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Extension of the exact steepest descent method to the middle convolution

By

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§ 1. Introduction

In this note I report my recent research [8] jointly done with Moteki on the extension of the exact steepest descent method to the middle convolution.

The exact steepest descent method was originally invented by Aoki, Kawai and the author in [1] to discuss Stokes phenomena for WKB solutions of an ordinary differential equation $P\psi = 0$ with polynomial coefficients. Its key idea is to consider the Laplace transform $\hat{P}\hat{\psi} = 0$ of $P\psi = 0$ with respect to the independent variable and to regard the inverse Laplace transform of a WKB solution of $\hat{P}\hat{\psi} = 0$ as a kind of integral representation of (the Borel sum of) a WKB solution of the original equation $P\psi = 0$. In [8] Moteki and I extend this method to the middle convolution, that is, we show that, if $P\psi = 0$ is obtained from $\tilde{P}\tilde{\psi} = 0$ via middle convolution (or, equivalently, Euler transform; see §3 for more details about the middle convolution and the Euler transform), the Euler transform of a WKB solution of $\tilde{P}\tilde{\psi} = 0$ provides us with the Borel sum of a WKB solution of the original equation $P\psi = 0$.

In this note, after reviewing the original exact steepest descent method in §2, I explain the core part of the joint paper [8] with Moteki on its extension to the middle convolution in §3. At the end of the note I also state a related result for the original exact steepest descent method which can be obtained by applying the reasoning developed in [8] to the original problem discussed in [1]. The final result will be used in my joint work [3] with Hirose, Kawai and Sasaki to discuss the structure of the Stokes geometry of a certain higher order ordinary differential equation that appears as the restriction of a holonomic system of confluent hypergeometric type with two independent variables.

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§ 2. Review of the exact steepest descent method

The exact steepest descent method deals with an ordinary differential equation with polynomial coefficients of the form

$$P\psi = \sum_{j=0}^{m} a_j(x)(\eta^{-1}\partial_x)^{m-j}\psi = 0 \tag{1}$$

and its WKB solution

$$\psi_j(x,\eta) = \exp\left(\eta \int^x \xi_j(x)\,dx\right) \sum_{n=0}^{\infty} \psi_{j,n}(x)\eta^{-n-1/2}. \tag{2}$$

Here $m \geq 2$ is a positive integer, $a_j(x)$ is a polynomial ($j = 0, \ldots, m$; $a_0(x) \equiv 1$), $\partial_x = \partial/\partial x$, and $\eta > 0$ denotes a large parameter. The function $\xi_j(x)$ ($j = 1, \ldots, m$) that describes the phase factor (i.e., the exponential factor) of a WKB solution is given by a root of the characteristic equation

$$p(x,\xi) = \sum_{j=0}^{m} a_j(x)\xi^{m-j} = 0. \tag{3}$$

Once a root $\xi_j(x)$ of (3) is fixed, higher order terms $\psi_{j,n}(x)$ ($n = 0, 1, \ldots$) of $\psi_j(x,\eta)$ are recursively determined. For the basic properties of WKB solutions we refer the reader to [7].

As WKB solutions do not converge, in the exact WKB analysis we employ the Borel resummation method to give an analytic meaning to them and, as its consequence, we observe that Stokes phenomena occur with WKB solutions on the so-called Stokes curves. In particular, to specify on which portions of (ordinary and new) Stokes curves a Stokes phenomenon does really occur is one of the most fundamental problems in the exact WKB analysis for higher order equations, i.e., in the case of $m \geq 3$ (cf. [4]). To attack this problem we make use of the (inverse) Laplace transform

$$\psi(x,\eta) = \int \exp(\eta x\xi)\hat{\psi}(\xi,\eta)\,d\xi \tag{4}$$

with respect to the independent variable in the exact steepest descent method.

To give a quick review of the method, let us first consider the case of Laplace type equations, i.e., the case where all the coefficients $a_j(x)$ is of degree at most 1. In this case the Laplace transform of (1) becomes a first order differential equation and hence can be solved explicitly. Using this explicit solution of the Laplace transform, we consider

$$\psi_j(x,\eta) = \int_{\Gamma_j} \exp(\eta x\xi)\hat{\psi}(\xi,\eta)\,d\xi = \int_{\Gamma_j} \exp\left(\eta\left(x\xi - \int^{\xi} x(\xi)d\xi\right)\right) R(\xi,\eta)\,d\xi, \tag{5}$$

where $x(\xi)$ is a root of the characteristic equation (3) with respect to x (that is, $p(x(\xi),\xi)=0$) and $\hat{\psi}(\xi,\eta)=\exp\Big(-\eta\displaystyle\int^{\xi}x(\xi)d\xi\Big)R(\xi,\eta)$ denotes the explicit solution of the Laplace transform. Note that the phase function $f(\xi)=x\xi-\displaystyle\int^{\xi}x(\xi)d\xi$ of the integral (5) has a saddle point at a root $\xi_j(x)$ of (3) and we take a steepest descent path Γ_j of $\Re f$ passing through a saddle point $\xi_j(x)$ as a path of integration for the inverse Laplace transform (5). Then, under this situation, we can verify that the right-hand side of (5) coincides with the Borel sum of a WKB solution (2) of the original equation (1) provided that the steepest descent path Γ_j does not hit another saddle point. In other words, a Stokes phenomenon occurs with $\psi_j(x,\eta)$ only when the steepest descent path Γ_j hits some saddle point. For the proof see [10].

The exact steepest descent method generalizes this analysis to an ordinary differential equation (1) with polynomial coefficients. When the degree of some coefficient $a_j(x)$ is greater than 1, the Laplace transform of (1) cannot be solved explicitly in general. In this situation, using a WKB solution

$$\hat{\psi}_k(\xi,\eta)=\exp\left(-\eta\int^{\xi}x_k(\xi)\,d\xi\right)\sum_{n=0}^{\infty}\hat{\psi}_{k,n}(\xi)\eta^{-n-1/2} \tag{6}$$

of the Laplace transformed equation $\hat{P}\hat{\psi}=0$ of (1), where $x_k(\xi)$ denotes a root of the characteristic equation (3) with respect to x, we consider

(7)
$$\psi=\int_{\Gamma_j}\exp(\eta x\xi)\hat{\psi}_k(\xi,\eta)\,d\xi=\int_{\Gamma_j}\exp\left(\eta\Big(x\xi-\int^{\xi}x_k(\xi)d\xi\Big)\right)\sum_{n=0}^{\infty}\hat{\psi}_{k,n}(\xi)\eta^{-n-1/2}\,d\xi$$

as a substitute for (5). Note that, similarly to the case of Laplace type equations, the phase function $f_k(\xi)=x\xi-\displaystyle\int^{\xi}x_k(\xi)d\xi$ of (7) has a saddle point at a root $\xi_j(x)$ of (3) also in the current situation and we again take a steepest descent path Γ_j of $\Re f_k$ passing through a saddle point $\xi_j(x)$ as a path of integration for (7). Of course, as the integrand $\hat{\psi}_k(\xi,\eta)$ of (7) does not converge, some modification becomes necessary. The most important observation done in [1] is that, reflecting the divergent character of $\hat{\psi}_k(\xi,\eta)$, we have to take bifurcated steepest descent paths also into account when the steepest descent path Γ_j crosses a Stokes curve of $\hat{P}$. More precisely, when Γ_j crosses a Stokes curve of $\hat{P}$ of type $k>k'$, we bifurcate a steepest descent path $\tilde{\Gamma}$ of $\Re f_{k'}$ passing through the crossing point (cf. Figure 1), and we repeat this bifurcation process of steepest descent paths until no further new crossing point of a steepest descent path and a Stokes curve appears. The totality of such steepest descent paths $\Gamma_j^{\text{exact}}:=\Gamma_j\cup\tilde{\Gamma}\cup\cdots$ is called "*an exact steepest descent path*" ("ESDP" for short) passing through a saddle point $\xi_j(x)$.

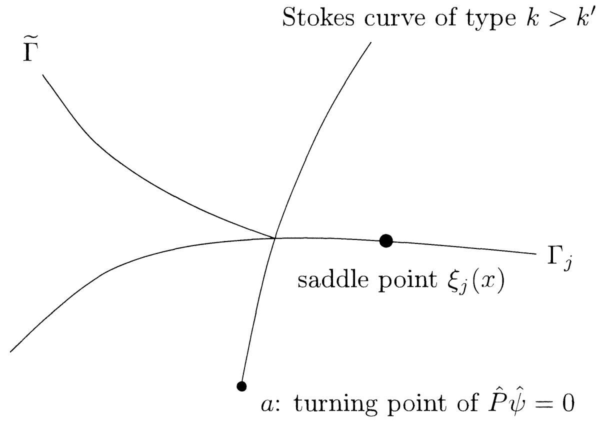

Figure 1. Bifurcation of a steepest descent path.

In [1] the integral of the form (7) along such an exact steepest descent path is examined in details. The main achievement of [1] can be summarized as the following "ESDP Ansatz":

ESDP Ansatz. *A Stokes phenomenon occurs with the Borel sum of a WKB solution* (2) *of* $P\psi = 0$ *if and only if an exact steepest descent path* Γ_j^{exact} *passing through a saddle point* $\xi_j(x)$ *hits another saddle point.*

§ 3. Extension of the exact steepest descent method to the middle convolution

In [8] Moteki and I extends the exact steepest descent method to the middle convolution. In this section I explain the main result of [8].

Let us start with recalling the definition of the middle convolution. The middle convolution was first introduced by Katz [6] in his study of local rigid systems and later reformulated as an operation on Fuchsian systems by Dettweiler-Reiter [2]. More recently Oshima has developed a systematic study of ordinary differential equations with polynomial coefficients and produced many remarkable results by using the middle convolution. (See, e.g., [9].) The following definition of the middle convolution, a modified version given by Iwaki-Koike [5] of the one formulated in [9], is more convenient in discussing Eq. (1) containing a large parameter η.

Definition 1. For a differential operator $\tilde{P}$ of the form (1) we define its middle convolution with a large parameter η by

$$P := mc_{\mu\eta}\tilde{P} = (\eta^{-1}\partial_x)^l \circ \mathrm{Ad}(\partial_x^{-\mu\eta})\tilde{P} = (\eta^{-1}\partial_x)^l \circ \partial_x^{-\mu\eta} \circ \tilde{P} \circ \partial_x^{\mu\eta}, \tag{8}$$

where μ is a non-zero complex constant and $l = \max\{\deg a_j + j - m\,; 0 \le j \le m\}$.

Note that, since $\mathrm{Ad}(\partial_x^{-\mu\eta})x^k = (x - \mu\eta\partial_x^{-1})^k$, $\mathrm{Ad}(\partial_x^{-\mu\eta})\tilde{P}$ contains negative powers of ∂_x. Thanks to the factor $(\eta^{-1}\partial_x)^l$, $P = mc_{\mu\eta}\tilde{P}$ becomes an ordinary differential operator of order $m+l$. Furthermore, if $\tilde{\psi}(x,\eta)$ is a solution of $\tilde{P}\tilde{\psi} = 0$, then a solution $\psi(x,\eta)$ of $P\psi = 0$ is provided by the Euler transform of $\tilde{\psi}$:

$$\psi(x,\eta) = \frac{1}{\Gamma(\mu\eta)} \int^x (x-z)^{\mu\eta-1}\tilde{\psi}(z,\eta)\, dz. \tag{9}$$

Now, the exact steepest descent method can be extended to the middle convolution (or the Euler transform) in the following manner. Suppose that an ordinary differential operator P of the form (1) is obtained from another operator $\tilde{P}$ of order m via middle convolution: $P = mc_{\mu\eta}\tilde{P}$. As in the case of the Laplace transform explained in §2, we take a WKB solution

$$\tilde{\psi}_k(z,\eta) = \exp\left(\eta \int^z \tilde{\xi}_k(z)\, dz\right) \sum_{n=0}^{\infty} \tilde{\psi}_{k,n}(z)\eta^{-n-1/2} \tag{10}$$

of $\tilde{P}\tilde{\psi} = 0$, where $\tilde{\xi}_k(z)$ is a root of the characteristic equation of $\tilde{P}\tilde{\psi} = 0$, and consider its Euler transform

$$\begin{aligned} \psi &= \int_{\Gamma_j} (x-z)^{\mu\eta-1}\tilde{\psi}_k(z,\eta)\, dz \\ &= \int_{\Gamma_j} \exp\left(\eta\Big(\mu\log(x-z) + \int^z \tilde{\xi}_k(z)dz\Big)\right) \sum_{n=0}^{\infty} \tilde{\psi}_{k,n}(z)\eta^{-n-1/2}\, \frac{dz}{x-z}. \end{aligned} \tag{11}$$

(We omit the Γ-factor for the sake of simplicity.) Let $g_k(z) = \mu\log(x-z) + \int^z \tilde{\xi}_k(z)dz$ be the phase function of the integral (11) and $z_j(x)$ denote a saddle point of it. In this situation we find that $\dfrac{\mu}{x - z_j(x)}$ satisfies the characteristic equation of $P\psi = 0$ and hence there exist $m+l$ saddle points of (11). In parallel to the case of the original exact steepest descent method, we take a steepest descent path Γ_j of $\Re g_k$ passing through a saddle point $z_j(x)$ and, further, consider the exact steepest descent path

$$\Gamma_j^{\mathrm{exact}} := \Gamma_j \cup \tilde{\Gamma} \cup \cdots \tag{12}$$

generated by Γ_j, that is, whenever Γ_j crosses a Stokes curve of $\tilde{P}$ of type $k > k'$, we bifurcate a steepest descent path $\tilde{\Gamma}$ of $\Re g_{k'}$ passing through the crossing point and consider the totality of such bifurcated steepest descent paths.

Then the main result of our joint paper [8] can be stated as follows:

Theorem 2. *Suppose that an ordinary differential operator* P *of* (1) *is obtained from another operator* $\tilde{P}$ *via middle convolution, i.e.,* $P = mc_{\mu\eta}\tilde{P}$. *For the operator* $\tilde{P}$ *we further assume that*

- $\tilde{P}$ *is of second order,*
- *all turning points of* $\tilde{P}$ *are simple,*
- *no Stokes curve of* $\tilde{P}$ *connects two turning points.*

Then, if an exact steepest descent path Γ_j^{exact} *passing through a saddle point* $z_j(x)$ *does not hit any other saddle point,*

$$\int_{\Gamma_j^{\text{exact}}} (x-z)^{\mu\eta-1}\tilde{\psi}_k(z,\eta)\,dz \tag{13}$$

defines a WKB solution $\psi_j(x,\eta)$ *of* $P\psi = 0$ *with exponential term* $\exp\Big(\eta \int^x \frac{\mu}{x-z_j(x)}\,dx\Big)$ *and it is Borel summable.*

Corollary 3. *Under the assumptions of Theorem 2, if an exact steepest descent path* Γ_j^{exact} *passing through a saddle point* $z_j(x)$ *does not hit any other saddle point, then no Stokes phenomenon occurs with the Borel sum of a WKB solution* $\psi_j(x,\eta)$ *of* $P\psi = 0$ *defined by* (13).

Remark. In the geometric situation described in Theorem 2 the Borel sum of (13) is given by

$$\int_{\Gamma_j} (x-z)^{\mu\eta-1}\tilde{\Psi}_k(z,\eta)\,dz + C\int_{\tilde{\Gamma}} (x-z)^{\mu\eta-1}\tilde{\Psi}_{k'}(z,\eta)\,dz + \cdots, \tag{14}$$

where $\tilde{\Psi}_k(z,\eta)$ is the Borel sum of a WKB solution $\tilde{\psi}_k(z,\eta)$ of $\tilde{P}\tilde{\psi} = 0$ and C denotes the Stokes coefficient that describes the Stokes phenomenon occurring with $\tilde{\Psi}_k(z,\eta)$ at a crossing point of Γ_j and a Stokes curve of $\tilde{P}\tilde{\psi} = 0$ of type $k > k'$. (Similarly, in the precise definition of (13) such Stokes coefficients should be also introduced in a suitable way.)

For the proof of Theorem 2 we refer the reader to [8]. Applying the reasoning employed in [8] to the original problem for the exact steepest descent method, we can also prove the following

Theorem 4. *Let* $\hat{P}\hat{\psi} = 0$ *be the Laplace transform of Eq.* (1). *We assume that*

- $\hat{P}$ *is of second order,*
- *all turning points of* $\hat{P}$ *are simple,*

- *no Stokes curve of* $\hat{P}$ *connects two turning points.*

Then, if an exact steepest descent path Γ_j^{exact} *passing through a saddle point* $\xi_j(x)$ *does not hit any other saddle point, a WKB solution*

$$\psi_j(x,\eta) = \int_{\Gamma_j^{\text{exact}}} \exp(\eta x\xi)\hat{\psi}_k(\xi,\eta)\,d\xi \tag{15}$$

of the original equation $P\psi = 0$ *is Borel summable.*

Theorem 4 guarantees that, in case $\hat{P}$ is of second order, the 'only if' part of the ESDP Ansatz is true under some suitable additional conditions. As is mentioned in Introduction, Theorem 4 will be used in my joint work with Hirose, Kawai and Sasaki on the analysis of a tangential system of some holonomic system with two independent variables (cf. [3]).

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