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# Relative Dolbeault cohomology and Sato hyperfunctions

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This is a slightly extended version of the talk I gave at the RIMS Joint Research "Microlocal analysis and asymptotic analysis". The contents are taken from [7] and [16] and we refer to these for details and more material. See also [6] and [18] for further development and applications.

## 1 Introduction

Čech-de Rham cohomology together with its integration theory has been effectively used in various problems related to localization of characteristic classes. Likewise we may develop the Čech-Dolbeault cohomology theory and on the way we naturally come up with the notion of relative Dolbeault cohomology. This cohomology turns out to be canonically isomorphic with the local (relative) cohomology of A. Grothendieck and M. Sato with coefficients in the sheaf of holomorphic forms so that it provides a handy way of expressing the latter.

In this article we present the theory of relative Dolbeault cohomology and give, as applications, simple explicit expressions of Sato hyperfunctions, some fundamental operations on them and related local duality theorems. Particularly noteworthy is that the integration of hyperfunctions in our framework, which is a descendant of the integration theory on the Čech-de Rham cohomology, is simply given as the usual integration of differential forms. Also the Thom class in relative de Rham cohomology plays an essential role in the scene of interaction between topology and analysis.

## 2 Relative Dolbeault cohomology

### 2.1 Relative cohomology

Let $\mathscr{S}$ be a sheaf of Abelian groups on a topological space $X$. For an open set $V$ in $X$, we denote by $\mathscr{S}(V)$ the group of sections on $V$. Also for an open subset $V' \subset V$ we denote by $\mathscr{S}(V,V')$ the group of sections on $V$ that vanish on $V'$. As reference cohomology theory we adopt the one via flabby resolution (cf. [2], [9]). Thus for an open set $X'$ in $X$, $H^q(X,X';\mathscr{S})$ denotes the $q$-the cohomology of the complex $\mathscr{F}^\bullet(X,X')$

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with $0 \to \mathscr{S} \to \mathscr{F}^\bullet$ a flabby resolution. It is uniquely determined modulo canonical isomorphisms, independently of the flabby resolution. Setting $S = X \smallsetminus X'$, it will also be denoted by $H^q_S(X;\mathscr{S})$. This cohomology in the first expression is referred to as the *relative cohomology* of $\mathscr{S}$ on $(X, X')$ (cf. [11]) and in the second expression the *local cohomology* of $\mathscr{S}$ on $X$ with support in $S$ (cf. [4]).

## 2.2 Čech-Dolbeault cohomology

Let $X$ be a complex manifold of dimension $n$. We denote by $\mathscr{E}^{(p,q)}_X$ and $\mathscr{O}^{(p)}_X$ the sheaves of $C^\infty$ $(p,q)$-forms and holomorphic $p$-forms on $X$. We denote $\mathscr{O}^{(0)}_X$ by $\mathscr{O}_X$. We also omit the suffix $X$ if there is no fear of confusion. Recall that the Dolbeault complex $(\mathscr{E}^{(p,\bullet)}, \bar\partial)$ gives a fine resolution of $\mathscr{O}^{(p)}$:

$$0 \longrightarrow \mathscr{O}^{(p)} \longrightarrow \mathscr{E}^{(p,0)} \xrightarrow{\bar\partial} \mathscr{E}^{(p,1)} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{E}^{(p,n)} \longrightarrow 0.$$

**Dolbeault cohomology:** The Dolbeault cohomology $H^{p,q}_{\bar\partial}(X)$ of $X$ of type $(p,q)$ is the $q$-th cohomology of the complex $(\mathscr{E}^{(p,\bullet)}(X), \bar\partial)$. The Dolbeault theorem says that there is an isomorphism

$$H^{p,q}_{\bar\partial}(X) \simeq H^q(X;\mathscr{O}^{(p)}). \tag{2.1}$$

Note that among the isomorphisms, there is a canonical one (cf. [16], [17]).

**Čech-Dolbeault cohomology:** The Čech-Dolbeault cohomology may be defined for an arbitrary covering of a complex manifold. Here we recall the case of coverings consisting of two open sets and refer to [15] and [16] for the general case and details.

Let $\mathcal{V} = \{V_0, V_1\}$ be an open covering of $X$ and set $V_{01} = V_0 \cap V_1$. We set

$$\mathscr{E}^{(p,q)}(\mathcal{V}) = \mathscr{E}^{(p,q)}(V_0) \oplus \mathscr{E}^{(p,q)}(V_1) \oplus \mathscr{E}^{(p,q-1)}(V_{01}).$$

Thus an element in $\mathscr{E}^{(p,q)}(\mathcal{V})$ is expressed by a triple $\xi = (\xi_0, \xi_1, \xi_{01})$. We define the differential

$$\bar\vartheta : \mathscr{E}^{(p,q)}(\mathcal{V}) \longrightarrow \mathscr{E}^{(p,q+1)}(\mathcal{V}) \quad \text{by} \quad \bar\vartheta(\xi_0, \xi_1, \xi_{01}) = (\bar\partial\xi_0, \bar\partial\xi_1, \xi_1 - \xi_0 - \bar\partial\xi_{01}).$$

Then we see that $\bar\vartheta \circ \bar\vartheta = 0$.

**Definition 2.2** The *Čech-Dolbeault cohomology* $H^{p,q}_{\bar\vartheta}(\mathcal{V})$ of $\mathcal{V}$ of type $(p,q)$ is the $q$-th cohomology of the complex $(\mathscr{E}^{(p,\bullet)}(\mathcal{V}), \bar\vartheta)$.

**Theorem 2.3** *The inclusion* $\mathscr{E}^{(p,q)}(X) \hookrightarrow \mathscr{E}^{(p,q)}(\mathcal{V})$ *given by* $\omega \mapsto (\omega|_{V_0}, \omega|_{V_1}, 0)$ *induces an isomorphism*

$$H^{p,q}_{\bar\partial}(X) \xrightarrow{\sim} H^{p,q}_{\bar\vartheta}(\mathcal{V}).$$

Note that the inverse is given by assigning to the class of $(\xi_0, \xi_1, \xi_{01})$ the class of $\rho_0\xi_0 + \rho_1\xi_1 - \bar\partial\rho_0 \wedge \xi_{01}$, where $\{\rho_0, \rho_1\}$ is a $C^\infty$ partition of unity subordinate to $\mathcal{V}$.

## 2.3 Relative Dolbeault cohomology

Let $X$ be as above and $S$ a closed set in $X$. Letting $V_0 = X \setminus S$ and $V_1$ a neighborhood of $S$ in $X$, we consider the coverings $\mathcal{V} = \{V_0, V_1\}$ and $\mathcal{V}' = \{V_0\}$ of $X$ and $X \setminus S$. We set

$$\mathscr{E}^{(p,q)}(\mathcal{V}, \mathcal{V}') = \{\, \xi \in \mathscr{E}^{(p,q)}(\mathcal{V}) \mid \xi_0 = 0 \,\} = \mathscr{E}^{(p,q)}(V_1) \oplus \mathscr{E}^{(p,q-1)}(V_{01}).$$

Then we see that $(\mathscr{E}^{(p,\bullet)}(\mathcal{V}, \mathcal{V}'), \bar{\vartheta})$ is a subcomplex of $(\mathscr{E}^{(p,\bullet)}(\mathcal{V}), \bar{\vartheta})$.

**Definition 2.4** The *relative Dolbeault cohomology* $H^{p,q}_{\bar{\vartheta}}(\mathcal{V}, \mathcal{V}')$ of $(\mathcal{V}, \mathcal{V}')$ of type $(p, q)$ is the $q$-th cohomology of the complex $(\mathscr{E}^{(p,\bullet)}(\mathcal{V}, \mathcal{V}'), \bar{\vartheta})$.

From the exact sequence of complexes

$$0 \longrightarrow \mathscr{E}^{p,\bullet}(\mathcal{V}, \mathcal{V}') \xrightarrow{j^*} \mathscr{E}^{p,\bullet}(\mathcal{V}) \xrightarrow{i^*} \mathscr{E}^{p,\bullet}(V_0) \longrightarrow 0,$$

where $j^*(\xi_1, \xi_{01}) = (0, \xi_1, \xi_{01})$ and $i^*(\xi_0, \xi_1, \xi_{01}) = \xi_0$, we have the following exact sequence:

$$\cdots \longrightarrow H^{p,q-1}_{\bar{\partial}}(V_0) \xrightarrow{\delta^*} H^{p,q}_{\bar{\vartheta}}(\mathcal{V}, \mathcal{V}') \xrightarrow{j^*} H^{p,q}_{\bar{\vartheta}}(\mathcal{V}) \xrightarrow{i^*} H^{p,q}_{\bar{\partial}}(V_0) \longrightarrow \cdots, \tag{2.5}$$

where $\delta^*$ assigns to the class of $\theta$ the class of $(0, -\theta)$. From the above and Theorem 2.3, we have:

**Proposition 2.6** *The cohomology $H^{p,q}_{\bar{\vartheta}}(\mathcal{V}, \mathcal{V}')$ is determined uniquely modulo canonical isomorphisms, independently of the choice of $V_1$.*

In view of the above we denote $H^{p,q}_{\bar{\vartheta}}(\mathcal{V}, \mathcal{V}')$ also by $H^{p,q}_{\bar{\vartheta}}(X, X \setminus S)$.

**Proposition 2.7 (Excision)** *For any open set $V$ containing $S$, there is a canonical isomorphism*

$$H^{p,q}_{\bar{\vartheta}}(X, X \setminus S) \simeq H^{p,q}_{\bar{\vartheta}}(V, V \setminus S).$$

The relative Dolbeault cohomology share all the fundamental properties with the relative (local) cohomology of $X$ with coefficients in $\mathscr{O}^{(p)}$. In fact we have (cf. [16]):

**Theorem 2.8 (Relative Dolbeault theorem)** *There is a cononical isomorphism*

$$H^{p,q}_{\bar{\vartheta}}(X, X \setminus S) \simeq H^{q}_{S}(X; \mathscr{O}^{(p)}).$$

We have the cup product and integration theory in Čech-Dolbeault cohomology, for which we come back in a special case.

# 3 Sato hyperfunctions

## 3.1 Hyperfunctions and hyperforms

Let $M$ be a real analytic manifold of dimension $n$ and $X$ its complexification. For a sheaf $\mathscr{S}$ on $X$, we denote by $\mathscr{H}^q_M(\mathscr{S})$ the sheaf defined by the presheaf $V \mapsto H^q_{M\cap V}(V, \mathscr{S})$. In fact it is supported on $M$ and may be thought of as a sheaf on $M$. We recall (cf. [8], [12]) that the sheaf of Sato hyperfunctions on $M$ is defined by

$$\mathscr{B}_M = \mathscr{H}^n_M(\mathscr{O}_X) \otimes_{\mathbb{Z}_M} or_{M/X},$$

where $or_{M/X} = \mathscr{H}^n_M(\mathbb{Z}_X)$ is the relative orientation sheaf, i.e., the orientation sheaf of the normal bundle $T_M X$. More generally we introduce the following:

**Definition 3.1** The sheaf of *p-hyperforms* on $M$ is defined by

$$\mathscr{B}^{(p)}_M = \mathscr{H}^n_M(\mathscr{O}^{(p)}_X) \otimes_{\mathbb{Z}_M} or_{M/X}.$$

It is what is referred to as the sheaf of $p$-forms with coefficients in hyperfunctions. Since $X$ is a complex manifold, it is always orientable. However the orientation we consider is not necessarily the "usual one". Here we say an orientation of $X$ is usual if $(x_1, y_1, \dots, x_n, y_n)$ is a positive coordinate system when $(z_1, \dots, z_n)$, $z_i = x_i + \sqrt{-1}y_i$, is a coordinate system on $X$. If $M$ is orientable, so is $T_M X$. Thus in this case, for any open set $U \subset M$, we have

$$\mathscr{B}^{(p)}_M(U) = H^n_U(V; \mathscr{O}^{(p)}_X) \otimes_{\mathbb{Z}_M(U)} H^n_U(V; \mathbb{Z}_X), \tag{3.2}$$

where $V$ is an open set in $X$ containing $U$ as a close set. We refer to such a $V$ a *complex neighborhood* of $U$ in $X$.

**Remark 3.3** In the above we used the fact that $M$ is purely $n$-codimensional in $X$ with respect to $\mathscr{O}^{(p)}_X$ and $\mathbb{Z}_X$. For the latter, this can be seen from the Thom isomorphism (cf. Subsection 5.1 below).

When we specify various orientations, we adopt the convention that the orientation of $T_M X$ followed by that of $M$ gives the orientation of $X$. Thus if we specify orientations of $X$ and $M$, the orientation of $T_M X$ is determined and we have a canonical isomorphism $or_{M/X} \simeq \mathbb{Z}_X$ so that we have canonical isomorphisms

$$\mathscr{B}^{(p)}_M \simeq \mathscr{H}^n_M(\mathscr{O}^{(p)}_X) \quad \text{and} \quad \mathscr{B}^{(p)}_M(U) \simeq H^n_U(V, \mathscr{O}^{(p)}_X) \ \text{ for any open set } U \subset M. \tag{3.4}$$

In the sequel, at some point the cohomology $H^n_U(V; \mathbb{Z}_X)$ is embedded in $H^n_U(V; \mathbb{C}_X)$, which is expressed by the relative de Rham cohomology, while $H^n_U(V; \mathscr{O}^{(p)}_X)$ will be expressed by the relative Dolbeault cohomology.

### 3.2 Hyperforms via relative Dolbeault cohomology

For simplicity we let $M = \mathbb{R}^n \subset \mathbb{C}^n = X$. We also orient $\mathbb{R}^n$ and $\mathbb{C}^n$ so that $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n, x_1, \ldots, x_n)$ are positive coordinate systems. Thus $(y_1, \ldots, y_n)$ is a positive coordinate system in the normal direction. Then for an open set $U \subset \mathbb{R}^n$ the space of $p$-hyperforms is given by (3.4). On the other hand, by Theorem 2.8 there is a canonical isomorphism

$$\mathscr{B}^{(p)}(U) \simeq H^{p,n}_{\bar\vartheta}(V, V \smallsetminus U).$$

In the sequel we identify $\mathscr{B}^{(p)}(U)$ with $H^{p,n}_{\bar\vartheta}(V, V \smallsetminus U)$ by the above isomorphism and give explicit expressions of hyperforms and some of the fundamental operations on them.

Letting $V_0 = V \smallsetminus U$ and $V_1$ a neighborhood of $U$ in $V$, we consider the open coverings $\mathcal{V} = \{V_0, V_1\}$ and $\mathcal{V}' = \{V_0\}$ of $V$ and $V \smallsetminus U$. Then $H^{p,n}_{\bar\vartheta}(V, V \smallsetminus U) = H^{p,n}_{\bar\vartheta}(\mathcal{V}, \mathcal{V}')$ and a $p$-hyperfom is represented by a pair $(\xi_1, \xi_{01})$ with $\xi_1$ a $(p, n)$-form on $V_1$, which is automatically $\bar\partial$-closed, and $\xi_{01}$ a $(p, n-1)$-form on $V_{01}$ such that $\xi_1 = \bar\partial\xi_{01}$ on $V_{01}$. We have the exact sequence (cf. (2.5))

$$H^{p,n-1}_{\bar\partial}(V) \longrightarrow H^{p,n-1}_{\bar\vartheta}(V \smallsetminus U) \xrightarrow{\delta^*} H^{p,n}_{\bar\vartheta}(V, V \smallsetminus U) \xrightarrow{j^*} H^{p,n}_{\bar\partial}(V).$$

By a theorem of Grauert [3], we may take as $V$ a Stein open set and, if we do this, we have $H^{p,n}_{\bar\partial}(V) \simeq H^n(V, \mathscr{O}^{(p)}) = 0$. Thus $\delta^*$ is surjective and every element in $H^{p,n}_{\bar\vartheta}(V, V \smallsetminus U)$ is represented by a cocycle of the form $(0, -\theta)$ with $\theta$ a $\bar\partial$-closed $(p, n-1)$-form on $V \smallsetminus U$.

In the case $n > 1$, $H^{p,n-1}_{\bar\partial}(V) \simeq H^{n-1}(V, \mathscr{O}^{(p)}) = 0$ and $\delta^*$ is an isomorphism. In the case $n = 1$, we have the exact sequence

$$H^{p,0}_{\bar\partial}(V) \longrightarrow H^{p,0}_{\bar\vartheta}(V \smallsetminus U) \xrightarrow{\delta^*} H^{p,1}_{\bar\vartheta}(V, V \smallsetminus U) \longrightarrow 0,$$

where $H^{p,0}_{\bar\vartheta}(V \smallsetminus U) \simeq H^0(V \smallsetminus U, \mathscr{O}^{(p)})$ and $H^{p,0}_{\bar\partial}(V) \simeq H^0(V, \mathscr{O}^{(p)})$. Thus, for $p = 0$, we recover the original expression of hyperfunctions by Sato in one dimensional case.

**Remark 3.5** Although a hyperform may be represented by a single differential form in most of the cases, it is important to keep in mind that it is represented by a pair $(\xi_1, \xi_{01})$ in general.

## 4 Some fundamental operations

Let $U$ be an open set in $\mathbb{R}^n$ and $V$ a complex neighborhood of $U$ in $\mathbb{C}^n$, as in Subsection 3.2.

**Multiplication by real analytic functions:** Let $\mathscr{A}(U)$ denote the space of real analytic functions on $U$. We define the multiplication

$$\mathscr{A}(U) \times H^{p,n}_{\bar\vartheta}(V, V \smallsetminus U) \longrightarrow H^{p,n}_{\bar\vartheta}(V, V \smallsetminus U)$$

by assigning to $(f, [\xi])$ the class of $(\tilde f\xi_1, \tilde f\xi_{01})$ with $\tilde f$ a holomorphic extension of $f$. Then the following diagram is commutative:

$$\begin{array}{ccc} \mathscr{A}(U) \times H^{p,n}_{\bar\vartheta}(V, V \smallsetminus U) & \longrightarrow & H^{p,n}_{\bar\vartheta}(V, V \smallsetminus U) \\ \wr\big| & & \wr\big| \\ \mathscr{A}(U) \times H^n_U(V, \mathscr{O}^{(p)}) & \longrightarrow & H^n_U(V, \mathscr{O}^{(p)}). \end{array}$$

**Partial derivatives :** We define the partial derivative

$$\frac{\partial}{\partial x_i} : H^{0,n}_{\bar\vartheta}(V, V \smallsetminus U) \longrightarrow H^{0,n}_{\bar\vartheta}(V, V \smallsetminus U)$$

as follows. Let $(\xi_1, \xi_{01})$ represent a hyperfunction on $U$. We write $\xi_1 = f\, d\bar z_1 \wedge \cdots \wedge d\bar z_n$ and $\xi_{01} = \sum_{j=1}^n g_j\, d\bar z_1 \wedge \cdots \wedge \widehat{d\bar z_j} \wedge \cdots \wedge d\bar z_n$. Then $\frac{\partial}{\partial x_i}[\xi]$ is represented by the cocycle

$$\Big(\frac{\partial f}{\partial z_i}\, d\bar z_1 \wedge \cdots \wedge d\bar z_n, \sum_{j=1}^n \frac{\partial g_j}{\partial z_i}\, d\bar z_1 \wedge \cdots \wedge \widehat{d\bar z_j} \wedge \cdots \wedge d\bar z_n\Big).$$

With this the following diagram is commutative:

$$\begin{array}{ccc} H^{0,n}_{\bar\vartheta}(V, V \smallsetminus U) & \xrightarrow{\frac{\partial}{\partial x_i}} & H^{0,n}_{\bar\vartheta}(V, V \smallsetminus U) \\ \wr\Big| & & \wr\Big| \\ H^n_U(V, \mathcal{O}) & \xrightarrow{\frac{\partial}{\partial z_i}} & H^n_U(V, \mathcal{O}). \end{array}$$

Thus for a differential operator $P(x, D)$, $P(x, D) : H^{0,n}_{\bar\vartheta}(V, V \smallsetminus U) \to H^{0,n}_{\bar\vartheta}(V, V \smallsetminus U)$ is well-defined.

**Differential :** We define the differential (cf. [16], here we denote $\partial$ by $d$)

$$d : H^{p,n}_{\bar\vartheta}(V, V \smallsetminus U) \longrightarrow H^{p+1,n}_{\bar\vartheta}(V, V \smallsetminus U) \tag{4.1}$$

by assigning to the class of $(\xi_1, \xi_{01})$ the class of $(-1)^n(\partial\xi_1, -\partial\xi_{01})$. Then the following diagram is commutative:

$$\begin{array}{ccc} H^{p,n}_{\bar\vartheta}(V, V \smallsetminus U) & \xrightarrow{d} & H^{p+1,n}_{\bar\vartheta}(V, V \smallsetminus U) \\ \wr\Big| & & \wr\Big| \\ H^n_U(V, \mathscr{O}^{(p)}) & \xrightarrow{d} & H^n_U(V, \mathscr{O}^{(p+1)}). \end{array}$$

We will see that this leads to the de Rham complex for hyperforms (cf. Subsection 5.3).

**Integration of hyperforms :** We take orientations of $\mathbb{R}^n$ and $\mathbb{C}^n$ as in Subsection 3.2. Let $K$ be a compact set in $U$. We define the space of $p$-hyperforms on $U$ with support in $K$ by the exact sequence

$$0 \longrightarrow \mathscr{B}^{(p)}_K(U) \longrightarrow \mathscr{B}^{(p)}(U) \longrightarrow \mathscr{B}^{(p)}(U \smallsetminus K) \longrightarrow 0.$$

Then we have:

**Proposition 4.2** *For any open set $V$ in $X$ containing $K$, there is a canonical isomorphism*

$$\mathscr{B}^{(p)}_K(U) \simeq H^{p,n}_{\bar\vartheta}(V, V \smallsetminus K).$$

Let $V$ be a complex neighborhood of $U$ and consider the coverings $\mathcal{V}_K = \{V_0, V_1\}$ and $\mathcal{V}'_K = \{V_0\}$, with $V_0 = V \smallsetminus K$ and $V_1$ a neighborhood of $K$ in $V$. Then we have a canonical identification $\mathscr{B}_K^{(p)}(U) = H_{\bar\vartheta}^{p,n}(\mathcal{V}_K, \mathcal{V}'_K)$. Let $R_1$ be a real $2n$-dimensional submanifold of $V_1$ with $C^\infty$ boundary $\partial R_1$ and set $R_{01} = -\partial R_1$. We define the integration

$$\int_U : \mathscr{B}_K^{(n)}(U) \longrightarrow \mathbb{C}$$

as follows. Noting that $u \in \mathscr{B}_K^{(n)}(U) = H_{\bar\vartheta}(\mathcal{V}_K, \mathcal{V}'_K)$ is represented by

$$\xi = (\xi_1, \xi_{01}) \in \mathscr{E}^{(n,n)}(\mathcal{V}_K, \mathcal{V}'_K) = \mathscr{E}^{(n,n)}(V_1) \oplus \mathscr{E}^{(n,n-1)}(V_{01}),$$

we define

$$\int_U u = \int_{R_1} \xi_1 + \int_{R_{01}} \xi_{01}.$$

It is not difficult to see that the definition does not depend on the choice of $\xi$.

**Local duality pairing:** Let $K$, $V$ and $V_1$ be as above. We have a pairing

$$H_{\bar\vartheta}^{p,q}(V, V \smallsetminus K) \times H_{\bar\partial}^{n-p,n-q}(V_1) \overset{\smile}{\longrightarrow} H_{\bar\vartheta}^{n,n}(V, V \smallsetminus K) \overset{\int}{\longrightarrow} \mathbb{C},$$

where the first arrow denotes the cup product. On the cocycle level, it is given by $((\xi_1, \xi_{01}), \eta) \mapsto (\xi_1 \wedge \eta, \xi_{01} \wedge \eta)$. If we set

$$H_{\bar\partial}^{p,q}[K] = \varinjlim_{V_1 \supset K} H_{\bar\partial}^{p,q}(V_1),$$

the above pairing induces a morphism

$$\bar{A} : H_{\bar\vartheta}^{p,q}(V, V \smallsetminus K) \longrightarrow H_{\bar\partial}^{n-p,n-q}[K]^* = \varinjlim_{V_1 \supset K} H_{\bar\partial}^{n-p,n-q}(V_1)^*, \tag{4.3}$$

which we call the $\bar\partial$-*Alexander morphism.* In the above we considered the algebraic duals, however in order to have the duality, we need to take topological duals.

**A theorem of Martineau:** The following theorem of A. Martineau [10] (also [5],[9]) may naturally be interpreted in our framework as one of the cases where the $\bar\partial$ - Alexander morphism is an isomorphism with topological duals so that the duality pairing is given by the cup product followed by integration as described above.

**Theorem 4.4** *Let $K$ be a compact set in $\mathbb{C}^n$ such that $H^q(K, \mathscr{O}^{(p)}) = 0$ for $q \geq 1$. Then for any open set $V$ in $\mathbb{C}^n$ containing $K$, $H_{\bar\vartheta}^{p,q}(V, V \smallsetminus K)$ and $H_{\bar\partial}^{n-p.n-q}[K]$ admits natural structures of FS and DFS spaces, respectively, and we have:*

$$\bar{A} : H_{\bar\vartheta}^{p,q}(V, V \smallsetminus K) \overset{\sim}{\longrightarrow} H_{\bar\partial}^{n-p,n-q}[K]' = \begin{cases} 0 & q \neq n \\ \mathscr{O}^{(n-p)}[K]' & q = n, \end{cases}$$

*where $'$ denotes the strong dual.*

The theorem is originally stated in terms of local cohomology for $p = 0$. In our framework the duality (in the case $q = n$) is described as follows. Let $V_0 = V \smallsetminus K$ and $V_1$ a neighborhood of $K$ in $V$ and consider the coverings $\mathcal{V}_K = \{V_0, V_1\}$ and $\mathcal{V}'_K = \{V_0\}$ of $V$ and $V \smallsetminus K$. Letting $R_1$ and $R_{01}$ be as before, the duality pairing is given, for a cocycle $(\xi_1, \xi_{01})$ in $\mathscr{E}^{(p,n)}(\mathcal{V}_K, \mathcal{V}'_K)$ and a holomorphic $(n-p)$-form $\eta$ near $K$, by

$$\int_{R_1} \xi_1 \wedge \eta + \int_{R_{01}} \xi_{01} \wedge \eta. \tag{4.5}$$

Note that the hypothesis $H^q(K, \mathscr{O}^{(p)}) = 0$, for $q \geq 1$, is fulfilled if $K$ is a subset of $\mathbb{R}^n$ by the theorem of Grauert.

Suppose $K \subset \mathbb{R}^n$ and denote by $\mathscr{A}^{(p)}$ the sheaf of real analytic $p$-forms on $\mathbb{R}^n$. Then we have

$$\mathscr{O}^{(p)}[K] = \varinjlim_{V_1 \supset K} \mathscr{O}^{(p)}(V_1) \simeq \varinjlim_{U_1 \supset K} \mathscr{A}^{(p)}(U_1) = \mathscr{A}^{(p)}[K],$$

where $V_1$ runs through neighborhoods of $K$ in $\mathbb{C}^n$ and $U_1 = V_1 \cap \mathbb{R}^n$.

**Corollary 4.6** *For any open set $U \subset \mathbb{R}^n$ containing $K$, the pairing*

$$\mathscr{B}_K^{(p)}(U) \times \mathscr{A}^{(n-p)}[K] \longrightarrow H_{\bar\vartheta}^{n,n}(V, V \smallsetminus K) \xrightarrow{\int} \mathbb{C}$$

*is topologically non-degenerate so that*

$$\mathscr{B}_K^{(p)}(U) \simeq \mathscr{A}^{(n-p)}[K]'.$$

**$\delta$-function :** We consider the case $K = \{0\} \subset \mathbb{R}^n$. We set

$$\Phi(z) = dz_1 \wedge \cdots \wedge dz_n \quad \text{and} \quad \Phi_i(z) = (-1)^{i-1} z_i \, dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_n.$$

The 0-Bochner-Martinelli form on $\mathbb{C}^n \smallsetminus \{0\}$ is defined as

$$\beta_n^0 = C'_n \frac{\sum_{i=1}^n \overline{\Phi_i(z)}}{\|z\|^{2n}}, \qquad C'_n = (-1)^{\frac{n(n-1)}{2}} \frac{(n-1)!}{(2\pi\sqrt{-1})^n}$$

so that

$$\beta_n = \beta_n^0 \wedge \Phi(z)$$

is the Bochner-Martinelli form on $\mathbb{C}^n \smallsetminus \{0\}$.

**Definition 4.7** The *$\delta$-function* is the element in

$$\mathscr{B}_{\{0\}}(\mathbb{R}^n) = H_{\bar\vartheta}^{0,n}(\mathbb{C}^n, \mathbb{C}^n \smallsetminus \{0\})$$

which is represented by

$$(0, -(-1)^{\frac{n(n+1)}{2}} \beta_n^0).$$

Recall the isomorphism in Corollary 4.6 in this case:

$$\mathscr{B}_{\{0\}}(\mathbb{R}^n) \simeq (\mathscr{A}_0^{(n)})',$$

where $\mathscr{A}_0^{(n)}$ denotes the stalk of $\mathscr{A}^{(n)}$ at 0.

**Proposition 4.8** *The $\delta$-function is the hyperfunction that assigns the value $h(0)$ to a representative $\omega = h(x)\Phi(x)$ of a germ in $\mathscr{A}_0^{(n)}$.*

**$\delta$-form:** We again consider the case $K = \{0\} \subset \mathbb{R}^n$.

**Definition 4.9** The *$\delta$-form* is the element in

$$\mathscr{B}^{(n)}_{\{0\}}(\mathbb{R}^n) = H^{n,n}_{\bar{\vartheta}}(\mathbb{C}^n, \mathbb{C}^n \smallsetminus \{0\})$$

which is represented by

$$(0, -(-1)^{\frac{n(n+1)}{2}} \beta_n).$$

Recall the isomorphism in Corollary 4.6 in this case:

$$\mathscr{B}^{(n)}_{\{0\}}(\mathbb{R}^n) \simeq (\mathscr{A}_0)'.$$

**Proposition 4.10** *The $\delta$-form is the $n$-hyperfom that assigns the value $h(0)$ to a representative $h(x)$ of a germ in $\mathscr{A}_0$.*

**Remark 4.11** If we orient $\mathbb{C}^n$ so that the usual coordinate system $(x_1, y_1, \ldots, x_n, y_n)$ is positive, the delta function $\delta(x)$ is represented by $(0, -\beta^0_n)$. Also, the delta form is represented by $(0, -\beta_n)$. Incidentally, it has the same expression as the Thom class of the trivial complex vector bundle of rank $n$ (cf. [13, Ch.III, Remark 4.6]).

# 5 Embedding of real analytic functions

Let $M$ be a real analytic manifold and $X$ its complexification. The embedding of the sheaf $\mathscr{A}$ of real analytic functions into the sheaf $\mathscr{B}$ of hyperfunctions on $M$ comes from the natural identification of 1 as a hyperfunction. Namely, from the canonical identification $\mathbb{Z}_M = or_{M/X} \otimes or_{M/X}$ and the canonical morphism $or_{M/X} = \mathscr{H}^n_M(\mathbb{Z}_X) \to \mathscr{H}^n_M(\mathscr{O}_X)$, we have a canonical morphism

$$\mathbb{Z}_M = or_{M/X} \otimes or_{M/X} \longrightarrow \mathscr{B}_M = \mathscr{H}^n_M(\mathscr{O}_X) \otimes or_{M/X}.$$

In fact it is injective and the image of 1 is the corresponding hyperfunction. In the sequel we try to find it explicitly in our framework. For this we consider the complexification $or^c_{M/X} = \mathscr{H}^n_M(\mathbb{C}_X)$ of $or_{M/X}$. Then the above morphism is extended to

$$\mathbb{C}_M = or^c_{M/X} \otimes or_{M/X} \longrightarrow \mathscr{B}_M. \tag{5.1}$$

We analyze the morphism $\mathscr{H}^n_M(\mathbb{C}_X) \to \mathscr{H}^n_M(\mathscr{O}_X)$ by making use of relative de Rham and relative Dolbeault cohomologies.

## 5.1 Relative de Rham cohomology

We refer to [1] and [13] for details on Čech-de Rham cohomology. For relative de Rham cohomology and the Thom class in this context, see [13].

Let $X$ be a $C^\infty$ manifold of dimension $m$. We denote by $\mathscr{E}^{(q)}_X$ the sheaf of $C^\infty$ $q$-forms on $X$. Recall that the de Rham complex $(\mathscr{E}^{(\bullet)}, d)$ gives a fine resolution of the constant sheaf $\mathbb{C} = \mathbb{C}_X$:

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathscr{E}^{(0)} \xrightarrow{d} \mathscr{E}^{(1)} \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{E}^{(m)} \longrightarrow 0.$$

The $q$-th de Rham cohomology $H^q_d(X)$ is the $q$-th cohomology of $(\mathscr{E}^{(\bullet)}(X), d)$. The de Rham theorem says that there is an isomorphism

$$H^q_d(X) \simeq H^q(X; \mathbb{C}_X).$$

Note that among the isomorphisms, there is a canonical one (cf. [17]).

The Čech-de Rham cohomology is defined as in the case of Čech-Dolbeault cohomology, replacing the Dolbeault complex by the de Rham complex. The differential $\bar{\vartheta}$ is now denoted by $D$. Likewise we may define the relative de Rham cohomology. Thus let $S$ be a closed set in $X$. Letting $V_0 = X \smallsetminus S$ and $V_1$ a neighborhood of $S$ in $X$, we consider the coverings $\mathcal{V} = \{V_0, V_1\}$ and $\mathcal{V}' = \{V_0\}$ of $X$ and $X \smallsetminus S$, as before. We set

$$\mathscr{E}^{(q)}(\mathcal{V}, \mathcal{V}') = \mathscr{E}^{(q)}(V_1) \oplus \mathscr{E}^{(q-1)}(V_{01})$$

and define

$$D : \mathscr{E}^{(q)}(\mathcal{V}, \mathcal{V}') \longrightarrow \mathscr{E}^{(q+1)}(\mathcal{V}, \mathcal{V}') \quad \text{by} \quad D(\sigma_1, \sigma_{01}) = (d\sigma_1, \sigma_1 - d\sigma_{01}).$$

**Definition 5.2** The $q$-th *relative de Rham cohomology* $H^q_D(\mathcal{V}, \mathcal{V}')$ is the $q$-th cohomology of the complex $(\mathscr{E}^{(\bullet)}(\mathcal{V}, \mathcal{V}'), D)$.

We may again show that it does not depend on the choice of $V_1$ and we denote it by $H^q_D(X, X \smallsetminus S)$. We have the relative de Rham theorem which says that there is a canonical isomorphism (cf. [14], [17]):

$$H^q_D(X, X \smallsetminus S) \simeq H^q(X, X \smallsetminus S; \mathbb{C}_X). \tag{5.3}$$

**Remark 5.4** The sheaf cohomology $H^q(X; \mathbb{Z}_X)$ is canonically isomorphic with the singular cohomology $H^q(X; \mathbb{Z})$ of $X$ with $\mathbb{Z}$-coefficients on finite chains and $H^q(X, X \smallsetminus S; \mathbb{Z}_X)$ is isomorphic with the relative singular cohomology $H^q(X, X \smallsetminus S; \mathbb{Z})$.

**Thom class :** Let $\pi : E \to M$ be an oriented $C^\infty$ real vector bundle of rank $l$ on a $C^\infty$ manifold $M$. We identify $M$ with the image of the zero section. Then we have the Thom isomorphism

$$T : H^{q-l}(M; \mathbb{Z}) \xrightarrow{\sim} H^q(E, E \smallsetminus M; \mathbb{Z}).$$

The *Thom class* $\Psi_E \in H^l(E, E \smallsetminus M; \mathbb{Z})$ of $E$ is the image of $[1] \in H^0(M; \mathbb{Z})$ by $T$.

The Thom isomorphism with $\mathbb{C}$-coefficients is expressed in terms of de Rham and relative de Rham cohomologies:

$$T : H^{q-l}_d(M) \xrightarrow{\sim} H^q_D(E, E \smallsetminus M).$$

Its inverse in given by the integration along the fibers of $\pi$ (cf. [13, Ch.II, Theorem 5.3]). Let $W_0 = E \smallsetminus M$ and $W_1 = E$ and consider the coverings $\mathcal{W} = \{W_0, W_1\}$ and $\mathcal{W}' = \{W_0\}$ of $E$ and $E \smallsetminus M$. Then, $H^q_D(E, E \smallsetminus M) = H^q_D(\mathcal{W}, \mathcal{W}')$ and we have:

**Proposition 5.5** *For the trivial bundle $E = \mathbb{R}^l \times M$, $\Psi_E$ is represented by the cocycle*

$$(0, -\psi_l) \qquad \textit{in} \ \ \mathscr{E}^{(l)}(\mathcal{W}, \mathcal{W}').$$

In the above $\psi_l$ is the angular form on $\mathbb{R}^l$, which is given by

$$\psi_l = C_l \frac{\sum_{i=1}^l \Phi_i(x)}{\|x\|^l}, \qquad \Phi_i(x) = (-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_l. \tag{5.6}$$

The constant $C_l$ is given by $\frac{(k-1)!}{2\pi^k}$ if $l = 2k$ and by $\frac{(2k)!}{2^l \pi^k k!}$ if $l = 2k+1$. The important fact is that it is closed and $\int_{S^{l-1}} \psi_l = 1$ for a usually oriented $(l-1)$-sphere in $\mathbb{R}^l \smallsetminus \{0\}$.

Let $X$ be a $C^\infty$ manifold of dimension $m$ and $M \subset X$ a closed submanifold of dimension $n$. Set $l = m - n$. If we denote by $T_M X$ the normal bundle of $M$ in $X$, by the tubular neighborhood theorem and excision, we have a canonical isomorphism

$$H^q(X, X \smallsetminus M; \mathbb{Z}) \simeq H^q(T_M X, T_M X \smallsetminus M; \mathbb{Z}).$$

Suppose $X$ and $M$ are oriented. Then $T_M X$ is orientable as a bundle. We orient it so that its orientation followed by that of $M$ gives the orientation of $X$. In this case the Thom class $\Psi_M \in H^l(X, X \smallsetminus M; \mathbb{Z})$ of $M$ in $X$ is defined to be the class corresponding to the Thom class of $T_M X$ under the above isomorphism for $q = l$. We also have the Thom isomorphism

$$T : H^{q-l}(M; \mathbb{Z}) \xrightarrow{\sim} H^q(X, X \smallsetminus M; \mathbb{Z}). \tag{5.7}$$

From this we see that $M$ is purely $l$-codimensional in $X$ with respect to $\mathbb{Z}_X$ and that the Thom class $\Psi_M$ may be thought of as the global section of $or_{M/X} \simeq \mathbb{Z}_M$ that gives the canonical generator at each point of $M$. Also for the complexification of the relative orientation sheaf $or^c_{M/X}$ and any open set $U$ in $M$, we have by (5.3),

$$or^c_{M/X}(U) \simeq H^l_U(V; \mathbb{C}_X) \simeq H^l_D(V, V \smallsetminus U), \tag{5.8}$$

where $V$ is an open set in $X$ containing $U$ as a closed set.

## 5.2 Relative de Rham and relative Dolbeault cohomologies

Let $X$ be a complex manifold of dimension $n$. We define $\rho^q : \mathscr{E}^{(q)} \longrightarrow \mathscr{E}^{(0,q)}$ by assigning to a $q$-form $\omega$ its $(0,q)$-component $\omega^{(0,q)}$. Then $\rho^{q+1}(d\omega) = \bar\partial(\rho^q \omega)$ and we have a morphism of complexes

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d, "\iota"] & \mathscr{E}^{(0)} \arrow[r, "d"] \arrow[d, "\rho^0"] & \mathscr{E}^{(1)} \arrow[r, "d"] \arrow[d, "\rho^1"] & \cdots \arrow[r, "d"] & \mathscr{E}^{(q)} \arrow[r, "d"] \arrow[d, "\rho^q"] & \cdots \\
0 \arrow[r] & \mathscr{O} \arrow[r] & \mathscr{E}^{(0,0)} \arrow[r, "\bar\partial"] & \mathscr{E}^{(0,1)} \arrow[r, "\bar\partial"] & \cdots \arrow[r, "\bar\partial"] & \mathscr{E}^{(0,q)} \arrow[r, "\bar\partial"] & \cdots .
\end{tikzcd}$$

Thus, for any open set $X'$ in $X$, there is a morphism $\rho^q : H^q_D(X, X') \to H^{0,q}_{\bar\vartheta}(X, X')$, which makes the following diagram commutative:

$$\begin{tikzcd}
H^q_D(X, X') \arrow[r, "\rho^q"] \arrow[d, "\wr"] & H^{0,q}_{\bar\vartheta}(X, X') \arrow[d, "\wr"] \\
H^q(X, X'; \mathbb{C}) \arrow[r, "\iota"] & H^q(X, X'; \mathscr{O}).
\end{tikzcd} \tag{5.9}$$

## 5.3 1 as a hyperfunction

Let $M$ and $X$ be as in the beginning of this section. We now try to find the image of 1 by the morphism (5.1). For simplicity we assume that $M$ is orientable. Let $U$ be a coordinate neighborhood in $M$ and $V$ a complex neighborhood of $U$ in $X$. We orient $X$ and $M$ so that the orientations give the ones for $V$ and $U$ as in Subsection 3.2. Then we have canonical isomorphisms

$$\mathbb{C}_M(U) \simeq \mathscr{H}^n_M(\mathbb{C}_X)(U) \simeq H^n_U(V;\mathbb{C}) \quad \text{and} \quad \mathscr{B}_M(U) = \mathscr{H}^n_M(\mathscr{O}_X)(U) \simeq H^n_U(V;\mathscr{O}).$$

Note that the first isomorphism above is the Thom isomorphism (5.7) with $\mathbb{C}$-coefficients for the pair $(V, U)$. Suppose $U$ is connected and let $\mathcal{V}$ and $\mathcal{V}'$ be coverings as in Subsection 3.2. Then we have the commutative diagram (cf. (5.8), (5.9)):

$$\begin{array}{ccccc} \mathbb{C} = H^0(U;\mathbb{C}) & \xrightarrow[T]{\sim} & H^n_U(V;\mathbb{C}) & \xrightarrow{\iota} & H^n_U(V;\mathscr{O}) \\ & & \wr\| & & \wr\| \\ & & H^n_D(\mathcal{V},\mathcal{V}') & \xrightarrow{\rho^n} & H^{0,n}_{\bar\vartheta}(\mathcal{V},\mathcal{V}'). \end{array}$$

The image of 1 by $T$ is the Thom class $\Psi_U$ which is represented by $\tau = (0, -\psi_n(y))$ in $\mathscr{E}^{(n)}(\mathcal{V},\mathcal{V}')$ with $\psi_n(y)$ the angular form on $\mathbb{R}^n_y$ (cf. (5.6)). Since $\rho^n(\tau) = (0, -\psi_n^{(0,n-1)})$, we have:

**Theorem 5.10** *As a hyperfunction, 1 is locally represented by the cocycle $(0, -\psi_n^{(0,n-1)})$ in $\mathscr{E}^{(0,n)}(\mathcal{V},\mathcal{V}')$, where $\psi_n^{(0,n-1)}$ is the $(0, n-1)$-component of $\psi_n(y)$.*

Using $y_i = 1/(2\sqrt{-1})(z_i - \bar z_i)$, we see that

$$\psi_n^{(0,n-1)} = (\sqrt{-1})^n C_n \frac{\sum_{i=1}^n (-1)^i (z_i - \bar z_i) d\bar z_1 \wedge \cdots \wedge \widehat{d\bar z_i} \wedge \cdots \wedge d\bar z_n}{\|z - \bar z\|^n}.$$

In particular, if $n = 1$,

$$\psi_1^{(0,0)} = \frac{1}{2}\frac{y}{|y|}.$$

**Embedding of real analytic forms into hyperforms:** Let $U$ and $V$ be as above. We define

$$\mathscr{A}^{(p)}(U) \longrightarrow \mathscr{B}^{(p)}(U) = H^{p,n}_{\bar\vartheta}(V, V \smallsetminus U)$$

by assigning to $\omega(x)$ in $\mathscr{A}^{(p)}(U)$ the class $[(0, -\psi_n^{(0,n-1)} \wedge \omega(z))]$, where $\psi_n^{(0,n-1)}$ is as above and $\omega(z)$ denotes the complexification of $\omega(x)$. Then it induces a sheaf monomorphism $\iota^{(p)} : \mathscr{A}^{(p)} \to \mathscr{B}^{(p)}$, which is compatible with the differentials $d$ of $\mathscr{A}^{(\bullet)}$ and $\mathscr{B}^{(\bullet)}$.

**de Rham complex for hyperforms:** Let $X$ be a complex manifold. Then we have the analytic de Rham complex

$$0 \longrightarrow \mathbb{C} \xrightarrow{\iota} \mathscr{O} \xrightarrow{d} \mathscr{O}^{(1)} \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{O}^{(n)} \longrightarrow 0$$

and the diagram (5.9) is extended to an isomorphism of complexes

$$\begin{array}{ccccccccccccc}
0 \longrightarrow & H^q_D(X,X') & \xrightarrow{\rho^q} & H^{0,q}_{\bar\vartheta}(X,X') & \xrightarrow{d} & H^{1,q}_{\bar\vartheta}(X,X') & \xrightarrow{d} \cdots \xrightarrow{d} & H^{n,q}_{\bar\vartheta}(X,X') & \longrightarrow 0 \\
& \wr\| & & \wr\| & & \wr\| & & \wr\| & \\
0 \longrightarrow & H^q(X,X';\mathbb{C}) & \xrightarrow{\iota} & H^q(X,X';\mathscr{O}) & \xrightarrow{d} & H^q(X,X';\mathscr{O}^{(1)}) & \xrightarrow{d} \cdots \xrightarrow{d} & H^q(X,X';\mathscr{O}^{(n)}) & \longrightarrow 0.
\end{array}$$

Let $M$ and $X$ be as above. Then from the fact that $M$ is purely $n$-codimensional in $X$ with respect to to $\mathbb{C}_X$ and $\mathscr{O}^{(p)}$, we see that the above complex for $X' = X \smallsetminus M$ leads to the following exact sequence of sheaves on $M$:

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathscr{B} \xrightarrow{d} \mathscr{B}^{(1)} \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{B}^{(n)} \longrightarrow 0.$$

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