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Laplace hyperfunctions from the viewpoint of Čech-Dolbeault cohomology

By

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Abstract

In this note, we introduce Laplace hyperfunctions from the view point of Čech-Dolbeault cohomology. Furthermore, we construct a Laplace transformation for a Čech-Dolbeault representation of a Laplace hyperfunction.

§ 1. Introduction

Recently, T. Suwa [12] and N. Honda [1] study the theory of Sato's hyperfunctions from the viewpoint of the Čech-Dolbeault cohomology. In their studies, a hyperfunction on $\mathbb{R}^n$ can be represented by a pair (τ_1, τ_{01}), where τ_1 is a $(0, n)$-form of C^∞-coefficients on $\mathbb{C}^n$ and τ_{01} is $(0, n-1)$-form of a C^∞-coefficients on $\mathbb{C}^n \setminus \mathbb{R}^n$. The one of advantages for such a presentation is that we can employ, in the theory of hyperfunctions, the similar techniques as those in the C^∞ category such as the partition of unity.

H. Komatsu ([5]-[11]) introduced the theory of Laplace hyperfunctions of one variable in order to consider the Laplace transform of a hyperfunction. The theory of Laplace hyperfunctions in several variables has been established by the author and N. Honda ([3],[4]). As we did in the hyperfunction theory, it is quite natural to study a Laplace hyperfunction from the viewpoint of the Čech-Dolbeault cohomology. In this note, we first describe a Laplace hyperfunction as a pair of C^∞ forms of exponential growth order at ∞ by using Čech-Dolbeault cohomology. Then, we define a Laplace transformation of a Čech-Dolbeault representative of a Laplace hyperfunction and its inverse Laplace

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transformation in our settings. For details, we refer the reader to the forthcoming paper [2].

At the end of this section, the author would like to express my gratitude to Professor Tatsuo Suwa and Professor Naofumi Honda for the valuable advises.

§ 2. Laplace hyperfunctions of a Čech-Dolbeault representation

Let $n \in \mathbb{N}$ and let M be an n-dimensional $\mathbb{R}$-vector space, and let E be a complexification $M \otimes_{\mathbb{R}} \mathbb{C}$ of M. We denote by $E^{\times}$ (resp. $M^{\times}$) the set $E \setminus 0$ (resp. $M \setminus 0$) and by $\mathbb{R}_+$ the set of positive real numbers. Let $\mathbb{D}_E$ (resp. $\mathbb{D}_M$) be the radial compacitfication $E \sqcup E^{\times}/\mathbb{R}_+$ (resp. $M \sqcup M^{\times}/\mathbb{R}_+$) of E (resp. M). We set $M_\infty := \mathbb{D}_M \setminus M$ and $E_\infty = \mathbb{D}_E \setminus E$. For a subset $T \subset \mathbb{D}_E$, the subset $N_\infty(T)$ in E_∞ is defined by

$$N_\infty(T) := E_\infty \setminus \overline{(E \setminus T)},$$

where the closure is taken in $\mathbb{D}_E$. Let U be an open subset in E. We also the open subset $\widehat{U}$ in $\mathbb{D}_E$ by

$$\widehat{U} := U \cup N_\infty(U).$$

We sometimes write $\widehat{\ }U$ instead of $\widehat{U}$.

Let V be an open subset in $\mathbb{D}_E$ and f a measurable function on $V \cap E$. We say that f is of exponential type (at ∞) on V if, for any compact subset K in V, there exists $H_K > 0$ such that $|\exp(-H_K|z|)\, f(z)|$ is essentially bounded on $K \cap E$. Let $\mathscr{Q}_{\mathbb{D}_E}(V)$ designate the set of C^∞ functions on $V \cap E$ whose higher derivatives are of exponential type. We denote by $\mathscr{Q}_{\mathbb{D}_E}$ the associated sheaf on $\mathbb{D}_E$ of the presheaf $\{\mathscr{Q}_{\mathbb{D}_E}(V)\}_V$ and by $\mathscr{Q}^{p,q}_{\mathbb{D}_E}$ the sheaf on $\mathbb{D}_E$ of $(p,\, q)$-forms with coefficients in $\mathscr{Q}_{\mathbb{D}_E}$. Set

$$\mathscr{Q}^k_{\mathbb{D}_E} = \bigoplus_{p+q=k} \mathscr{Q}^{p,q}_{\mathbb{D}_E}.$$

Now we define the de-Rham complex $\mathscr{Q}^{\bullet}_{\mathbb{D}_E}$ on $\mathbb{D}_E$ with coefficients in $\mathscr{Q}_{\mathbb{D}_E}$ by

$$0 \longrightarrow \mathscr{Q}^0_{\mathbb{D}_E} \xrightarrow{d} \mathscr{Q}^1_{\mathbb{D}_E} \xrightarrow{d} \dots \xrightarrow{d} \mathscr{Q}^{2n}_{\mathbb{D}_E} \longrightarrow 0,$$

and the Dolbeault complex $\mathscr{Q}^{p,\bullet}_{\mathbb{D}_E}$ on $\mathbb{D}_E$ by

$$0 \longrightarrow \mathscr{Q}^{p,0}_{\mathbb{D}_E} \xrightarrow{\bar\partial} \mathscr{Q}^{p,1}_{\mathbb{D}_E} \xrightarrow{\bar\partial} \dots \xrightarrow{\bar\partial} \mathscr{Q}^{p,n}_{\mathbb{D}_E} \longrightarrow 0.$$

Let $\mathscr{O}^{\exp}_{\mathbb{D}_E}$ denote the sheaf of holomorphic functions of exponential type (at ∞) on $\mathbb{D}_E$.

We have the following proposition and theorem.

Proposition 2.1. *Both the canonical morphisms of complexes below are quasi-isomorphic:*

$$\mathbb{C}_{\mathbb{D}_E} \longrightarrow \mathscr{Q}^{\bullet}_{\mathbb{D}_E}, \qquad \mathscr{O}^{\exp,(p)}_{\mathbb{D}_E} \longrightarrow \mathscr{Q}^{p,\bullet}_{\mathbb{D}_E}.$$

Theorem 2.2. *Assume that $V \cap E$ is Stein and that V is regular at ∞. Then we have the quasi-isomorphism*

$$\mathscr{O}^{\exp,(p)}_{\mathbb{D}_E}(V) \longrightarrow \mathscr{Q}^{p,\bullet}_{\mathbb{D}_E}(V).$$

We have the edge of the wedge theorem for holomorphic functions of exponential type.

Theorem 2.3 ([3], Theorem 3.12). *The complexes $\mathbf{R}\Gamma_{\mathbb{D}_M}(\mathscr{O}^{\exp,(p)}_{\mathbb{D}_E})$ and $\mathbf{R}\Gamma_{\mathbb{D}_M}(\mathbb{Z}_{\mathbb{D}_E})$ are concentrated in degree n. Furthermore, $\mathrm{H}^n_{\mathbb{D}_M}(\mathbb{Z}_{\mathbb{D}_E})$ is isomorphic to $\mathbb{Z}_{\mathbb{D}_M}$.*

Define

$$\mathscr{B}^{\exp,(p)}_{\mathbb{D}_M} := \mathrm{H}^n_{\mathbb{D}_M}(\mathscr{O}^{\exp,(p)}_{\mathbb{D}_E}) \otimes_{\mathbb{Z}_{\mathbb{D}_M}} or_{\mathbb{D}_E/\mathbb{D}_M},$$

$$or_{\mathbb{D}_E/\mathbb{D}_M} := \mathrm{H}^n_{\mathbb{D}_M}(\mathbb{Z}_{\mathbb{D}_E}).$$

By Theorem 2.3, we have

$$\mathscr{B}^{\exp,(p)}_{\mathbb{D}_M}(\Omega) = \mathrm{H}^n_{\mathbb{D}_M}(V;\, \mathscr{O}^{\exp,(p)}_{\mathbb{D}_E}) \ \otimes_{\mathbb{Z}_{\mathbb{D}_M}(\Omega)}\ \mathrm{H}^n_{\mathbb{D}_M}(V;\, \mathbb{Z}_{\mathbb{D}_E})$$

for any open subset Ω in $\mathbb{D}_M$. Here V is an open subset in $\mathbb{D}_E$ with $V \cap \mathbb{D}_M = \Omega$.

We can construct the boundary value map in a functorial way.

Theorem 2.4. *Let U be an open subset in $\mathbb{D}_E$ which satisfies $\mathbb{D}_M \subset \overline{U}$. Assume that U is cohomologically trivial in $\mathbb{D}_E$. Then we have the boundary value map*

$$b_U : \mathscr{O}^{\exp}_{\mathbb{D}_E}(U) \longrightarrow \mathscr{B}^{\exp}_{\mathbb{D}_M}(\mathbb{D}_M).$$

Set $V_0 = \mathbb{D}_E \setminus \mathbb{D}_M$, $V_1 = \mathbb{D}_E$ and $V_{01} = V_0 \cap V_1$. Then define the coverings

$$\mathcal{V} = \{V_0,\, V_1\}, \qquad \mathcal{V}' = \{V_{01}\}.$$

Let $\mathscr{Q}^{p,\bullet}_{\mathbb{D}_E}(\mathcal{V},\, \mathcal{V}')$ denote the Čech-Dolbeault complex to the pair $(\mathcal{V},\, \mathcal{V}')$ of coverings with coefficients in $\mathscr{Q}_{\mathbb{D}_E}$, i.e.,

$$0 \longrightarrow \mathscr{Q}^{p,0}_{\mathbb{D}_E}(\mathcal{V},\, \mathcal{V}') \xrightarrow{\ \overline{\vartheta}\ } \mathscr{Q}^{p,1}_{\mathbb{D}_E}(\mathcal{V},\, \mathcal{V}') \xrightarrow{\ \overline{\vartheta}\ } \dots \xrightarrow{\ \overline{\vartheta}\ } \mathscr{Q}^{p,n}_{\mathbb{D}_E}(\mathcal{V},\, \mathcal{V}') \longrightarrow 0.$$

Here

$$\mathscr{Q}^{p,k}_{\mathbb{D}_E}(\mathcal{V},\, \mathcal{V}') = \mathscr{Q}^{p,\,k}_{\mathbb{D}_E}(V_1) \oplus \mathscr{Q}^{p,\,k-1}_{\mathbb{D}_E}(V_{01}),$$

$$\overline{\vartheta}(\xi_1, \xi_{01}) = (\overline{\partial}\xi_1, \xi_1|_{V_{01}} - \overline{\partial}\xi_{01}).$$

Let $\mathscr{Q}^{\bullet}_{\mathbb{D}_E}(\mathcal{V}, \mathcal{V}')$ denote the Čech - de-Rham complex to the pair $(\mathcal{V}, \mathcal{V}')$ of coverings with coefficients in $\mathscr{Q}_{\mathbb{D}_E}$.

$$0 \longrightarrow \mathscr{Q}^0_{\mathbb{D}_E}(\mathcal{V}, \mathcal{V}') \xrightarrow{D} \mathscr{Q}^1_{\mathbb{D}_E}(\mathcal{V}, \mathcal{V}') \xrightarrow{D} \dots \xrightarrow{D} \mathscr{Q}^{2n}_{\mathbb{D}_E}(\mathcal{V}, \mathcal{V}') \longrightarrow 0,$$

where D is used to denote the differential of this complex.

Theorem 2.5. *We have the canonical quasi-isomorphisms:*

$$\mathbf{R}\Gamma_{\mathbb{D}_M}(\mathbb{D}_E; \mathscr{O}^{\exp,(p)}_{\mathbb{D}_E}) \simeq \mathscr{Q}^{p,\bullet}_{\mathbb{D}_E}(\mathcal{V}, \mathcal{V}'), \qquad \mathbf{R}\Gamma_{\mathbb{D}_M}(\mathbb{D}_E; \mathbb{C}_{\mathbb{D}_E}) \simeq \mathscr{Q}^{\bullet}_{\mathbb{D}_E}(\mathcal{V}, \mathcal{V}').$$

In what follows, we constantly use the notations below:

$$\mathrm{H}^{p,k}_{\overline{\vartheta},\mathscr{Q}}(\mathcal{V}, \mathcal{V}') := \mathrm{H}^k(\mathscr{Q}^{p,\bullet}_{\mathbb{D}_E}(\mathcal{V}, \mathcal{V}')), \qquad \mathrm{H}^k_{D,\mathscr{Q}}(\mathcal{V}, \mathcal{V}') := \mathrm{H}^k(\mathscr{Q}^{\bullet}_{\mathbb{D}_E}(\mathcal{V}, \mathcal{V}')).$$

Hence we have

$$\mathscr{B}^{\exp,(p)}_{\mathbb{D}_M}(\mathbb{D}_M) \simeq \mathrm{H}^{p,n}_{\overline{\vartheta},\mathscr{Q}}(\mathcal{V}, \mathcal{V}') \otimes_{\mathbb{Z}_{\mathbb{D}_M}(\mathbb{D}_M)} or_{\mathbb{D}_M/\mathbb{D}_E}(\mathbb{D}_M).$$

§ 3. Laplace transformation

Let $(z_1 = x_1 + \sqrt{-1}y, \cdots, z_n = x_n + \sqrt{-1}y_n)$ be the coordinates system of E. We fix the orientation of M and E so that $\{dx_1, dx_2, \dots, dx_n\}$ gives the positive orientation on M, and $\{dy_1, \dots, dy_n, dx_1, \dots, dx_n\}$ give the one on E. Let M^* and E^* be dual vector spaces of M and E respectively. Then we also define the radial compactification $\mathbb{D}_{M^*}$ (resp. $\mathbb{D}_{E^*}$) of M^* (resp. E^*).

Let Ω be an open subset in $\mathbb{D}_{E^*}$ and f a holomorphic function on Ω. We say that f is of infra-exponential type (at ∞) on Ω if, for any compact set $K \subset \Omega$ and any $\epsilon > 0$, there exists $C > 0$ such that

$$|f(\zeta)| \le Ce^{\epsilon|\zeta|} \qquad (\zeta \in K \cap E^*).$$

Let $\mathscr{O}^{\inf}_{\mathbb{D}_{E^*}}$ designate the set of holomorphic functions of infra-exponential type on $\mathbb{D}_E$. and let $\mathscr{A}^{\exp}_{\mathbb{D}_M}$ denote the sheaf of real analytic functions of exponential type. Let $j : M \hookrightarrow \mathbb{D}_M$ be the canonical inclusion. The sheaf $\mathscr{V}^{\exp}_{\mathbb{D}_M}$ of real analytic volumes of exponential type is given by

$$\mathscr{V}^{\exp}_{\mathbb{D}_M} = \mathscr{O}^{\exp,(n)}_{\mathbb{D}_E}\Big|_{\mathbb{D}_M} \otimes_{\mathbb{Z}_{\mathbb{D}_M}} or_{\mathbb{D}_M}.$$

Here $or_{\mathbb{D}_M} := j_* or_M$.

Definition 3.1. Let D be a subset in $\mathbb{D}_E$ and non-zero $\zeta_0 \in E^*_\infty$. We say that D is properly contained in a half space (of $\mathbb{D}_E$) with direction ζ_0 if there exists a point $a \in E$ such that

$$\overline{D} \setminus \{a\} \subset {}^\frown\{z \in E;\, \mathrm{Re}\langle z-a,\, \zeta_0\rangle > 0\}. \tag{3.1}$$

holds. In the similar way, for $\xi_0 \in M^*_\infty$, we say that a subset D in $\mathbb{D}_M$ is properly contained in a half space (of $\mathbb{D}_M$) of direction ξ_0 if if there exists a point $a \in M$ such that

$$\overline{D} \setminus \{a\} \subset {}^\frown\{x \in M;\, \langle x-a,\, \xi_0\rangle > 0\}. \tag{3.2}$$

Let K be a closed subset in $\mathbb{D}_M$ properly contained in a half space of $\mathbb{D}_M$ with direction $\xi_0 \in M^*_\infty$ and V its open neighborhood of K in $\mathbb{D}_E$. Set $U := \mathbb{D}_M \cap V$ and

$$\mathcal{V}_K := \{V_0 := V \setminus K,\, V_1 := V\}, \qquad \mathcal{V}'_K := \{V_0\}.$$

Then we have

$$\Gamma_K(U;\, \mathscr{B}^{\exp}_{\mathbb{D}_M} \otimes_{\mathscr{A}^{\exp}_{\mathbb{D}_M}} \mathscr{V}^{\exp}_{\mathbb{D}_M}) \simeq \mathrm{H}^{n,n}_{\bar{\vartheta},\mathscr{Q}}(\mathcal{V}_K,\, \mathcal{V}_K{}') \underset{\mathbb{Z}_{\mathbb{D}_M}(U)}{\otimes} or_{\mathbb{D}_M/\mathbb{D}_E}(U) \underset{\mathbb{Z}_{\mathbb{D}_M}(U)}{\otimes} or_M(U \cap M).$$

Let $u \otimes a_{\mathbb{D}_M/\mathbb{D}_E} \otimes a_M \in \Gamma_K(U;\, \mathscr{B}^{\exp}_{\mathbb{D}_M} \otimes_{\mathscr{A}^{\exp}_{\mathbb{D}_M}} \mathscr{V}^{\exp}_{\mathbb{D}_M})$, and let $\nu = (\nu_1,\, \nu_{01}) \in \mathscr{Q}^{n,n}_{\mathbb{D}_E}(\mathcal{V}_K,\, \mathcal{V}_K{}')$ be a representative of u, i.e., $u = [\nu]$. For this element, we define the Laplace transformation as follows.

Definition 3.2. The Laplace transformation of u is defined by

$$L(u)(\zeta) := \int_D e^{-z\zeta}\nu_1 - \int_{\partial D} e^{-z\zeta}\nu_{01},$$

where D is a contractible open subset in $\mathbb{D}_E$ with (partially) smooth boundary such that $K \subset D \subset \overline{D} \subset V$ and it is properly contained in a half space of $\mathbb{D}_E$ with direction ξ_0.

Note that $L(u)$ is independent of the choices of a representative ν of u and D of the integral. Let Γ be a proper closed cone in M and $a \in M$. We denote by $\Gamma^\circ \subset E^*$ the dual open cone of Γ in E^*. Assume that $K = \overline{\{a\} + \Gamma}$. Then we have the following proposition.

Proposition 3.3. *$e^{a\zeta}L(u)$ belongs to $\mathscr{O}^{\inf}_{\mathbb{D}_{E^*}}(N_\infty(\Gamma^\circ))$.*

§ 4. Inverse transformation

To construct a inverse Laplace transformation, we prepare some definitions. Let T be a real analytic manifold and set

$$Y := T \times \mathbb{D}_E, \qquad Y_\infty = T \times (\mathbb{D}_E \setminus E).$$

We denote by $p_T : Y \to T$ (resp. $p_{\mathbb{D}_E} : Y \to \mathbb{D}_E$) the canonical projection to T (resp. $\mathbb{D}_E$).

Let W be an open subset Y and $f(t,z)$ a measurable function on $W \setminus Y_\infty$. We say that $f(t,z)$ is of exponential type on W if, for any compact subset K in W, there exists $H_K > 0$ such that $|\exp(-H_K|z|)\, f(t,z)|$ is essentially bounded on $K \setminus Y_\infty$.

Now we introduce the set $\mathscr{L}\mathscr{Q}_Y(W)$ consisting of a measurable function $f(t,z)$ on $W \setminus Y_\infty$ which satisfies the following conditions:

1. For almost every t_0 in $p_T(W)$, $f(t_0,z)$ is a C^∞ function of the variables z on $(p_T^{-1}(t_0) \cap W) \setminus Y^\infty$.
2. Any higher derivative of $f(t,z)$ with respect to the variables z is of exponential type on W.

Let $\mathscr{L}\mathscr{Q}_Y^k$ denotes the sheaf on Y of k-forms with respect to the variables in $\mathbb{D}_E$, and let us define the de-Rham complex $\mathscr{L}\mathscr{Q}_Y^\bullet$ by

$$0 \longrightarrow \mathscr{L}\mathscr{Q}_Y^0 \xrightarrow{d_{\mathbb{D}_E}} \mathscr{L}\mathscr{Q}_Y^1 \xrightarrow{d_{\mathbb{D}_E}} \cdots \xrightarrow{d_{\mathbb{D}_E}} \mathscr{L}\mathscr{Q}_Y^{2n} \longrightarrow 0,$$

where $d_{\mathbb{D}_E}$ is the differential on $\mathbb{D}_E$. We denote by $\mathscr{L}_{loc,T}^\infty$ the sheaf of L_{loc}^∞-functions on T. We have the following two propositions.

Proposition 4.1. *We have the quasi-isomorphism*

$$p_T^{-1}\mathscr{L}_{loc,T}^\infty \longrightarrow \mathscr{L}\mathscr{Q}_Y^\bullet.$$

Proposition 4.2. *The complex* $\mathbf{R}\Gamma_{p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)}(p_T^{-1}\mathscr{L}_{loc,T}^\infty)$ *is concentrated in degree* n, *and we have the canonical isomorphism*

$$\tilde{p}_T^{-1}\mathscr{L}_{loc,T}^\infty \otimes_{\mathbb{Z}_Y} or_{p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)/Y} \longrightarrow \mathrm{H}^n_{p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)}(p_T^{-1}\mathscr{L}_{loc,T}^\infty),$$

where $\tilde{p}_T : p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M) = T \times \mathbb{D}_M \to T$ *is the canonical projection.*

Let Γ be an $\mathbb{R}_+$-conic proper open subset in M and $a \in M$.
Let $f \in e^{-a\zeta}\mathscr{O}_{\mathbb{D}_{E^*}}^{\inf}(N_\infty(\Gamma^\circ))$. Then, by the definition, we can easily see:

1. There exists a continuous function $\psi : (N_\infty(\Gamma^\circ) \cap M_\infty^*) \times [0,\infty) \to \mathbb{R}_{\geq 0}$ such that, for each $\xi_* \in (N_\infty(\Gamma^\circ) \cap M_\infty^*)$, the function $\psi(\xi_*, t)$ is an infra-linear function of the variable t and f is holomorphic on an open subset $W \cap E^*$, where
(4.1)
$$W := \frown \left\{\zeta = t\xi_* + \sqrt{-1}\eta;\ \eta \in M^* \setminus \{0\},\ \xi_* \in (N_\infty(\Gamma^\circ) \cap M_\infty^*),\ t > \psi(\xi_*, |\eta|)\right\}.$$
Here we identify M_∞^* with $S^{n-1} \subset M^*$.

2. There exists a continuous infra-linear function $\varphi(t)$ on $[0,\infty)$ such that

$$|f(\zeta)| \le e^{-\operatorname{Re}(a\zeta)+\varphi(|\zeta|)} \qquad (\zeta \in W \cap E^*). \tag{4.2}$$

We also define an n-dimensional chain in E^* by

$$\gamma^* := \{\zeta = \xi + \sqrt{-1}\eta \in E^*;\ \eta \in M^* \setminus \{0\},\ \xi = \psi_{\xi_0}(|\eta|)\,\xi_0\},$$

where ξ_0 is a unit vector in $N_\infty(\Gamma^\circ) \cap M^*_\infty$ and $\psi_{\xi_0}(t)$ is a continuous infra-linear function on $[0,\infty)$ with $\psi_{\xi_0}(t) > \psi(\xi_0, t)$ $(t \in [0,\infty))$. Set $T = S^{n-1}$ and $Y = S^{n-1} \times \mathbb{D}_E$. Define coverings

$$\mathcal{W} = \{W_0 = Y \setminus p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M),\ W_1 = Y\}, \qquad \mathcal{W}' = \{W_0\}.$$

Recall the isomorphisms

$$\Gamma(T;\, \mathscr{L}^\infty_{loc,T}) = \Gamma(Y;\, \tilde{p}_T^{-1}\mathscr{L}^\infty_{loc,T})$$

$$\xrightarrow{\sim} \mathrm{H}^n_{p_{\mathbb{D}_E}^{-1}(\mathbb{D}_M)}(Y;\, p_T^{-1}\mathscr{L}^\infty_{loc,T}) = \mathrm{H}^n(\mathscr{L}\mathscr{Q}^\bullet_Y(\mathcal{W},\mathcal{W}')),$$

and set

$$\Omega := {}^{\frown}\{(\eta, z) \in S^{n-1} \times E;\ \langle \eta, \operatorname{Im} z\rangle > 0\} \subset Y.$$

Let $j : \Omega \to Y$ be the canonical open inclusion. Then we can take a special $\omega = (\omega_1, \omega_{01}) \in \mathscr{L}\mathscr{Q}^n_Y(\mathcal{W},\mathcal{W}'))$ satisfying the following conditions:

1. $D_{\mathbb{D}_E}\omega = 0$ and $[\omega]$ is the image of a constant function $1 \in \Gamma(T;\, \mathscr{L}^\infty_{loc,T})$ through the above isomorphisms.
2. We have

$$\operatorname{Supp}_{W_1}(\omega_1) \subset \Omega, \quad \operatorname{Supp}_{W_{01}}(\omega_{01}) \subset \Omega.$$

Now we define the inverse Laplace transformation.

Definition 4.3. The inverse Laplace transform L^{-1} is given by

$$L^{-1}_\omega(f) := \frac{\nu_{\mathbb{D}_M}}{(2\pi\sqrt{-1})^n}\Big(\int_{\gamma^*} f(\zeta)\,\rho(\omega_1)(\frac{\eta}{|\eta|}, z)\,e^{\zeta z}\,d\zeta,$$

$$\int_{\gamma^*} f(\zeta)\,\rho(\omega_{01})(\frac{\eta}{|\eta|}, z)\,e^{\zeta z}\,d\zeta\Big).$$

Here $\zeta = \xi + \sqrt{-1}\eta$ and $\nu_{\mathbb{D}_M} = dz \otimes a_{\mathbb{D}_M/\mathbb{D}_E} \in \mathscr{V}^{\exp}_{\mathbb{D}_M}(\mathbb{D}_M)$, where $a_{\mathbb{D}_M/\mathbb{D}_E} \in or_{\mathbb{D}_M/\mathbb{D}_E}(\mathbb{D}_M)$ is determined by the orientation of γ^* through the isomorphism $or_{\sqrt{-1}M^*} \simeq or_{\mathbb{D}_{M^*}/\mathbb{D}_{E^*}} \simeq or_{\mathbb{D}_M/\mathbb{D}_E}$.

We have the following proposition.

Proposition 4.4. *We have*

1. *The integration* $L_\omega^{-1}(f)$ *converges and it belongs to* $\mathscr{Q}^{n,n}_{\mathbb{D}_E}(\mathcal{V}, \mathcal{V}') \otimes \mathscr{V}^{\exp}_{\mathbb{D}_M}$. *Furthermore,* $\overline{\vartheta}(L_\omega^{-1}(f)) = 0$ *holds.*
2. $[L_\omega^{-1}(f)]$ *does not depend on the choices of* ω. *It also independent of* ξ_0 *and* ψ_{ξ_0} *which appear in the definition of* γ^*.
3. *The support of* $[L^{-1}(f)]$ *is contained in* $K := \overline{\{a\} + \Gamma} \subset \mathbb{D}_M$.

We have the Laplace inversion formula.

Theorem 4.5. *We have*

$$L \circ L^{-1} = \mathrm{id}, \qquad L^{-1} \circ L = \mathrm{id}\,.$$

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