrm{s}}\frac{\partial\hat{\psi}}{\partial t_{\mathrm{s}}} + \sum_{m\in\mathbb{N}} D^m(\lambda(x)z_0)\cdot\frac{\partial\hat{\psi}}{\partial z_m} = \hat{F}(t_{\mathrm{d}}, t_{\mathrm{s}}, x, \hat{\psi}, D\hat{\psi}), \tag{$\hat{\Psi}$}$$

with a given function

$$\hat{F}(t_{\rm d}, t_{\rm s}, x, z_0, z_1) = a(t_{\rm d}, x)t_{\rm s} + \lambda(x)z_0 + \sum_{k\geq 2}\hat{F}_k(t_{\rm d}, t_{\rm s}, x, z_0, z_1), \tag{3.4}$$

$$\hat{F}_k(t_{\rm d}, t_{\rm s}, x, z_0, z_1) = \sum_{i+j_0+j_1=k} F_{i,j_0,j_1}(t_{\rm d}, x)t_{\rm s}^i z_0^{j_0} z_1^{j_1}.$$

and an unknown

$$\begin{aligned}\hat{\psi}(t_{\rm d}, t_{\rm s}, x, z) &= \sum_{k\geq 1}\hat{\psi}_k(t_{\rm d}, t_{\rm s}, x, z_0, \dots, z_{k-1}) \\ &= \sum_{k\geq 1}\sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N}^k,\\ i+|j|=k}} \psi_{i,j}(t_{\rm d}, x)t_{\rm s}^i z_0^{j_0}\cdots z_{k-1}^{j_{k-1}},\end{aligned} \tag{3.5}$$

where F_{i,j_0,j_1} and $\psi_{i,j}$ are given in (3.1) and in (3.3), respectively. For a solution $\hat{\psi}$ to the equation $(\hat{\Psi})$, we can show that $\psi(t, x, z) := \hat{\psi}(t, t, x, z)$ solves the equation (Ψ).

Theorem 3.1. *Let $\beta(x)$ be a germ of a holomorphic function in x in a neighborhood of $x = 0$. Then, there exists a unique formal solution $\hat{\psi}$ of form (3.5) to the equation $(\hat{\Psi})$, satisfying $\psi_{0,1}(t_{\rm d}, x) = \beta(x)$.*

Remark. For any formal solution $\hat{\psi}$ of form (3.5) to the equation $(\hat{\Psi})$, $\psi_{0,1}(t_{\rm d}, x)$ is necessarily independent of $t_{\rm d}$.

Let us give the idea of the proof.

The equation $(\hat{\Psi})$ reads, for the homogeneous part of degree 1,

$$\Big(t_{\rm d}\frac{\partial}{\partial t_{\rm d}} + 1 - \lambda(x)\Big)\psi_{1,0}(t_{\rm d}, x)t_{\rm s} + t_{\rm d}\frac{\partial}{\partial t_{\rm d}}\psi_{0,1}(t_{\rm d}, x)z_0 = a(t_{\rm d}, x)t_{\rm s},$$

or equivalently

$$\Big(t_{\rm d}\frac{\partial}{\partial t_{\rm d}} + 1 - \lambda(x)\Big)\psi_{1,0}(t_{\rm d}, x) = a(t_{\rm d}, x), \quad t_{\rm d}\frac{\partial}{\partial t_{\rm d}}\psi_{0,1}(t_{\rm d}, x) = 0,$$

and for the higher degree parts,

$$\begin{aligned}&\sum_{i+|j|=k}\Big\{t_{\rm d}\frac{\partial}{\partial t_{\rm d}} + i + \lambda(x)\big(|j| - 1\big)\Big\}\psi_{i,j}(t_{\rm d}, x)t^i z^j \\ &+ \sum_{i+|j|=k}\sum_{m\in\mathbb{N}}\sum_{p=0}^{m-1} j_m\lambda_{m,p}(x)\psi_{i,j}(t_{\rm d}, x)t^i(z_p/z_m)z^j \\ &= \text{terms determined by } \hat{F} \text{ and } \{\hat{\psi}_\ell\}_{1\leq \ell<k}.\end{aligned}$$

Therefore, we have $\psi_{0,1}(t_{\mathrm{d}}, x) = \beta(x)$ for an arbitrary holomorphic function $\beta(x)$, and the other $\psi_{i,j}$ are determined uniquely in a recursion according to the order for (i, j) determined by $i + |j|$ and $\sum_p (p+1)j_p$, since $\left(t_{\mathrm{d}}\frac{\partial}{\partial t_{\mathrm{d}}} + \mu(x)\right)$ with $\mathrm{Re}\,\mu(x) > 0$ admits an inverse

$$f(t_{\mathrm{d}}, x) \mapsto \int_0^1 q^{\mu(x)-1} f(qt_{\mathrm{d}}, x)\,dq,$$

for $C_t^1\mathcal{O}_x$-functions.

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