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Stability of Transonic Shock Solutions for One-Dimensional Euler-Poisson Equations

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1 Introduction and Problems

The hydrodynamical model of semiconductor devices or plasmas is described by the nonlinear system

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u} + p\mathbf{I}_n) = \rho\nabla\Phi, \\ (\rho\mathcal{E})_t + \operatorname{div}(\rho\mathcal{E}\mathbf{u} + p\mathbf{u}) = \rho\mathbf{u}\cdot\nabla\Phi, \\ \Delta\Phi = \rho - b(x), \end{cases} \tag{1.1}$$

called the *Euler-Poisson system* (see [36]). In the system above, $\mathbf{u}, \rho$, p, and $\mathcal{E}$ represent the macroscopic particle velocity, electron density, pressure, and the total energy density, respectively. The electric potential Φ is generated by the Coulomb force of particles. The fixed positive function $b(\mathrm{x}) > 0$ represents the density of fixed, positively charged background ions. In fact, the system (1.11) can also be used to model the biological transport of ions in channel proteins [4]. The system (1.11) is closed with the aid of definition of specific total energy and the equation of state

$$\mathcal{E} = \frac{|\mathbf{u}|^2}{2} + \mathfrak{e} \quad \text{and} \quad p = p(\rho, \mathfrak{e}), \tag{1.2}$$

respectively, where $\mathfrak{e}$ is the internal energy. In this paper, we consider the case for which the pressure p and *the enthalpy* $\imath = \mathfrak{e} + \frac{p}{\rho}$ are given by

$$p(\rho, S) = S\rho^\gamma \quad \text{and} \quad \imath(\rho, S) = \frac{\gamma}{\gamma-1}S\rho^{\gamma-1}, \tag{1.3}$$

respectively, where we follow the notations in gas dynamics to call the constant $\gamma > 1$ the adiabatic constant and the quantity $\ln S$ entropy. One of the interesting phenomenon for the system (1.11) is the electric field can provide more stablizing effect. Mathematically speaking, when $b(x) \equiv b_0$ for some constant b_0, the associated linearized system around trivial steady state $(\rho, \mathbf{u}, S) = (b_0, \mathbf{0}, \bar{S})$ where $\bar{S}$ is a constant state, is a Klein-Gordon type system (equation) which has faster dispersive decay than the wave equations which correspond to the linearized Euler system. With the aid of this faster dispersive decay and the nice structure of the system (1.11), the global classical solutions of the system (1.11) with small and smooth irrotational data were established in [14, 23, 20, 15] in the cases with different spatial dimensions.

A natural problem is to see whether there are some other physically nontrivial steady states. If there are some nontrivial steady solutions, can we prove the stability of these solutions? The steady state of the system (1.11) is governed by the following steady Euler-Poisson system

$$\begin{cases} \mathrm{div}_{\mathrm{x}}(\rho\mathbf{u}) = 0, \\ \mathrm{div}_{\mathrm{x}}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = \rho\nabla_{\mathrm{x}}\Phi, \\ \mathrm{div}_{\mathrm{x}}(\rho\mathbf{u}\mathscr{B}) = \rho\mathbf{u}\cdot\nabla_{\mathrm{x}}\Phi, \\ \Delta_{\mathrm{x}}\Phi = \rho - b(\mathrm{x}), \end{cases} \tag{1.4}$$

where *Bernoulli's function* $\mathscr{B}$ is given by

$$\mathscr{B}(\rho, |\mathbf{u}|, S) = \frac{|\mathbf{u}|^2}{2} + \mathfrak{e} + \frac{p}{\rho} = \frac{|\mathbf{u}|^2}{2} + \frac{\gamma}{\gamma-1}S\rho^{\gamma-1}. \tag{1.5}$$

There are several issues to make the system (1.4) complicated. The first is that (1.4) is a mixed type system, and its type depends on *the Mach number* M which is given by $M = \frac{|\mathbf{u}|}{c(\rho)}$ for $c(\rho) = \sqrt{p'(\rho)}$. Here, c is called the *local sound speed*. If $M < 1$, then (1.4) can be decomposed into a nonlinear elliptic system and homogeneous transport equations, and the flow is said *subsonic*. If $M > 1$, on the other hand, (1.4) can be decomposed into a nonlinear hyperbolic-elliptic coupled system and homogeneous transport equations at best, and the flow is said *supersonic*. The second issue is that the last equation in (1.4), which is a *Poisson equation*, has a nonlocal effect to the other equations in (1.4), and it makes the fluid variables $\rho, \mathbf{u}$ and electric potential Φ interact in a highly nonlinear way. Also, physical boundary conditions such as fixed exit pressure give nonlinear boundary conditions for the system (1.4).

In the previous works, some pure subsonic or supersonic solutions are obtained for both one-dimensional and multidimensional cases (cf. [9, 10, 38]). For a viscous approximation of transonic solutions in two dimensional case for the equations of semiconductors, see [13]. However, there have been only a few results for the transonic flows. In the following, we list several results which are closely related to the present paper. First, a boundary value problem for (1.9) was discussed in [1] for a linear pressure function of the form $p(\rho) = k\rho$, furthermore, the boundary conditions read $\rho(0) = \rho(L) = \bar{\rho}$ with $\bar{\rho}$ being a subsonic state and the density of the background charge satisfied $0 < b < \rho_s$. The solution in [1] may contain transonic shock. On the other hand, since the boundary conditions and the pressure function are special in [1], it is desired to consider the more general boundary conditions with more general equation of states. In [44], a phase plane analysis is given for system (1.9). However, no transonic shock solutions were constructed in [44]. A transonic solution which may contain transonic shocks was constructed by Gamba (cf. [12]) by using a vanishing viscosity limit method. However, the solutions as the limit of vanishing viscosity may contain boundary layers. Therefore, the question of well-posedness of the boundary value problem for the inviscid problem can not be answered by the vanishing viscosity method. Moreover, the structure of the solutions constructed by the vanishing viscosity method in [12] is shown to be of bounded total variation and possibly contain more than one transonic shock.

In [33], the authors considered one dimensional solutions of (1.4) with a constant background charge $b(x) = b_0 > 0$. In particular, in some case there are might be several transonic shock solution when the boundary conditions are prescribed. Hence it is natural to single out the physical solutions. Our basic idea is to study structural stability and dynamical stability of these solutions. Furthermore, the Euler-Poisson system has one dimensional smooth transonic solutions which does not happen for one dimensional Euler system. Our another aim is to study the stability for these smooth transonic solutions.

It is inevitable to consider small perturbations of one dimensional transonic shocks in multidimensional domain, but there are very few known results about multidimensional solutions of Euler-Poisson system(cf.[12, 13]). Comparing with extensive studies and recent significant progress on transonic shock solutions of the Euler system(see [6, 8, 46] and references therein), stability problems for multidimensional transonic flows of the Euler-Poisson system are essentially open. The main difference of the Euler-Poisson system from the Euler system is that the Poisson equation for electric potential is coupled with the other equations in the Euler-Poisson system. And, this makes it hard to analyze even one dimensional solution of the Euler-Poisson system. In fact, one dimensional flow of the Euler-Poisson system behaves very differently from the one of the Euler system(cf.[33]). And it is even harder to study multidimensional transonic flow of the Euler-Poisson system due to nonlinear interaction between the electric potential Φ and all the other fluid variables.

As the first step to investigate stability of multidimensional transonic flow of the Euler-Poisson system, we study the unique existence and stability of subsonic flows of steady Euler-Poisson system under perturbations of the exit pressure and electric potential difference on non-insulated boundary.

There have been quite a few results about existence of subsonic solution of hydrodynamic equations, which are the Euler-Poisson system with relaxation, under smallness assumptions on the flow velocity for both unsteady and steady cases(see [9, 10, 16, 35]). Here we focus on the system without relaxation.

1.1 Stbility problems in one dimensional setting

First, consider the boundary value problem for

$$\begin{cases} (\rho u)_x = 0, \\ (p(\rho) + \rho u^2)_x = \rho E, \\ E_x = \rho - b(x) \end{cases} \tag{1.6}$$

in an interval $0 \leq x \leq L$ with the boundary conditions:

$$(\rho, u, E)(0) = (\rho_l, u_l, E_l), \quad \rho(L) = \rho_r. \tag{1.7}$$

If one denotes $\rho_l u_l = J$, then the velocity is given by

$$u = J/\rho\,. \tag{1.8}$$

Thus the boundary value problem (1.6) and (1.7) reduce to

$$\begin{cases} (p(\rho) + \frac{J^2}{\rho})_x = \rho E, \\ E_x = \rho - b(x), \end{cases} \tag{1.9}$$

with the boundary conditions:

$$(\rho, E)(0) = (\rho_l, E_l), \quad \rho(L) = \rho_r. \tag{1.10}$$

There is a unique solution $\rho = \rho_s$ for the equation $p'(\rho) = J^2/\rho^2$, which is the sonic state (recall that $J = \rho u$). The flow is called supersonic (respectively subsonic) if

$$p'(\rho) < J^2/\rho^2, \ i.e.\ \rho < \rho_s \text{ (respectively } p'(\rho) > J^2/\rho^2, \ i.e.\ \rho > \rho_s).$$

Definition 1. *A piecewise smooth solution (ρ, E) with $\rho > 0$ of (8) (or equivalently (2) with $u = \frac{J}{\rho}$) is said to be a transonic shock solution, if it is separated by a shock discontinuity, and of the form*

$$(\rho, E) = \begin{cases} (\rho_{sup}, E_{sup})(x), & 0 < x < x_0, \\ (\rho_{sub}, E_{sub})(x), & x_0 < x < L, \end{cases}$$

satisfying the Rankine-Hugoniot conditions

$$p(\rho_{sup}(x_0-)) + \frac{J^2}{\rho_{sup}(x_0-)} = p(\rho_{sub}(x_0+)) + \frac{J^2}{\rho_{sub}(x_0+)}, \quad E_{sup}(x_0-) = E_{sub}(x_0+),$$

and is supersonic behind the shock and subsonic ahead of the shock, i.e.,

$$\rho_{sup}(x_0-) < \rho_s < \rho_{sub}(x_0+), \ \textit{and} \ \ \rho_{sub}(x) > \rho_s, \ \textit{for } x \in [x_0, L].$$

Problem 1. *Is the one dimensional steady transonic shock solution structurally stable when we perturb the boundary conditions or the background charge $b(x)$?*

Second, we would like to investigate the dynamical stability of the steady transonic shock solutions. For a given steady transonic shock solution, one can extend $(\bar{\rho}_-, \bar{E}_-)$ $((\bar{\rho}_+, \bar{E}_+))$ to be a smooth supersonic solution of (1.4) on $[0, x_0+\delta]$ $([x_0-\delta, L])$ for some $\delta > 0$, which coincides with $(\bar{\rho}_-, \bar{E}_-)$ $((\bar{\rho}_+, \bar{E}_+))$ on $[0, x_0]$ $([x_0, L])$.

We consider the initial boundary value problem of system of one dimensional Euler-Poisson equations:

$$\begin{aligned} &\rho_t + (\rho u)_x = 0, \\ &(\rho u)_t + (p(\rho) + \rho u^2)_x = \rho E, \\ &E_x = \rho - b(x), \end{aligned} \tag{1.11}$$

with the initial data

$$(\rho, u, E)(0, x) = (\rho_0, u_0, E_0)(x), \tag{1.12}$$

and the boundary condition

$$(\rho, u, E)(t, 0) = (\rho_l, \frac{\bar{J}}{\rho_l}, E_l), \quad \rho(t, L) = \rho_r, \tag{1.13}$$

where ρ_l, E_l and ρ_r are the same as in (1.10).

We assume that the initial data are of the form

$$(\rho_0, u_0)(x) = \begin{cases} (\rho_{0-}, u_{0-})(x), & \text{if } 0 < x < \tilde{x}_0, \\ (\rho_{0+}, u_{0+})(x), & \text{if } \tilde{x}_0 < x < L, \end{cases} \tag{1.14}$$

and

$$E_0(x) = E_l + \int_0^x (\rho_0(s) - b(s))ds \tag{1.15}$$

which is a small perturbation of $(\bar{\rho}, \bar{u}, \bar{E})$. Moreover, (ρ_0, u_0, E_0) is assumed to satisfy the Rankine-Hogoniot conditions as $x = \tilde{x}_0$,

$$\begin{aligned} &\big((p(\rho_{0+}) + \rho_{0+}u_{0+}^2 - (p(\rho_{0-}) + \rho_{0-}u_{0-}^2)\big) \cdot (\rho_{0+} - \rho_{0-})(\tilde{x}_0) \\ =&(\rho_{0+}u_{0+} - \rho_{0-}u_{0-})^2(\tilde{x}_0). \end{aligned} \tag{1.16}$$

Problem 2. *Is the solution for the unsteady problem convergent to the associated steady states?*

1.2 Problem on subsonic flows with zero vorticity

Fix an open, bounded and connected set $\Lambda \subset \mathbb{R}^{n-1} (n \geq 2)$ with a smooth boundary $\partial\Lambda$, and define a nozzle $\mathcal{N}$ by

$$\mathcal{N} := \{\mathrm{x} = (\mathrm{x}', x_n) \in \mathbb{R}^n : \mathrm{x}' \in \Lambda, \;\; x_n \in (0, L)\} \subset \mathbb{R}^n. \tag{1.17}$$

The nozzle boundary $\partial\mathcal{N}$ consists of the entrance $\Gamma_0 = \Lambda \times \{0\}$, the exit $\Gamma_L = \Lambda \times \{L\}$, and the insulated boundary $\Gamma_w = \partial\Lambda \times (0, L)$. In order to study the system (1.4) in a multidimensional domains $\mathcal{N}$ with arbitrary cross-section Λ, we consider isentropic irrotational flow where $S =$ *constant* and the velocity $\mathbf{u}$ of the flow is represented by

$$\mathbf{u} = \nabla\varphi \tag{1.18}$$

for a scalar function φ which is called a *velocity potential* function. Then we write $p = p(\rho)$. By (1.4) and (1.18), the second equation in (1.4) can be rewritten as

$$\rho\nabla(\mathscr{B} - \Phi) = 0 \tag{1.19}$$

for

$$\mathscr{B} = \frac{1}{2}|\nabla\varphi|^2 + \int_{k_0}^{\rho} \frac{p'(\varrho)}{\varrho} d\varrho. \tag{1.20}$$

For $\rho > 0$, (1.19) implies

$$\mathscr{B} - \Phi \equiv K_0 \tag{1.21}$$

for a constant K_0. Without loss of generality, we choose $K_0 = 0$. Set

$$h(\rho) := \int_{k_0}^{\rho} \frac{p'(\varrho)}{\varrho} d\varrho. \tag{1.22}$$

Then, the equation (1.21) with $K_0 = 0$ implies that $h(\rho) = \Phi - \frac{1}{2}|\nabla\varphi|^2$. Hence one can rewrite (1.22) as $\rho = h^{-1}(\Phi - \frac{1}{2}|\nabla\varphi|^2)$. We use this expression to reduce (1.4) to a nonlinear system of second order equations for φ and Φ:

$$\mathrm{div}\big(\rho(\Phi, |\nabla\varphi|^2)\nabla\varphi\big) = 0, \tag{1.23}$$

$$\Delta\Phi = \rho(\Phi, |\nabla\varphi|^2) - b \tag{1.24}$$

with $\rho > 0$ given by

$$\rho(\Phi, |\nabla\varphi|^2) = h^{-1}(\Phi - \frac{1}{2}|\nabla\varphi|^2) \tag{1.25}$$

provided that h^{-1} is well defined. If we regard (1.23) as an equation for φ, then it is mixed type. More precisely, (1.23) is elliptic if and only if

$$|\nabla\varphi|^2 < p'(\rho)\,(subsonic) \tag{1.26}$$

and is *hyperbolic* if and only if

$$|\nabla\varphi|^2 > p'(\rho)\,(supersonic). \tag{1.27}$$

The system of (1.23) and (1.24) becomes a quasilinear elliptic system if (1.26) holds, and a hyperbolic-elliptic coupled system if (1.27) holds.

Our interest is on stability of subsonic solution under perturbations of exit pressure and electric potential difference on non-insulated boundary from a fixed point. So the boundary conditions are formulated as follows. First, for a given function p_{ex} on Γ_L, set

$$p\big(\rho(\Phi, |\nabla\varphi|^2)\big) = p_{ex} \quad \text{on} \quad \Gamma_L. \tag{1.28}$$

On the wall boundary, slip/insulated boundary conditions for φ and Φ are prescribed as follows:

$$\partial_{\mathbf{n}_w}\varphi = \partial_{\mathbf{n}_w}\Phi = 0 \ \text{ on } \ \Gamma_w \tag{1.29}$$

where $\mathbf{n}_w$ is the unit inward normal vector on Γ_w. We fix a point x_0 on Γ_L, and prescribe the electric potential difference between two points $\mathrm{x} \in \Gamma_0 \cup \Gamma_L$ and x_0 as follows:

$$\Phi(\mathrm{x}) - \Phi(\mathrm{x}_0) = \begin{cases} \bar{\Phi}_{en}(\mathrm{x}) & \text{for } \mathrm{x} \in \Gamma_0 \\ \bar{\Phi}_{ex}(\mathrm{x}) & \text{for } \mathrm{x} \in \Gamma_L \end{cases}. \tag{1.30}$$

In (1.30), the value of $\Phi(\mathrm{x}_0)$ is uniquely determined by (1.21) and one point boundary condition for the Bernoulli's function:

$$\mathscr{B}(\mathrm{x}_0) = \mathscr{B}_0. \tag{1.31}$$

Finally, homogeneous Dirichlet boundary condition for φ is imposed at the entrance:

$$\varphi = 0 \quad \text{on} \quad \Gamma_0. \tag{1.32}$$

Problem 3. *Fix a point* $\bar{\mathrm{x}} = (\bar{x}_1, 0) \in \Gamma_0$. *Find a solution* $(\rho, \mathbf{u}, p, \Phi)$ *to* (1.4) *in* $\mathcal{N}$ *satisfying the boundary conditions* (1.28)-(1.31).

Suppose the vorticity of the flow is not zero, can we also prove that subsonic solutions are structurally stable under multidimensional perturbations of boundary conditions? More precisely, our main concern is to solve the following problem.

Problem 4. *Fix a point* $\bar{\mathrm{x}} = (\bar{x}_1, 0) \in \Gamma_0$*. Find a solution* $(\rho, \mathbf{u}, p, \Phi)$ *to* (1.4) *in* $\mathcal{N}$ *satisfying the boundary conditions*

$$S = S_{en}, \quad \mathscr{B} = \mathscr{B}_{en}, \quad \mathbf{u} \cdot \boldsymbol{\tau} = 0 \quad on \ \Gamma_0, \tag{1.33}$$

$$\Phi - \Phi(\bar{\mathrm{x}}) = \Phi_{bd} \ \ on \ \Gamma_0 \cup \Gamma_L, \tag{1.34}$$

$$\mathbf{u} \cdot \mathbf{n}_w = \partial_{\mathbf{n}_w} \Phi = 0 \ \ on \ \Gamma_w, \tag{1.35}$$

$$p = p_{ex} \ \ on \ \Gamma_L. \tag{1.36}$$

2 Main Results

In this section, we give our main results for the study on stability of transonic shock solutions under one dimensional perturbations, subsonic solutions under multidimensional perturbations.

2.1 Stability of transonic shocks in one dimensional setting

Theorem 1. *([32]) Let* $J > 0$ *be a constant, and let* b_0 *satisfy*

$$0 < \min_{x \in [0,L]} b_0(x) \leq \max_{x \in [0,L]} b_0(x) < \rho_s \tag{2.1}$$

and (ρ_l, E_l) *be a supersonic state* $(0 < \rho_l < \rho_s)$*,* ρ_r *be a subsonic state* $(\rho_r > \rho_s)$*. If the boundary value problem (1.9) and (1.10) admits a unique transonic shock solution* $(\rho^{(0)}, E^{(0)})$ *for the case when* $b(x) = b_0(x)$ $(x \in [0, L])$ *with a single transonic shock located at* $x = x_0 \in (0, L)$ *satisfying*

$$E^{(0)}(x_0+) = E^{(0)}(x_0-) > 0, \tag{2.2}$$

then there exists $\epsilon_0 > 0$ *such that if*

$$\|b - b_0\|_{C^0[0,L]} = \epsilon \leq \epsilon_0, \tag{2.3}$$

then the boundary problem (1.9) and (1.10) admits a unique transonic shock solution $(\tilde{\rho}, \tilde{E})$ *with a single transonic shock locating at some* $\tilde{x}_0 \in [x_0 - C\epsilon, x_0 + C\epsilon]$ *for some constant* $C > 0$*.*

Remark 1. *When* $b_0(x) \equiv const$*, it should be noted that there are a large class of boundary data which ensure the existence and uniqueness of the transonic shock solutions satisfying the assumptions in Theorem 1 , see [33].*

Our next result is about dynamical stability of transonic shock in one dimensional setting.

Theorem 2. *([32]) Let $(\bar{\rho}, \bar{u}, \bar{E})$ be a steady transonic shock solution to system (1.11). Moreover, there exists a $\delta > 0$ (δ depends on $(\bar{\rho}, \bar{u}, \bar{E})$) such that*

$$\bar{E}_-(x_0) = \bar{E}_+(x_0) > -\delta. \tag{2.4}$$

Then there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \leq \varepsilon_0$, if the initial data (ρ_0, u_0, E_0) satisfy (1.15), (1.16), in the sense that

$$\begin{aligned} &|x_0 - \tilde{x}_0| + \|(\rho_{0+}, u_{0+}) - (\bar{\rho}_+, \bar{u}_+)\|_{H^{k+2}([\check{x}_0, L])} \\ &+ \|(\rho_{0-}, u_{0-}) - (\bar{\rho}_-, \bar{u}_-)\|_{H^{k+2}([0, \hat{x}_0])} < \varepsilon, \end{aligned} \tag{2.5}$$

for some integer $k \geq 15$ and some small $\varepsilon > 0$, where $\check{x}_0 = \min\{x_0, \tilde{x}_0\}$ and $\hat{x}_0 = \max\{x_0, \tilde{x}_0\}$, and the $k+2$-th order compatibility conditions at $x = 0$, $x = x_0$ and $x = L$, then the initial boundary value problem (1.11), (1.12) and (1.13) admits a unique piecewise smooth entropy solution $(\rho, u, E)(x, t)$ for $(t, x) \in [0, \infty) \times [0, L]$ containing a single transonic shock $x = s(t)$ ($0 < s(t) < L$) with $s(0) = \tilde{x}_0$.

Furthermore, there exist $T_0 > 0$ and $\lambda > 0$ such that

$$(\rho_-, u_-, E_-)(t, x) = (\bar{\rho}_-, \bar{u}_-, \bar{E}_-)(x), \quad \text{for } 0 \leq x < s(t),$$

for $t > T_0$ and

$$\|(\rho_+, u_+,)(\cdot, t) - (\bar{\rho}_+, \bar{u}_+)(\cdot)\|_{W^{k-7,\infty}(s(t),L)} + \|E_+(\cdot, t) - \bar{E}_+(\cdot)\|_{W^{k-6,\infty}(s(t),L)} \leq C\varepsilon e^{-\lambda t},$$

$$\sum_{m=0}^{k-6} |\partial_t^m (s(t) - x_0)| \leq C\varepsilon e^{-\lambda t},$$

for $t \geq 0$, where $(\bar{\rho}_\pm, \bar{u}_\pm, \bar{E}_\pm)$ are the solutions of the Euler-Poisson equations in the associated regions.

The condition (2.4) is used to prove the exponential dynamical stability of the steady transonic shock solutions. When this condition is violated, we have the following linear instability results for some special cases.

Theorem 3. *There exist $L > 0$ and a linearly unstable transonic shock solution $(\bar{\rho}, \bar{u}, \bar{E})$ satisfying*

$$\bar{E}_-(x_0) = \bar{E}_+(x_0) < -C \tag{2.6}$$

for some positive constant C.

Several remarks are in order concerning Theorems 1-3.

Remark 2. *In both Theorem 1 and Theorem 2, the results are also true if we impose small perturbations for the boundary conditions (1.10).*

Remark 3. *It follows from the results in [33] and Theorem 1, that the background transonic shock solution in Theorem 2 does exist. Moreover, we do not assume that $b(x)$ is a small perturbation of a constant in both Theorems 1 and 2, which may have large variations.*

Remark 4. *In [43], the local-in-time stability of transonic shock solutions for the Cauchy problem of (1.11) is considered by assuming the existence of steady transonic shocks. Here, we prove the global-in -time exponential stability for the initial boundary value problem.*

Remark 5. *The compatibility conditions for the initial boundary value problems for hyperbolic equations were discussed in detail in [40, 34, 37].*

Remark 6. *In Theorem 2, the regularity assumption is not optimal. By adapting the method in [37], less regularity assumption than that in (2.5) will be enough . However, our proof only involves the elementary weighted energy estimates rather than paradifferential calculus.*

Remark 7. *The results here was used to prove stability for transonic shocks in quasi-one-dimensional nozzles.*

2.2 Stability of subsonic potential flows

For a bounded connected open set $\Omega \subset \mathbb{R}^n$, let Γ be a closed portion of $\partial\Omega$. For $\mathrm{x}, \mathrm{y} \in \Omega$, set

$$\delta_{\mathrm{x}} := \mathrm{dist}(\mathrm{x}, \Gamma) \quad \text{and} \quad \delta_{\mathrm{x},\mathrm{y}} := \min(\delta_{\mathrm{x}}, \delta_{\mathrm{y}}).$$

For $k \in \mathbb{R}$, $\alpha \in (0,1)$ and $m \in \mathbb{Z}^+$, define the standard Hölder norms by

$$\|u\|_{m,0,\Omega} := \sum_{0\le|\beta|\le m} \sup_{\mathrm{x}\in\Omega} |D^\beta u(\mathrm{x})|, \quad [u]_{m,\alpha,\Omega} := \sum_{|\beta|=m} \sup_{\mathrm{x},\mathrm{y}\in\Omega,\mathrm{x}\neq\mathrm{y}} \frac{|D^\beta u(\mathrm{x}) - D^\beta u(\mathrm{y})|}{|\mathrm{x}-\mathrm{y}|^\alpha},$$

and the weighted Hölder norms by

$$\|u\|^{(k,\Gamma)}_{m,0,\Omega} := \sum_{0\le|\beta|\le m} \sup_{\mathrm{x}\in\Omega} \delta_{\mathrm{x}}^{\max(|\beta|+k,0)} |D^\beta u(\mathrm{x})|,$$

$$[u]^{(k,\Gamma)}_{m,\alpha,\Omega} := \sum_{|\beta|=m} \sup_{\mathrm{x},\mathrm{y}\in\Omega,\mathrm{x}\neq\mathrm{y}} \delta_{\mathrm{x},\mathrm{y}}^{\max(m+\alpha+k,0)} \frac{|D^\beta u(\mathrm{x}) - D^\beta u(\mathrm{y})|}{|\mathrm{x}-\mathrm{y}|^\alpha},$$

$$\|u\|_{m,\alpha,\Omega} := \|u\|_{m,0,\Omega} + [u]_{m,\alpha,\Omega}, \quad \|u\|^{(k,\Gamma)}_{m,\alpha,\Omega} := \|u\|^{(k,\Gamma)}_{m,0,\Omega} + [u]^{(k,\Gamma)}_{m,\alpha,\Omega},$$

where D^β denotes $\partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n}$ for a multi-index $\beta = (\beta_1, \cdots, \beta_n)$ with $\beta_j \in \mathbb{Z}_+$ and $|\beta| = \sum_{j=1}^n \beta_j$. $C^{m,\alpha}_{(k,\Gamma)}(\Omega)$ denotes the completion of the set of all smooth functions whose $\|\cdot\|^{(k,\Gamma)}_{m,\alpha,\Omega}$ norms are finite.

The main result for stability of subsonic potential flows for the Euler-Poisson system is as follows.

Theorem 4. *([2]) Let $\mathcal{N}$ be as in* (1.17)*. Fix $b_0 > 0$ and $L > 0$, and let the parameter set $\mathfrak{P}_0$ be as in Proposition 8. Given $(\bar{\Phi}_{en,0}, \mathscr{B}_{0,0}, p_{ex,0}) \in \mathfrak{P}_0$, let (φ_0, Φ_0) be the corresponding background solution. Assume that $b, (\bar{\Phi}_{en}, \bar{\Phi}_{ex}, p_{ex})$ and $\mathscr{B}_0$ are given as small perturbations of b_0, $(\bar{\Phi}_{en,0}, 0, \bar{p}_{ex,0})$ and $\mathscr{B}_{0,0}$, respectively, in the following sense:*

$$\begin{aligned} &\|b - b_0\|_{\alpha,\mathcal{N}} \leq \sigma, \quad |\mathscr{B}_0 - \mathscr{B}_{0,0}| \leq \sigma \\ &\|\bar{\Phi}_{en} - \bar{\Phi}_{en,0}\|_{2,\alpha,\Gamma_0} + \|\bar{\Phi}_{ex}\|_{2,\alpha,\Gamma_L} + \|p_{ex} - p_{ex,0}\|_{\alpha,\Gamma_L} \leq \sigma \end{aligned} \tag{2.7}$$

for a small constant $\sigma > 0$ to be specified below. Also, suppose that $\bar{\Phi}_{en}$ and $\bar{\Phi}_{ex}$ satisfy the compatibility conditions

$$\partial_{\mathbf{n}_w} \bar{\Phi}_{en} = 0 \ \ on \ \ \overline{\Gamma}_0 \cap \overline{\Gamma}_w, \quad \partial_{\mathbf{n}_w} \bar{\Phi}_{ex} = 0 \ \ on \ \ \overline{\Gamma}_L \cap \overline{\Gamma}_w. \tag{2.8}$$

Then, for any given $\alpha \in (0,1)$, there exists a constant $\bar{\sigma} > 0$ depending on $b_0, L, \bar{\Phi}_{en,0}, \mathscr{B}_{0,0}, p_{ex,0}$ and α such that wherever $\sigma \in (0, \bar{\sigma}]$, if the boundary data and b satisfy (2.7) *and* (2.8)*, then the nonlinear system* (1.23)-(1.24) *with boundary conditions* (1.28)-(1.31) *has a unique solution $(\varphi, \Phi) \in [C^{1,\alpha}(\overline{\mathcal{N}}) \cap C^{2,\alpha}(\mathcal{N})]^2$ satisfying the following properties:*

(a) The equations in (1.23)-(1.24) *form a uniformly elliptic system in $\mathcal{N}$. Equivalently, the solution (φ, Φ) satisfies the inequality*

$$p'(\rho(\Phi, |\nabla\varphi|^2)) - |\nabla\varphi|^2 \geq \bar{\nu} > 0 \quad \tilde{x}in \quad \overline{\mathcal{N}}$$

for a positive constant $\bar{\nu}$, i.e., the flow governed by (φ, Φ) is subsonic;

(b) (φ, Φ) satisfy the estimate

$$\|\varphi - \varphi_0\|^{(-1-\alpha,\Gamma_0\cup\Gamma_L)}_{2,\alpha,\mathcal{N}} + \|\Phi - \Phi_0\|^{(-1-\alpha,\Gamma_0\cup\Gamma_L)}_{2,\alpha,\mathcal{N}} \leq C\sigma, \tag{2.9}$$

for σ in (2.7)*. The constants $\bar{\nu}$ and C depend only on $b_0, L, \bar{\Phi}_{en,0}, \mathscr{B}_{0,0}, p_{ex,0}, n, \Lambda$ and α.*

Remark 8. *We point out that the boundary conditions* (1.28)–(1.32) *are physically measurable. In one-dimensional solutions, they correspond to prescribing the pressure(or equivalently prescribing the density) at both ends of the nozzle.*

When the flow has nonzero vorticity, our main results for Problem 4 is as follows.

Theorem 5. *([3]) Suppose that $(\bar{\rho}, \bar{\mathbf{u}}, \bar{p}, \Phi_0)$ is the subsonic background solution in $\mathcal{N}$ associated with parameters $b_0 > 0$, $S_0 > 0$, $J_0 > 0$, $\rho_0 > \rho_c$, E_0, and L. Assume that Φ_{bd} satisfies the* *compatibility condition*

$$\partial_{x_2}\Phi_{bd} = 0 \quad on \quad \overline{\Gamma_w} \cap (\overline{\Gamma_0} \cup \overline{\Gamma_L}). \tag{2.10}$$

(a) *(Existence) There exists a $\sigma_1 > 0$ depending on the data and α so that if*

$$\omega_1(b) + \omega_2(S_{en}, \mathscr{B}_{en}) + \omega_3(\Phi_{bd}, p_{ex}) \le \sigma_1, \tag{2.11}$$

where

$$\begin{aligned} &\omega_1(b) := \|b - b_0\|_{\alpha,\mathcal{N}}, \quad \omega_2(S_{en}, \mathscr{B}_{en}) := \|(S_{en}, \mathscr{B}_{en}) - (S_0, \mathscr{B}_0)\|_{1,\alpha,\Gamma_0}, \\ &\omega_3(\Phi_{bd}, p_{ex}) := \|\Phi_{bd} - \Phi_0\|^{(-1-\alpha,\partial(\Gamma_0\cup\Gamma_L))}_{2,\alpha,\Gamma_0\cup\Gamma_L} + \|p_{ex} - \bar{p}\|^{(-\alpha,\partial\Gamma_L)}_{1,\alpha,\Gamma_L}, \end{aligned} \tag{2.12}$$

then the boundary value problem (1.4) *with* (1.33)–(1.36) *has a solution* $(\rho, \mathbf{u}, p, \Phi)$ *satisfying*

$$\begin{aligned} &\|(\rho, \mathbf{u}, p) - (\bar{\rho}, \bar{\mathbf{u}}, \bar{p})\|^{(-\alpha,\Gamma_0\cup\Gamma_L)}_{1,\alpha,\mathcal{N}} + \|\Phi - \Phi_0\|^{(-1-\alpha,\Gamma_0\cup\Gamma_L)}_{2,\alpha,\mathcal{N}} \\ &\le C\left(\omega_1(b) + \omega_2(S_{en}, \mathscr{B}_{en}) + \omega_3(\Phi_{bd}, p_{ex})\right) \end{aligned} \tag{2.13}$$

where the constant C depends only on the data and α.

(b) *(Uniqueness) There exists a $\sigma_2 > 0$ depending on the data, α, and μ such that if*

$$\omega_1(b) + \omega_2(S_{en}, \mathscr{B}_{en}) + \omega_3(\Phi_{bd}, p_{ex}) + \omega_4(S_{en}, \mathscr{B}_{en}, \Phi_{bd}) \le \sigma_2, \tag{2.14}$$

with $\alpha \in (\frac{1}{2}, 1)$ and $\mu \in (2, \infty)$ where

$$\omega_4(S_{en}, \mathscr{B}_{en}, \Phi_{bd}) := \|(S_{en}, \mathscr{B}_{en} - \Phi_{bd}) - (S_0, \mathscr{B}_0)\|_{W^{2,\mu}(\Gamma_0)}, \tag{2.15}$$

then the solution $(\rho, \mathbf{u}, p, \Phi)$ obtained in (a) is unique.

Remark 9. *We can also prove the stability of subsonic flows under small perturbations of the nozzle boundary.*

3 Key ingredients for the proof

In this section, we give the key ingredients for the proof of main results.

3.1 Stability of transonic shocks in one dimensional setting

In this subsection, we give the key points for the analysis on one dimensional solutions of the Euler-Poisson system.

3.1.1 structural stability of one dimensional transonic shock solutions

The following monotone relation between the shock position and the downstream density plays a crucial role for the proof of Theorem 1.

Lemma 6. *Let* $(\rho^{(1)}, E^{(1)})$ *and* $(\rho^{(2)}, E^{(2)})$ *be two transonic shock solutions of (1.9), and* $(\rho^{(i)}, E^{(i)})(i = 1, 2)$ *are defined as follows*

$$(\rho^{(i)}, E^{(i)}) = \begin{cases} (\rho^{(i)}_{sup}, E^{(i)}_{sup}), \ \textit{for} \ \ 0 < x < x_i, \\ (\rho^{(i)}_{sub}, E^{(i)}_{sub}), \ \textit{for} \ \ x_i < x < L, \end{cases}$$

where

$$\rho^{(i)}_{sup} < \rho_s < \rho^{(i)}_{sub} \quad \textit{for} \ \ i = 1, 2.$$

Moreover, they satisfy the same upstream boundary conditions,

$$\rho^{(1)}(0) = \rho^{(2)}(0) = \rho_l, \ \ E^{(1)}(0) = E^{(2)}(0) = E_l.$$

If $b < \rho_s$*,* $x_1 < x_2$ *and* $E^{(2)}_{sup}(x_1) > 0$*, then*

$$\rho^{(1)}(L) > \rho^{(2)}(L).$$

The lemma is proved by the comparison principle for ODEs.

3.1.2 Dynamical Stability of Transonic Shock Solutions

The dynamical stability of transonic shock depends on the following two facts. First, the perturbations to the left vanish when time is large because of the supersonic boundary conditions on the left. Second, the perturbations to the right decay due to the absorption at the shock. The proof of the latter property is not straightforward. In addition to the usual technical difficulties from the quasilinear structure there is a fundamental difficulty that the problem involves a free boundary (shock) on the left of the subsonic region. The key is to prove decay for the linearized problem.

Let $(\bar{\rho}, \bar{u}, \bar{E})$ be a steady transonic shock solution satisfying (2.4). Suppose that the initial data (ρ_0, u_0, E_0) satisfies (2.5) and the $k + 2$-th order compatibility conditions. It follows from the argument in [27] that there exists a piecewise smooth solution containing a single shock $x = s(t)$ (with $s(0) = \tilde{x}_0$) satisfying the Rankine-Hogoniot conditions and Lax geometric shock condition, of the Euler-Poisson equations on $[0, \bar{T}]$ for some $\bar{T} > 0$, which can be written as

$$(\rho, u, E)(x, t) = \begin{cases} (\rho_-, u_-, E_-), \ \text{if } 0 < x < s(t), \\ (\rho_+, u_+, E_+), \ \text{if } s(t) < x < L. \end{cases} \tag{3.1}$$

Note that, when $t > T_0$ for some $T_0 > 0$, (ρ_-, u_-, E_-) will depend only on the boundary conditions at $x = 0$. Moreover, when ε is small, by the standard lifespan argument, we have $T_0 < \bar{T}$ (cf. [27]). Therefore,

$$(\rho_-, u_-, E_-) = (\bar{\rho}_-, \bar{u}_-, \bar{E}_-) \text{ for } t > T_0. \tag{3.2}$$

In the following, for simplicity of the presentation, we may assume $T_0 = 0$ without loss of generality. As long as we have the local stability, we need only to obtain uniform estimates in the region $x > s(t), t > 0$. For this purpose, we will formulate an initial boundary value problem in this region. First, the Rankine-Hugoniot conditions for (3.1) read

$$[\rho u] = [\rho]s'(t), \ \ [\rho u^2 + p] = [\rho u]s'(t), \tag{3.3}$$

where $[f] = f(s(t)+, t) - f(s(t)-, t)$, so $[p + \rho u^2] \cdot [\rho] = [\rho u]^2$. Thus, by implicit function theorem, we have

$$(J_+ - \bar{J})(t, s(t)) = \mathscr{A}_1((\rho_+ - \bar{\rho}_+)(t, s(t)), s(t) - x_0) \tag{3.4}$$

where $\mathscr{A}_1$ regarded as a function of two variables satisfies

$$\mathscr{A}_1(0,0) = 0, \quad \frac{\partial \mathscr{A}_1}{\partial(\rho_+ - \bar{\rho}_+)} = -\frac{p'(\bar{\rho}_+) - \frac{\bar{J}^2}{\bar{\rho}_+^2}}{2\bar{J}/\bar{\rho}_+}(x_0), \quad \frac{\partial \mathscr{A}_1}{\partial(s - x_0)} = -\frac{(\bar{\rho}_+ - \bar{\rho}_-)\bar{E}_+}{2\bar{J}/\bar{\rho}_+}(x_0).$$

Substituting (3.4) into the first equation in (3.3) yields

$$s'(t) = \mathscr{A}_2(\rho_+ - \bar{\rho}_+, s(t) - x_0) \tag{3.5}$$

where $\mathscr{A}_2$ satisfies $\mathscr{A}_2(0,0) = 0$ and

$$\frac{\partial \mathscr{A}_2}{\partial(\rho_+ - \bar{\rho}_+)} = -\frac{p'(\bar{\rho}_+) - \bar{J}^2/\bar{\rho}_+^2}{2\bar{u}_+(\bar{\rho}_+ - \bar{\rho}_-)}(x_0), \quad \frac{\partial \mathscr{A}_2}{\partial(s(t) - x_0)} = -\frac{\bar{E}_+}{2\bar{u}_+}(x_0).$$

Set $Y = E_+(x,t) - \bar{E}_+(x)$. Then

$$Y_t = \bar{J} - J_+, \quad Y_x = \rho_+ - \bar{\rho}_+.$$

Therefore, it follows from the second equation in the Euler-Poisson equations (1.11) that

$$\partial_{tt} Y + \partial_x \left(p(\bar{\rho}_+) + \frac{\bar{J}^2}{\bar{\rho}_+} - p(\bar{\rho}_+ + Y_x) - \frac{(\bar{J} - Y_t)^2}{\bar{\rho}_+ + Y_x} \right) + \bar{E}_+ \partial_x Y + \bar{\rho}_+ Y + Y Y_x = 0. \tag{3.6}$$

One has

$$\mathscr{L}_0 Y = \partial_{tt} Y - \partial_x((p'(\bar{\rho}_+) - \frac{\bar{J}^2}{\bar{\rho}_+^2})\partial_x Y) + \partial_x(\frac{2\bar{J}}{\bar{\rho}_+}\partial_t Y) + \bar{E}_+\partial_x Y + \bar{\rho}_+ Y. \tag{3.7}$$

Furthermore, the Rankine-Hugoniot conditions (3.4) and (3.5) can be written as

$$Y_t = -\mathscr{A}_1(Y_x, s(t) - x_0), \tag{3.8}$$

and

$$s' = \mathscr{A}_2(Y_x, s - x_0), \tag{3.9}$$

respectively. Moreover, direct computation yields

$$s(t) - x_0 = \mathscr{A}_3(Y(t, s(t))), \tag{3.10}$$

where $\mathscr{A}_3(0) = 0$ and

$$\frac{\partial \mathscr{A}_3}{\partial Y} = \frac{1}{\bar{\rho}_-(x_0) - \bar{\rho}_+(x_0)}.$$

Combining (3.8) and (3.10) together yields

$$\partial_t Y = \mathscr{A}_4(Y_x, Y), \quad \text{at } x = s(t), \tag{3.11}$$

where

$$\mathscr{A}_4(0,0) = 0, \quad \frac{\partial \mathscr{A}_4}{\partial Y_x} = \frac{c^2(\bar{\rho}_+)(x_0) - \bar{u}_+^2(x_0)}{2\bar{u}_+(x_0)}, \quad \frac{\partial \mathscr{A}_4}{\partial Y} = -\frac{\bar{E}_+(x_0)}{2\bar{u}_+(x_0)}.$$

Note that on the right boundary, $x = L$, Y satisfies

$$\partial_x Y = 0, \quad \text{at } x = L. \tag{3.12}$$

Our goal is to derive uniform estimates for Y and s which satisfy (3.6), (3.10) (3.11) and (3.12).

Introduce the transformation

$$\tilde{t} = t, \quad \tilde{x} = (L - x_0)\frac{x - s(t)}{L - s(t)} + x_0, \quad \sigma(\tilde{t}) = s(t) - x_0,$$

to transform the problem to the fixed domain $[x_0, L]$. Using this transformation, we have new system and boundary conditions on the fixed domain. The linearized problem for this transformed problem can be summarized as follows

$$\begin{cases} \mathscr{L}_0 Y = 0, \quad x_0 < x < L,\ t > 0, \\ \partial_x Y = \dfrac{2\bar{u}_+}{p'(\bar{\rho}_+) - \bar{u}_+^2}(x_0)\partial_x Y + \dfrac{\bar{E}_+}{p'(\bar{\rho}_+) - \bar{u}_+^2}(x_0)Y, \quad \text{at } x = x_0, \\ \partial_x Y = 0, \quad \text{at } x = L, \\ Y(0,x) = h_1(x),\ Y_t(0,x) = h_2(x),\ x_0 < x < L. \end{cases} \tag{3.13}$$

In this subsection, we study the linearized problem.

Theorem 7. *Assume that $\bar{E}_+$ satisfies (2.4). Let Y be a smooth solution of the linearized problem* (3.13)*. Then there exist $\alpha_0 \in (0,1)$ and $T > 0$ such that*

$$\hat{\varphi}_k(Y, t+T) < \alpha_0 \hat{\varphi}_k(Y,t) \qquad \text{for } t \geq 0, \tag{3.14}$$

where $\hat{\varphi}_k$ is defined as $\hat{\varphi}_k(Y,t) = \sum_{m=0}^k \varphi_m(Y,t)$, where

$$\begin{aligned} \varphi_m(Y,t) &= \frac{\bar{E}_+}{\bar{\rho}_+}(x_0)(\partial_t^m Y)^2(t,x_0) \\ &+ \int_{x_0}^L \frac{1}{\bar{\rho}_+}\left\{(\partial_t^{m+1}Y)^2 + \left(p'(\bar{\rho}_+) - \frac{\bar{J}_+^2}{\bar{\rho}_+^2}\right)(\partial_x \partial_t^m Y)^2 + \bar{\rho}_+ \partial_t^m Y^2\right\}(t,x)dx. \end{aligned}$$

The key ingredients for the proof of this theorem is that we first obtain an energy estimate by choosing suitable multiplier and then combine Rauch-Taylor type estimate and spectral analysis to give the decay of the energy of the problem.

Remark 10. *When $\bar{E}_+$ satisfies (2.4), it follows from the Sobolev inequality that there exists a constant $C > 0$ such that*

$$\varphi_m(Y,t) > C\int_{x_0}^L \left\{(\partial_t^{m+1}Y)^2 + (\partial_x \partial_t^m Y)^2 + \partial_t^m Y^2\right\}(t,x)dx.$$

This is the key reason that we can handle the case that $\bar{E}_+(x_0)$ is a negative number with small magnitude.

3.2 Stability of multidimensional subsonic flows

If we fix b as a constant $b_0 > 0$ in the equation (1.24) then the equations (1.23) and (1.24) become invariant under translation. So if the boundary data $\bar{\Phi}_{en}$, $\bar{\Phi}_{ex}$ and p_{ex} are all constants, then one may look for a solution (φ, Φ) as functions of x_n only for $x_n \in (0, L)$. We note that $\bar{\Phi}_{ex} = 0$ if $\bar{\Phi}_{ex}$ is a constant.

Proposition 8 (*One dimensional subsonic flow*)**.** *Fix constants $b_0 > 0$ and $L > 0$. Then there exists a nonempty set $\mathfrak{P}_0$ of parameters in $\mathbb{R}^2 \times \mathbb{R}^+$ so that for any $(\bar{\Phi}_{en,0}, \mathscr{B}_{0,0}, p_{ex,0}) \in \mathfrak{P}_0$, if $(\bar{\Phi}_{en}, \mathscr{B}_0, p_{ex}) = (\bar{\Phi}_{en,0}, \mathscr{B}_{0,0}, p_{ex,0})$ then the system of* (1.23) *and* (1.24) *in $\mathcal{N}$ with the boundary conditions* (1.28)–(1.32) *has a unique C^2 one-dimensional solution (φ, Φ) in $\overline{\mathcal{N}}$ satisfying the inequalities $\rho(\Phi, |\nabla\varphi|^2) > 0$ and $|\nabla\varphi|^2 < p'(\rho(\Phi, |\nabla\varphi|^2))$ in $\overline{\mathcal{N}}$.*

We fix $(\bar{\Phi}_{en,0}, \mathscr{B}_{0,0}, p_{ex,0}) \in \mathfrak{P}_0$, and let (φ_0, Φ_0) be the corresponding background solution. Let b, $(\bar{\Phi}_{en}, \bar{\Phi}_{ex}, p_{ex})$ satisfy the estimates (2.7) for $\sigma \in (0, \bar{\sigma})$ with $\bar{\sigma}$ to be determined later.

For $(z, \mathbf{q}) = (z, q_1, \cdots, q_n) \in \mathbb{R} \times \mathbb{R}^n$ and for ρ defined by (1.25), set

$$\mathbf{A}(z, \mathbf{q}) = (A_1, \cdots, A_n)(z, \mathbf{q}) = \rho(z, |\mathbf{q}|^2)\mathbf{q}, \quad B(z, \mathbf{q}) = \rho(z, |\mathbf{q}|^2). \tag{3.15}$$

Suppose that $(\varphi, \Phi) \in [C^2(\mathcal{N})]^2$ satisfy the equations in (1.23)-(1.24) in $\mathcal{N}$, and set

$$(\psi, \Psi) := (\varphi, \Phi) - (\varphi_0, \Phi_0). \tag{3.16}$$

Since (φ_0, Φ_0) also satisfies (1.23)-(1.24), one has

$$L_1(\psi, \Psi) := \mathrm{div}\left(\sum_{j=1}^{n} \partial_{q_j} A_i(\Phi_0, D\varphi_0)\partial_j\psi + \Psi\partial_z\mathbf{A}(\Phi_0, D\varphi_0)\right) = \mathrm{div}\mathbf{F}(\mathrm{x}, \Psi, D\psi) \tag{3.17}$$

and

$$L_2(\psi, \Psi) := \Delta\Psi - \partial_z B(\Phi_0, D\varphi_0)\Psi - \partial_{\mathbf{q}} B(\Phi_0, D\varphi_0) \cdot D\psi = f(\mathrm{x}, \Psi, D\psi) + b_0 - b(\mathrm{x}) \tag{3.18}$$

where $\mathbf{F}$ and f are higher order terms.

Furthermore, (ψ, Ψ) defined by (3.16) satisfy

$$\begin{aligned} &\psi = 0 \quad \text{on} \quad \Gamma_0, \\ &\Psi = \begin{cases} (\mathscr{B}_0 - \mathscr{B}_{0,0}) + (\bar{\Phi}_{en} - \bar{\Phi}_{en,0}) =: \Psi_{en} & \text{on } \Gamma_0 \\ (\mathscr{B}_0 - \mathscr{B}_{0,0}) + \bar{\Phi}_{ex} =: \Psi_{ex} & \text{on } \Gamma_L \end{cases}, \\ &\partial_{\mathbf{n}_w}\psi = \partial_{\mathbf{n}_w}\Psi = 0 \quad \text{on} \quad \Gamma_w \end{aligned} \tag{3.19}$$

$$p(B(\Phi_0 + \Psi, \nabla\varphi_0 + \nabla\psi)) - p(B(\Phi_0, \nabla\varphi_0)) = p_{ex} - p_{ex,0} \quad \text{on } \Gamma_{ex} \tag{3.20}$$

for $B(z, \mathbf{q})$ defined by (3.15).

Lemma 9. *The linear boundary value problem* (3.17)–(3.20) *has a unique weak solution $(v, W) \in [H^1(\mathcal{N})]^2$, and $(\tilde{v}, \tilde{W}) := (v, W) - (0, W_{bd})$ satisfy*

$$\|\tilde{v}\|_{H^1(\mathcal{N})} + \|\tilde{W}\|_{H^1(\mathcal{N})} \le C_H \tag{3.21}$$

for a constant C_H depending only on the data.

One of the key ingredients for the proof of this lemma is the following observation for the system,

$$\eta\partial_z \mathbf{A}(\Psi_0, D\varphi_0)\cdot D\xi + \eta\partial_{\mathbf{q}} B(\Psi_0, D\varphi_0)\cdot D\xi = 0 \quad \text{for any} \quad (\xi,\eta)\in\mathcal{H}. \tag{3.22}$$

3.2.1 Euler-Poisson system with voriticity

We apply the Helmholtz decomposition to rewrite the velocity field $\mathbf{u}$ as a summation of a gradient and a divergence free field. Given velocity field $\mathbf{u} = (u,v)$, denote curl$\mathbf{u} = \partial_{x_2}u - \partial_{x_1}v$. Then one can find a function ψ satisfying

$$\begin{cases} \Delta\psi = \text{curl}\,\mathbf{u} \ \ \tilde{x}in \quad \mathcal{N}, \\ \partial_{x_1}\psi = 0 \ \ \tilde{x}on \ \ \Gamma_0\cup\Gamma_L, \quad \psi = 0 \ \text{ on } \ \Gamma_w. \end{cases} \tag{3.23}$$

Then, it is easy to see that

$$\text{curl}(\mathbf{u} - \nabla^\perp\psi) = 0$$

where $\nabla^\perp\psi = (\partial_{x_2}\psi, -\partial_{x_1}\psi)$. Therefore, there exists a function φ satisfying $\nabla\varphi = \mathbf{u} - \nabla^\perp\psi$. Hence the velocity field $\mathbf{u}$ can be expressed as

$$\mathbf{u} = \nabla\varphi + \nabla^\perp\psi. \tag{3.24}$$

It follows from (1.5) and (3.24) that the density ρ can be written as

$$\rho = H(S, \mathscr{K} + \Phi - \frac{1}{2}|\nabla\varphi + \nabla^\perp\psi|^2) \tag{3.25}$$

where the function $H(S,\zeta)$ is defined by

$$H(S,\zeta) = \left[\frac{(\gamma-1)\zeta}{\gamma\mathfrak{p}\exp(\frac{S}{c_v})}\right]^{\frac{1}{\gamma-1}}. \tag{3.26}$$

The straightforward computations for (1.4) give

$$u(v_{x_1} - u_{x_2}) = T(\rho,S)S_{x_2} - \mathscr{K}_{x_2} \tag{3.27}$$

where T is the temperature defined by

$$T(\rho,S) = \frac{c_v\mathfrak{p}}{\gamma-1}\exp(\frac{S}{c_v})\rho^{\gamma-1}.$$

The system (1.4) can be written as the following nonlinear system for $(\varphi, \psi, \Phi, S, \mathscr{K})$:

$$\mathrm{div}\big(H(S, \mathscr{K} + \Phi - \frac{1}{2}|\nabla\varphi + \nabla^{\perp}\psi|^2)(\nabla\varphi + \nabla^{\perp}\psi)\big) = 0, \tag{3.28}$$

$$\Delta\Phi = H(S, \mathscr{K} + \Phi - \frac{1}{2}|\nabla\varphi + \nabla^{\perp}\psi|^2) - b, \tag{3.29}$$

$$\Delta\psi = -\frac{T(\rho, S)\partial_{x_2}S - \partial_{x_2}\mathscr{K}}{\partial_{x_1}\varphi + \partial_{x_2}\psi}, \tag{3.30}$$

$$\rho(\nabla\varphi + \nabla^{\perp}\psi)\cdot\nabla S = \rho(\nabla\varphi + \nabla^{\perp}\psi)\cdot\nabla\mathscr{K} = 0, \tag{3.31}$$

where $\rho = H(S, \mathscr{K} + \Phi - \frac{1}{2}|\nabla\varphi + \nabla^{\perp}\psi|^2)$.

The following lemma guarantees that there exists a solutions $\mathcal{W}$ of the problem

$$V\cdot\nabla\mathcal{W} = 0 \quad \text{in} \quad \mathcal{N}, \quad \mathcal{W} = \mathcal{W}_{en} \quad \text{on} \quad \Gamma_0, \tag{3.32}$$

where

$$\mathrm{div}V = 0 \quad \text{in} \quad \mathcal{N}, \quad V\cdot\mathbf{n}_w = 0 \quad \text{on} \quad \Gamma_w. \tag{3.33}$$

Lemma 10. *Suppose that a vector field* $\mathbf{V} = (V_1, V_2)$ *satisfies* (3.33) *and the estimate*

$$\|\mathbf{V}\|_{1,\alpha,\mathcal{N}}^{(-\alpha,\Gamma_0\cup\Gamma_L)} \leq K_0 \tag{3.34}$$

for a constant $K_0 > 0$. *In addition, assume that there exists a constant* $\nu^* > 0$ *satisfying*

$$V_1 \geq \nu^* \quad \textit{in} \quad \overline{\mathcal{N}}. \tag{3.35}$$

Then (3.32) *has a unique solution* $\mathcal{W} \in (C^{1,\alpha}(\overline{\mathcal{N}}))^2$ *satisfying*

$$\|\mathcal{W} - \mathcal{W}_0\|_{1,\alpha,\mathcal{N}} \leq C^*\|\mathcal{W}_{en} - \mathcal{W}_0\|_{1,\alpha,\Gamma_0}, \tag{3.36}$$

where the constant C^* *depends only on* L, ν^*, K_0 *and* α.

4 Future work

In order to study the structural stability of transonic shock solutions under multidimensional perturbations, we will first investigate the structural stability of supersonic solutions for the Euler-Poisson system. Furthermore, since the Euler-Poisson system has nontrivial one dimensional smooth transonic solutions, we would like to study the stability of these smooth transonic solution. Finally, the stability of transonic shock solutions is a significant problem to be studied.

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