数理解析研究所講究録 第2107巻 2019年 125-138

A new existence proof for gravity-capillary solitary water waves

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1 Introduction

The classical water-wave problem concerns the two-dimensional, irrotational flow of a perfect fluid of unit density subject to the forces of gravity and surface tension. In dimensionless coordinates the fluid occupies the domain $D_\eta = \{(x,y) : x \in \mathbb{R},\, y \in (0, 1+\eta(x,t))\}$, where (x,y) are the usual Cartesian coordinates and $\eta > -1$ is a function of the spatial coordinate x and time t. In terms of an Eulerian velocity potential $\varphi(x,y,t)$, the mathematical problem is to solve Laplace's equation

$$\varphi_{xx} + \varphi_{yy} = 0, \qquad 0 < y < 1+\eta, \tag{1}$$

with boundary conditions

$$\varphi_y = 0, \qquad y = 0, \tag{2}$$

$$\eta_t = \varphi_y - \eta_x\varphi_x, \qquad y = 1+\eta, \tag{3}$$

$$\varphi_t = -\tfrac{1}{2}\varphi_x^2 - \tfrac{1}{2}\varphi_y^2 - \eta + \beta\left[\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right]_x, \qquad y = 1+\eta, \tag{4}$$

in which $\beta > 0$ is a dimensionless constant called the *Bond number*. Equation (2) is the condition that water cannot permeate the rigid horizontal boundary at $y = 0$, while (3), (4) are respectively the kinematic and dynamic conditions at the free surface. *Travelling waves* are solutions of the form $\eta(x,t) = \eta(x-ct)$, $\varphi(x,y,z) = \varphi(x-ct,y)$, while *solitary waves* are non-trivial travelling waves which satisfy the asymptotic conditions $\eta(x) \to 0$ as $x \to \pm\infty$; they correspond to localised disturbances of permanent form which move from left to right with constant speed c.

Let us focus on strong surface tension ($\beta > 1/3$). In the classical weakly nonlinear approach one makes the Ansatz

$$c^2 = 1 - \varepsilon^2, \qquad \eta(x) = \varepsilon^2\rho(\varepsilon x)$$

for travelling water waves and finds that to leading order ρ satisfies the Korteweg-de Vries equation

$$\rho - (\beta - \tfrac{1}{3})\rho_{xx} + \tfrac{3}{2}\rho^2 = 0; \tag{5}$$

this equation admits an explicit solitary wave of depression given by the formula

$$\rho^\star(x) = -\mathrm{sech}^2\left(\frac{x}{2(\beta - \frac{1}{3})^{1/2}}\right)$$

(see Benjamin [1]). The use of (5) to predict the existence of solitary waves of depression was rigorously justified by Kirchgässner [5]. Kirchgässner's method is based upon sophisticated spatial dynamics and centre-manifold reduction techniques (and has subsequently been refined by several authors). This note presents an alternative proof which is elementary in the sense that its main ingredients are the contraction-mapping principle and implicit-function theorem.

It is possible to formulate the water-wave problem (1)–(4) in terms of the variables η and $\Phi = \varphi|_{y=\eta}$ (see Zakharov [6] and Craig & Sulem [3]). The Zakharov-Craig-Sulem formulation of the water-wave problem is

$$\begin{aligned} &\eta_t - G(\eta)\Phi = 0, \\ &\Phi_t - \beta\left[\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right]_x + \eta + \frac{1}{2}\Phi_x^2 - \frac{(G(\eta)\Phi + \eta_x\Phi_x)^2}{2(1+\eta_x^2)} = 0, \end{aligned}$$

where $G(\eta)\Phi = \varphi_y - \eta_x\varphi_x\big|_{y=\eta}$ and φ is the (unique) solution of the boundary-value problem

$$\begin{aligned} &\varphi_{xx} + \varphi_{yy} = 0, && 0 < y < 1 + \eta, \\ &\varphi = \Phi, && y = 1 + \eta, \\ &\varphi_y = 0, && y = 0. \end{aligned}$$

Travelling waves are solutions of the form $\eta(x,t) = \eta(x - ct)$, $\Phi(x,t) = \Phi(x - ct)$; they satisfy

$$-c\eta_x - G(\eta)\Phi = 0, \tag{6}$$

$$-c\Phi_x - \beta\left[\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right]_x + \eta + \frac{1}{2}\Phi_x^2 - \frac{(G(\eta)\Phi + \eta_x\Phi_x)^2}{2(1+\eta_x^2)} = 0. \tag{7}$$

Using (6), one finds that $\Phi = -cG(\eta)^{-1}\eta_x$, and inserting this formula into (7) yields the equation

$$\mathcal{K}(\eta) - c^2\mathcal{L}(\eta) = 0, \tag{8}$$

where

$$\mathcal{K}(\eta) = -\beta\left[\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right]_x + \eta, \quad \mathcal{L}(\eta) = -\frac{1}{2}(K(\eta)\eta)^2 + \frac{(\eta_x - \eta_x K(\eta)\eta)^2}{2(1+\eta_x^2)} + K(\eta)\eta = 0 \tag{9}$$

and $K(\eta)\xi = -(G(\eta)^{-1}\xi_x)_x$. Note the equivalent definition $K(\eta)\xi = -(\varphi|_{y=1+\eta})_x$, where φ is the solution of the boundary-value problem

$$\varphi_{xx} + \varphi_{yy} = 0, \qquad 0 < y < 1 + \eta, \tag{10}$$

$$\varphi_y - \eta_x \varphi_x = \xi_x, \qquad y = 1 + \eta, \tag{11}$$

$$\varphi_y = 0, \qquad y = 0 \tag{12}$$

(which is unique up to an additive constant); the operator K is carefully studied in Section 2 below.

The key to the existence theory in the present paper is a splitting of η into two parts. The dominant part η_1 has spectrum near the origin, and thus corresponds to a long wave; it satisfies a perturbation of the Korteweg-de Vries equation. The spectrum of the secondary part η_2 is on the other hand bounded away from the origin, and it can be determined as a function of η_1. To this end, denote the Fourier transform $\mathcal{F}(\eta)$ of η by $\hat{\eta}$, let χ be the characteristic function of the set $B_\delta(0)$ and define

$$\eta_1 = \chi(D)\eta, \qquad \eta_2 = (1 - \chi(D))\eta,$$

where $m(D)$ is the Fourier-multiplier operator induced by the bounded function m (so that $\mathcal{F}(m(D)\eta) = m\hat{\eta}$). It follows that the support of $\hat{\eta}_1$ is contained in the neighbourhood $B_\delta(0)$ of the origin, while the support of $\hat{\eta}_2$ lies outside this set. Writing $c^2 = 1 - \varepsilon^2$ and decomposing (8) into

$$\chi(D)(\mathcal{K}(\eta_1+\eta_2) - (1-\varepsilon^2)\mathcal{L}(\eta_1+\eta_2) = 0, \quad (1-\chi(D))(\mathcal{K}(\eta_1+\eta_2) - (1-\varepsilon^2)\mathcal{L}(\eta_1+\eta_2) = 0,$$

one finds that the second equation can be solved for η_2 as a function of η_1 for sufficiently small values of ε; substituting $\eta_2 = \eta_2(\eta_1)$ into the first yields the reduced equation

$$\chi(D)(\mathcal{K}(\eta_1 + \eta_2(\eta_1)) - (1 - \varepsilon^2)\mathcal{L}(\eta_1 + \eta_2(\eta_1)) = 0$$

for η_1. Finally, the scaling

$$\eta_1(x) = \varepsilon^2 \rho(\varepsilon x)$$

transforms the reduced equation into a perturbation of (5) (see Sections 4–6).

The existence theory is completed in Section 6, where it is demonstrated that the reduced equation for ρ indeed has a solution which is a perturbation of the Korteweg-de Vries solitary wave of depression. The key step is a nondegeneracy result for the solitary-wave solution of (5) which allows one to apply a suitable version of the implicit-function theorem.

2 The operator K

The boundary-value problem (10)–(12) is handled using the change of variable

$$y' = \frac{y}{1+\eta}, \qquad u(x, y') = \varphi(x, y),$$

which maps $\Sigma_\eta = \{(x,y)\colon x \in \mathbb{R}, 0 < y < 1+\eta(x)\}$ to the strip $\Sigma = \mathbb{R} \times (0,1)$. Dropping the primes, one finds that (10)–(12) are transformed into

$$u_{xx} + u_{yy} = \partial_x F_1(\eta, u) + \partial_y F_3(\eta, u), \qquad 0 < y < 1, \tag{13}$$

$$u_y = 0, \qquad y = 0, \tag{14}$$

$$u_y = F_3(\eta, u) + \xi_x, \qquad y = 1, \tag{15}$$

where

$$F_1(\eta, u) = -\eta u_x + y\eta_x u_y, \qquad F_3(\eta, u) = \frac{\eta u_y}{1+\eta} + y\eta_x u_x - \frac{y^2}{1+\eta}\eta_x^2 u_y,$$

and $K(\eta)\xi = -u_x|_{y=1}$. We study this boundary-value problem in the spaces

$$\mathcal{Z} = \{\eta \in \mathcal{S}'(\mathbb{R}) : \|\eta\|_{\mathcal{Z}} := \|\hat{\eta}_1\|_{L^1(\mathbb{R})} + \|\eta_2\|_2 < \infty\}$$

and

$$H^2_\star(\Sigma) = \{u \in L^2_{\mathrm{loc}}(\Sigma) : \|u\|^2_{\star,2} := \|u_x\|_1^2 + \|u_y\|_1^2 < \infty\}$$

for η and u.

Lemma 1 *For each $\xi \in H^{3/2}(\mathbb{R})$ and sufficiently small $\eta \in \mathcal{Z}$ the boundary-value problem* (13)–(15) *admits a solution u which is unique up to an additive constant and satisfies $u \in H^2_\star(\Sigma)$. Furthermore, the mapping $\mathcal{Z} \to \mathcal{L}(H^{3/2}(\mathbb{R}), H^2_\star(\Sigma))$ given by $\eta \mapsto (\xi \mapsto u)$ is analytic at the origin.*

Proof. For each F_1, $F_3 \in H^1(\Sigma)$ and $\xi \in H^{3/2}(\mathbb{R})$ the equations

$$u_{xx} + u_{yy} = \partial_x F_1 + \partial_y F_3, \qquad 0 < y < 1, \tag{16}$$

$$u_y = F_3(\eta, u), \qquad y = 0, \tag{17}$$

$$u_y = F_3(\eta, u) + \xi_x, \qquad y = 1, \tag{18}$$

admit a unique solution $u = U(F_1, F_3, \xi)$ given by the explicit formula

$$U(F_1, F_3, \xi) = \mathcal{F}^{-1}\left[\int_0^1 \left(\mathrm{i}kG(y,\tilde{y})\hat{F}_1 - G_{\tilde{y}}(y,\tilde{y})\hat{F}_3\right)\,\mathrm{d}\tilde{y}\right],$$

in which

$$G(y,\tilde{y}) = \begin{cases} -\dfrac{\cosh|k|y\cosh|k|(1-\tilde{y})}{|k|\sinh|k|}, & 0 \le y \le \tilde{y} \le 1, \\[2ex] -\dfrac{\cosh|k|\tilde{y}\cosh|k|(1-y)}{|k|\sinh|k|}, & 0 \le \tilde{y} \le y \le 1; \end{cases}$$

it follows from this formula that

$$\|U(F_1, F_3, \xi)\|_{2,\star} \lesssim \|F_1\|_1 + \|F_3\|_1 + \|\xi\|_{3/2}$$

(cf. Buffoni, Groves, Wahlén & Sun [2, Appendix A]).

Define

$$T : H^2_\star(\Sigma) \times \mathcal{Z} \times H^{3/2}(\mathbb{R}) \to H^2_\star(\Sigma)$$

by

$$T(u, \eta, \xi) = u - U(F_1(\eta, u), F_3(\eta, u), \xi),$$

and note that the solutions of (13)–(15) are precisely the zeros of $T(\cdot, \eta, \xi)$. Using the estimates

$$\begin{aligned}\|\eta^n w\|_{H^1(\Sigma)} &\lesssim \|\eta\|^n_{1,\infty}\|w\|_{H^1(\Sigma)} \\ &\lesssim (\|\eta_1\|_{1,\infty} + \|\eta_2\|_2)^n\|w\|_{H^1(\Sigma)},\end{aligned}$$

$$\begin{aligned}\|y\eta_x w\|_{H^1(\Sigma)} &\lesssim (\|\eta_{1x}\|_{1,\infty}\|w\|_{H^1(\Sigma)} + \|\eta_{2x}w\|_{L^2(\Sigma)} + \|\eta_{2x}w_x\|_{L^2(\Sigma)} + \|\eta_{2xx}w\|_{L^2(\Sigma)}) \\ &\lesssim (\|\eta_{1x}\|_{1,\infty} + \|\eta_{2x}\|_\infty)\|w\|_{H^1(\Sigma)} + \|\eta_{2xx}\|_0\|w\|_{H^1(\Sigma)} \\ &\lesssim (\|\eta_{1x}\|_{1,\infty} + \|\eta_2\|_2)\|w\|_{H^1(\Sigma)},\end{aligned}$$

$$\begin{aligned}\|y^2\eta^n\eta_x^2 w\|_{H^1(\Sigma)} &\lesssim \|\eta\|^n_{1,\infty}(\|\eta_{1x}\|^2_{1,\infty}\|w\|_{H^1(\Sigma)} + \|\eta^2_{2x}w\|_{L^2(\Sigma)} + \|\eta^2_{2x}w_x\|_{L^2(\Sigma)} + \|\eta_{2x}\eta_{2xx}w\|_{L^2(\Sigma)}) \\ &\lesssim \|\eta\|^n_{1,\infty}((\|\eta_{1x}\|_{1,\infty} + \|\eta_{2x}\|_\infty)^2\|w\|_{H^1(\Sigma)} + \|\eta_{2x}\|_\infty\|\eta_{2xx}\|_0\|w\|_{H^1(\Sigma)}) \\ &\lesssim (\|\eta_1\|_{2,\infty} + \|\eta_2\|_2)^{n+2}\|w\|_{H^1(\Sigma)}\end{aligned}$$

and

$$\|\eta_1\|_{2,\infty} + \|\eta_2\|_2 \lesssim \|\hat{\eta}_1\|_{L^1(\mathbb{R})} + \|\eta_2\|_2 = \|\eta\|_{\mathcal{Z}}$$

(uniformly in n), one finds that the mappings $H^2_\star(\Sigma) \times \mathcal{Z} \to H^1(\Sigma)$ given by $(\eta, u) \mapsto F_1(\eta, u)$ and $(\eta, u) \mapsto F_1(\eta, u)$ are analytic at the origin; it follows that T is also analytic at the origin. Furthermore $T(0, 0, 0) = 0$ and $\mathrm{d}_1T[0, 0, 0] = I$ is an isomorphism. By the analytic implicit-function theorem there exist open neighbourhoods N_1 and N_2 of the origin in $\mathcal{Z}$ and $H^{3/2}(\mathbb{R})$ and an analytic function $v : N_1 \times N_2 \to H^2_\star(\Sigma)$ such that

$$T(v(\eta, \xi), \eta, \xi) = 0.$$

Since v is linear in ξ one can take N_2 to be the whole space $H^{3/2}(\mathbb{R})$. □

Corollary 2 *The mapping* $K(\cdot)\colon \mathcal{Z} \to \mathcal{L}(H^{3/2}(\mathbb{R}), H^{1/2}(\mathbb{R}))$ *is analytic at the origin.*

Corollary 3 *The formulae* (9) *define functions* $\mathcal{K}$, $\mathcal{L} : \mathcal{Z} \to L^2(\mathbb{R})$ *which are analytic at the origin and satisfy* $\mathcal{K}(0) = \mathcal{L}(0) = 0$.

3 Taylor expansions

In the obvious notation, write

$$\mathcal{K}(\eta) = \sum_{j=0}^{\infty} \mathcal{K}_{2j+1}(\eta), \qquad \mathcal{K}_{\mathrm{r}}(\eta) = \sum_{j=1}^{\infty} \mathcal{K}_{2j+1}(\eta),$$

and note that

$$\mathcal{K}_1(\eta) = \eta - \beta\eta_{xx}.$$

Similarly, write

$$K(\eta) = \sum_{j=0}^{\infty} K_j(\eta), \quad K_{\mathrm{nl}}(\eta) = \sum_{j=1}^{\infty} K_j(\eta), \quad K_{\mathrm{r}}(\eta) = \sum_{j=2}^{\infty} K_j(\eta).$$

Proposition 4 *One has the explicit representations*

$$K_0\xi = |D| \coth |D|\xi, \qquad K_1(\eta)\xi = -(\eta\xi_x)_x - K_0(\eta K_0\xi).$$

Proof. Note that $K_0\xi = -u_x^0|_{y=1}$, $K_1\xi = -u_x^1|_{y=1}$, where

$$\begin{aligned}
&u_{xx}^0 + u_{yy}^0 = 0, && u_{xx}^1 + u_{yy}^1 = (-\eta u_x^0 + y\eta_x u_y^0)_x + (\eta u_y^0 + y\eta_x u_x^0)_y, && 0 < y < 1,\\
&u_y^0 = 0, && u_y^1 = 0, && y = 0,\\
&u_y^0 = \xi_x, && u_y^1 = \eta u_y^0 + \eta_x u_x^0, && y = 1,
\end{aligned}$$

so that

$$\widehat{u^0} = \frac{\cosh(|k|y)}{|k| \sinh |k|}\hat{\xi}$$

and $u^1 = y\eta u_y^0 + v^1$, where

$$\begin{aligned}
&v_{xx}^1 + v_{yy}^1 = 0, && 0 < y < 1,\\
&v_y^1 = 0, && y = 0,\\
&v_y^1 = (\eta u_x^0)_x, && y = 1,
\end{aligned}$$

so that

$$\widehat{u^1} = \frac{\cosh(|k|y)}{|k| \sinh |k|}\widehat{\eta u_x^0} = -\frac{\cosh(|k|y)}{|k| \sinh |k|}\widehat{\eta K_0 \xi}.$$

□

Finally, write

$$\mathcal{L}(\eta) = \sum_{n=1}^{\infty} \mathcal{L}_n(\eta), \qquad \mathcal{L}_{\mathrm{r}}(\eta) = \sum_{n=3}^{\infty} \mathcal{L}_n(\eta),$$

and note that

$$\mathcal{L}_1(\eta) = K_0\eta, \qquad \mathcal{L}_2(\eta) = -\frac{1}{2}(K_0\eta)^2 + \frac{1}{2}\eta_x^2 + K_1(\eta)\eta = \frac{1}{2}\eta_x^2 - \frac{1}{2}(\eta^2)_{xx} - \frac{1}{2}(K_0\eta)^2 - K_0(\eta K_0\eta);$$

clearly

$$\mathcal{L}_2(\eta) = m(\{\eta\}^2), \quad \mathrm{d}\mathcal{L}_2[\eta](v) = 2m(\eta, v),$$

where

$$m(u,v) = \frac{1}{2}\big(u_x v_x - (K_0 u)(K_0 v) - (uv)_{xx} - K_0(uK_0 v + vK_0 u)\big).$$

Proposition 5 *The estimate $\|m(u,v)\|_0 \lesssim \|u\|_{\mathcal{Z}}\|v\|_2$ holds for each u, $v \in H^2(\mathbb{R})$.*

Proof. Estimate

$$\begin{aligned}
\|u_x v_x\|_0 &\lesssim (\|u_{1x}\|_\infty + \|u_{2x}\|_\infty)\|v_x\|_0 \lesssim (\|\hat{u}_1\|_{L^1(\mathbb{R})} + \|u_2\|_2)\|v\|_2 = \|u\|_{\mathcal{Z}}\|v\|_2,\\
\|K_0 u K_0 v\|_0 &\lesssim (\|K_0 u_1\|_\infty + \|K_0 u_2\|_\infty)\|K_0 v\|_0 \lesssim (\|\hat{u}_1\|_{L^1(\mathbb{R})} + \|K_0 u_2\|_1)\|v\|_1 \lesssim \|u\|_{\mathcal{Z}}\|v\|_2,\\
\|(uv)_{xx}\|_0 &\lesssim \|uv\|_2 \lesssim (\|u_1\|_{2,\infty} + \|u_2\|_2)\|v\|_2 \lesssim \|u\|_{\mathcal{Z}}\|v\|_2,\\
\|K_0(uK_0 v)\|_0 &\lesssim \|uK_0 v\|_1 \lesssim (\|u_1\|_{1,\infty} + \|u_2\|_1)\|v\|_1 \lesssim \|u\|_{\mathcal{Z}}\|v\|_2,\\
\|K_0(vK_0 u)\|_0 &\lesssim \|vK_0 u\|_1 \lesssim (\|K_0 u_1\|_{1,\infty} + \|K_0 u_2\|_1)\|v\|_1 \lesssim \|u\|_{\mathcal{Z}}\|v\|_2,
\end{aligned}$$

where the inequality

$$|k| \coth |k| \lesssim 1 + |k|$$

has been used. □

The next lemma gives estimates for $\mathcal{K}_{\mathrm{r}}(\eta)$ and $\mathcal{L}_{\mathrm{r}}(\eta)$ for $\eta \in U$, where

$$U = \{\eta \in H^2(\mathbb{R}) : \|\eta\|_{\mathcal{Z}} < M\}$$

and M is a sufficiently small positive constant (note that U is an open neighbourhood of the origin in $H^2(\mathbb{R})$ since $H^2(\mathbb{R})$ is continuously embedded in $\mathcal{Z}$).

Lemma 6 *The estimates*

$$\begin{aligned}
\|\mathcal{K}_{\mathrm{r}}(\eta)\|_0, \|\mathcal{L}_{\mathrm{r}}(\eta)\|_0 &\lesssim \|\eta\|_{\mathcal{Z}}^2\|\eta\|_2,\\
\|\mathrm{d}\mathcal{K}_{\mathrm{r}}[\eta](v)\|_0, \|\mathrm{d}\mathcal{L}_{\mathrm{r}}[\eta](v)\|_0 &\lesssim \|\eta\|_{\mathcal{Z}}^2\|v\|_2 + \|\eta\|_{\mathcal{Z}}\|\eta\|_2\|v\|_{\mathcal{Z}}
\end{aligned}$$

hold for each $\eta \in U$ and v, $w \in H^2(\mathbb{R}^2)$.

Proof. Note that

$$\mathcal{L}_{\mathrm{r}}(\eta) = -K_0\eta K_{\mathrm{nl}}(\eta)\eta - \frac{1}{2}(K_{\mathrm{nl}}(\eta)\eta)^2 + K_{\mathrm{r}}(\eta)\eta - \frac{\eta_x^4}{2(1+\eta_x^2)} + \frac{\eta_x^2}{2(1+\eta_x^2)}\big((K(\eta)\eta)^2 - 2K(\eta)\eta\big)$$

and examine this formula and its derivatives as above, using the further estimates

$$\|K(\eta)\eta\|_{1/2} \lesssim \|\eta\|_{3/2}, \quad \|K_{\mathrm{nl}}(\eta)\eta\|_{1/2} \lesssim \|\eta\|_{\mathcal{Z}}\|\eta\|_{3/2}, \quad \|K_{\mathrm{r}}(\eta)\eta\|_{1/2} \lesssim \|\eta\|_{\mathcal{Z}}^2\|\eta\|_{3/2},$$

$$\|\mathrm{d}K[\eta](v)\eta\|_{1/2}, \|\mathrm{d}K_{\mathrm{nl}}[\eta](v)\eta\|_{1/2} \lesssim \|v\|_{\mathcal{Z}}\|\eta\|_{3/2}, \quad \|\mathrm{d}K_{\mathrm{r}}[\eta](v)\eta\|_{1/2} \lesssim \|\eta\|_{\mathcal{Z}}\|v\|_{\mathcal{Z}}\|\eta\|_{3/2}.$$

The corresponding estimates for $\mathcal{K}_{\rm r}$ are obtained by examining the explicit formula

$$\mathcal{K}_{\rm r}(\eta) = \left(1 - \frac{1}{(1+\eta_x^2)^{3/2}}\right)\eta_{xx}. \qquad \square$$

4 The reduction procedure

Write $c^2 = 1 - \varepsilon^2$, decompose $\mathcal{X} = H^2(\mathbb{R}^2)$ into the direct sum of $\mathcal{X}_1 = \chi(D)\mathcal{X}$ and $\mathcal{X}_2 = (1-\chi(D))\mathcal{X}$ and observe that $\eta \in U$ satisfies (8) if and only if

$$g(D)\eta_1 + \varepsilon^2 K_0\eta_1 + \chi(D)\mathcal{N}(\eta_1+\eta_2) = 0, \tag{19}$$

$$g(D)\eta_2 + \varepsilon^2 K_0\eta_2 + (1-\chi(D))\mathcal{N}(\eta_1+\eta_2) = 0, \tag{20}$$

where $g(k) = 1 + \beta|k|^2 - |k|\coth|k|$ and

$$\mathcal{N}(\eta) := \mathcal{K}_{\rm r}(\eta) - (1-\varepsilon^2)(\mathcal{L}_2(\eta) + \mathcal{L}_{\rm r}(\eta)).$$

Equation (20) may be written in the form

$$\eta_2 = (1-\chi(D))g(D)^{-1}\mathcal{A}(\eta_1,\eta_2) \tag{21}$$

with

$$\mathcal{A}(\eta_1,\eta_2) = \varepsilon^2 K_0\eta_2 + \mathcal{N}(\eta_1+\eta_2). \tag{22}$$

Proposition 7 *The mapping $(1-\chi(D))g(D)^{-1}$ defines a bounded linear operator $L^2(\mathbb{R}^2) \to H^2(\mathbb{R}^2)$.*

Proof. Note that $g(k) \gtrsim |k|^2$ for $|k| \geq \delta$. $\square$

We proceed by solving (21) for η_2 as a function of η_1 using the following following fixed-point theorem, which is proved by a straightforward application of the contraction mapping principle.

Theorem 8 *Let $\mathcal{Y}_1$, $\mathcal{Y}_2$ be Banach spaces, Y_1, Y_2 be closed sets in, respectively, $\mathcal{Y}_1$, $\mathcal{Y}_2$ containing the origin and $G\colon Y_1 \times Y_2 \to \mathcal{Y}_2$ be a smooth function. Suppose that there exists a continuous function $r\colon Y_1 \to [0,\infty)$ such that*

$$\|G(y_1,0)\| \leq \tfrac{1}{2}r, \quad \|\mathrm{d}_2G[y_1,y_2]\| \leq \tfrac{1}{3}$$

for each $y_2 \in \bar{B}_r(0) \subseteq Y_2$ and each $y_1 \in Y_1$.

Under these hypotheses there exists for each $y_1 \in Y_1$ a unique solution $y_2 = y_2(y_1)$ of the fixed-point equation $y_2 = G(y_1,y_2)$ satisfying $y_2(y_1) \in \bar{B}_r(0)$. Moreover $y_2(y_1)$ is a smooth function of $y_1 \in Y_1$ and in particular satisfies the estimate

$$\|\mathrm{d}y_2[y_1]\| \leq 2\|\mathrm{d}_1G[y_1,y_2(y_1)]\|.$$

We apply Theorem 8 to equation (21) with $\mathcal{Y}_1 = \mathcal{X}_1$, $\mathcal{Y}_2 = \mathcal{X}_2$, equipping $\mathcal{X}_1$ with the scaled norm

$$|||\eta||| := \left(\int_{\mathbb{R}} \left(1 + \varepsilon^{-2}(\beta - \tfrac{1}{3})k^2\right) |\hat{\eta}(k)|^2 \,\mathrm{d}k\right)^{1/2}$$

and $\mathcal{X}_2$ with the usual norm for $H^2(\mathbb{R})$, and taking $Y_1 = X_1$, $Y_2 = X_2$, where

$$X_1 = \{\eta_1 \in \mathcal{X}_1 \colon |||\eta_1||| \leq R_1\}, \qquad X_2 = \{\eta_2 \in \mathcal{X}_2 \colon \|\eta_2\|_2 \leq R_2\};$$

the function G is given by the right-hand side of (21). Using the following proposition one can guarantee that $\|\hat{\eta}_1\|_{L^1(\mathbb{R}^2)} < M/2$ for all $\eta_1 \in X_1$ for an arbitrarily large value of R_1; the value of R_2 is then constrained by the requirement that $\|\eta_2\|_2 < M/2$ for all $\eta_2 \in X_2$.

Proposition 9 *The estimate* $\|\hat{\eta}_1\|_{L^1(\mathbb{R}^2)} \lesssim \varepsilon^{1/2}|||\eta_1|||$ *holds for each* $\eta_1 \in \mathcal{X}_1$.

Proof. Observe that

$$\int_{\mathbb{R}} |\hat{\eta}_1(k)| \,\mathrm{d}k_1 \,\mathrm{d}k_2 = \int_{\mathbb{R}} \frac{(1+\varepsilon^{-2}k^2)^{1/2}}{(1+\varepsilon^{-2}k^2)^{1/2}} |\hat{\eta}_1(k)| \,\mathrm{d}k \lesssim |||\eta_1||| I_1^{1/2},$$

where

$$I_1 = \int_{\operatorname{supp}\chi} \frac{1}{1+\varepsilon^{-2}k^2} \,\mathrm{d}k = 2\varepsilon \int_0^{\delta/\varepsilon} \frac{1}{1+s^2} \,\mathrm{d}s \leq 2\varepsilon \int_0^{\infty} \frac{1}{1+s^2} \,\mathrm{d}s = 2\pi\varepsilon. \qquad \square$$

The next step is estimate each term appearing in the formula for $\mathcal{A}$; note in particular that

$$\|\eta\|_{\mathcal{Z}} \lesssim \varepsilon^{1/2}|||\eta_1||| + \|\eta_2\|_2, \qquad \|\eta\|_3 \lesssim |||\eta_1||| + \|\eta_2\|_2$$

for each $\eta \in H^2(\mathbb{R})$.

Lemma 10 *The estimates*

(i) $\|\mathcal{A}(\eta_1, \eta_2)\|_0 \lesssim \varepsilon^{1/2}|||\eta_1|||^2 + \varepsilon^{1/2}|||\eta_1||| \|\eta_2\|_2 + |||\eta_1||| \|\eta_2\|_2^2 + \|\eta_2\|_2^2 + \varepsilon^2\|\eta_2\|_2$,

(ii) $\|\mathrm{d}_1\mathcal{A}[\eta_1, \eta_2]\|_{\mathcal{L}(\mathcal{X}_1, L^2(\mathbb{R}))} \lesssim \varepsilon^{1/2}|||\eta_1||| + \varepsilon^{1/2}\|\eta_2\|_2 + \|\eta_2\|_2^2$,

(ii) $\|\mathrm{d}_2\mathcal{A}[\eta_1, \eta_2]\|_{\mathcal{L}(\mathcal{X}_2, L^2(\mathbb{R}))} \lesssim \varepsilon^{1/2}|||\eta_1||| + |||\eta_1||| \|\eta_2\|_2 + \|\eta_2\|_2 + \varepsilon^2$,

hold for each $\eta_1 \in X_1$ *and* $\eta_2 \in X_2$.

Theorem 11 *Equation* (21) *has a unique solution* $\eta_2 \in X_2$ *which depends smoothly upon* $\eta_1 \in X_1$ *and satisfies the estimates*

$$\|\eta_2(\eta_1)\|_2 \lesssim \varepsilon^{1/2}|||\eta_1|||^2, \qquad \|\mathrm{d}\eta_2[\eta_1]\|_{\mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)} \lesssim \varepsilon^{1/2}|||\eta_1|||.$$

Proof. Choosing R_2 and ε sufficiently small, one finds $r > 0$ such that $\|G(\eta_1, 0)\|_2 \le r/2$ and $\|\mathrm{d}_2 G[\eta_1, \eta_2]\|_{\mathcal{L}(\mathcal{X}_2,\mathcal{X}_2)} \le 1/3$ for $\eta_1 \in X_1$, $\eta_2 \in X_2$, and Theorem 8 asserts that equation (21) has a unique solution $\eta_2 \in X_2$ which depends smoothly upon $\eta_1 \in X_1$. More precise estimates are obtained by choosing $C > 0$ so that $\|G(\eta_1, 0)\|_2 \le C\varepsilon^{1/2}|||\eta_1|||^2$ for $\eta_1 \in X_1$ and writing $r(\eta) = 2C\varepsilon^{1/2}|||\eta_1|||^2$, so that

$$\|\mathrm{d}_1 G[\eta_1, \eta_2]\|_{\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)} \lesssim \varepsilon^{1/2}|||\eta_1|||, \quad \|\mathrm{d}_2 G[\eta_1, \eta_2]\|_{\mathcal{L}(\mathcal{X}_2,\mathcal{X}_2)} \lesssim 1$$

for $\eta_1 \in X_1$ and $\eta_2 \in \overline{B}_{r(\eta_1)}(0) \subseteq X_2$, and the stated estimates for $\eta_2(\eta_1)$ follow from Theorem 8. □

5 The reduced equation for η_1

The next step is to show that the reduced equation for η_1 is given by

$$g(D)\eta_1 + \varepsilon^2\eta_1 + \chi(D)\left[\tfrac{3}{2}\eta_1^2 + \underline{\mathcal{O}}(\varepsilon^{5/2}|||\eta_1|||^2) + \underline{\mathcal{O}}(\varepsilon|||\eta_1|||^3)\right] = 0,$$

where the symbol $\underline{\mathcal{O}}(\varepsilon^\gamma|||\eta_1|||^r)$ (with $\gamma \ge 0$, $r \ge 1$) denotes a smooth function $\mathcal{R} : X_1 \to L^2(\mathbb{R})$ which satisfies the estimates

$$\|\mathcal{R}(\eta_1)\|_0 \lesssim \varepsilon^\gamma|||\eta_1|||^r, \quad \|\mathrm{d}\mathcal{R}[\eta_1]\|_{\mathcal{L}(\mathcal{X}_1,L^2(\mathbb{R}))} \lesssim \varepsilon^\gamma|||\eta_1|||^{r-1}$$

for each $\eta \in X_1$.

Proposition 12 *The estimates*

$$\|\eta_{1x}\|_0 = \mathcal{O}(\varepsilon|||\eta_1|||), \qquad \|K_0\eta_1\|_0 = \eta_1 + \mathcal{O}(\varepsilon|||\eta_1|||)$$

hold for each $\eta_1 \in X_1$.

Proof. Note that

$$\|\eta_{1x}\|_0^2 = \||k|\hat{\eta}_1\|_0^2 \le \varepsilon^2|||\eta_1|||^2$$

and

$$\|(K_0 - I)\eta_1\|_0^2 = \|(|k|\coth|k| - 1)\,\hat{\eta}_1\|_0^2 \lesssim \||k|^2\hat{\eta}_1\|_0^2 \lesssim \varepsilon^2|||\eta_1|||^2.$$ □

Lemma 13 *The estimate*

$$\mathcal{L}_2(\eta_1 + \eta_2(\eta_1)) = -\tfrac{3}{2}\eta_1^2 + \underline{\mathcal{O}}(\varepsilon^{3/2}|||\eta_1|||^2) + \underline{\mathcal{O}}(\varepsilon|||\eta_1|||^3)$$

holds for each $\eta_1 \in X_1$.

Proof. Using Proposition 5 and Theorem 11, one finds that

$$\mathcal{L}_2(\eta_1+\eta_2(\eta_1)) = m(\{\eta_1\}^2) + 2m(\eta_1,\eta_2(\eta_1)) + m(\{\eta_2(\eta_1)\}^2) = m(\{\eta_1\}^2) + \mathcal{O}(\varepsilon|||\eta_1|||^3);$$

furthermore

$$\begin{aligned}
&\|\eta_{1x}^2\|_0 \le \|\eta_{1x}\|_\infty \|\eta_{1x}\|_0 = \mathcal{O}(\varepsilon^{3/2}|||\eta_1|||^2),\\
&\|\eta_1\eta_{1x}\|_0 \le \|\eta_1\|_\infty \|\eta_{1x}\|_0 = \mathcal{O}(\varepsilon^{3/2}|||\eta_1|||^2),\\
&\|\eta_1\eta_{1xx}\|_0 \le \|\eta_1\|_\infty \|\eta_{1xx}\|_0 = \mathcal{O}(\varepsilon^{3/2}|||\eta_1|||^2),\\
&\|(K_0\eta_1)^2-\eta_1^2\|_0 \le \|K_0\eta_1+\eta_1\|_\infty \|(K_0-I)\eta_1\|_0 = \mathcal{O}(\varepsilon^{3/2}|||\eta_1|||^2),\\
&\|\eta_1 K_0\eta_1-\eta_1^2\|_0 = \|\eta_1(K_0\eta_1-\eta_1)\|_0 \le \|\eta_1\|_\infty \|(K_0\eta_1-\eta_1)\|_0 = \mathcal{O}(\varepsilon^{3/2}|||\eta_1|||^2),\\
&\|(K_0-I)(\eta_1K_0\eta_1)\|_0 \lesssim \varepsilon|||\eta_1 K_0\eta_1||| = \mathcal{O}(\varepsilon^{3/2}|||\eta_1|||^2),
\end{aligned}$$

so that

$$m(\{\eta_1\}^2) = -\tfrac{3}{2}\eta_1^2 + \mathcal{O}(\varepsilon^{3/2}|||\eta_1|||^2).$$

The estimate for the derivative is obtained in a similar fashion. □

Lemma 14 *The estimates* $\mathcal{K}_{\mathrm{r}}(\eta_1+\eta_2(\eta_1))$, $\mathcal{L}_{\mathrm{r}}'(\eta_1+\eta_2) = \underline{\mathcal{O}}(\varepsilon|||\eta_1|||^3)$ *hold for each* $\eta_1 \in X_1$.

Proof. This result follows from Lemma 6 and Theorem 11. □

Finally, note that

$$\begin{aligned}
\varepsilon^2\mathcal{L}_2(\eta_1+\eta_2(\eta_1)) &= -\tfrac{3}{2}\underbrace{\varepsilon^2\eta_1^2}_{=\underline{\mathcal{O}}(|||\eta_1|||^2)} + \underline{\mathcal{O}}(\varepsilon^{3/2}|||\eta_1|||^2) + \underline{\mathcal{O}}(\varepsilon|||\eta_1|||^3).
\end{aligned}$$

6 The reduced equation for ρ

Write

$$\eta_1(x) = \varepsilon^2\rho(\varepsilon x),$$

so that $\rho \in B_R(0) \subseteq \chi(\varepsilon D)H^1(\mathbb{R})$ solves the equation

$$g(\varepsilon D)\rho + \varepsilon^2\rho + \chi(\varepsilon D)\left[\tfrac{3}{2}\varepsilon^2\rho^2 + \underline{\mathcal{O}}_0(\varepsilon^{7/2}\|\rho\|_1^2)\right] = 0 \tag{23}$$

(note that $|||\eta||| = \varepsilon^{3/2}\|\rho\|_1$). Here $R > 0$ is chosen so that $R_1 \le \varepsilon^{3/2}R$ and the symbol $\underline{\mathcal{O}}_s(\varepsilon^\gamma\|\rho\|_1^r)$ (with $\gamma \ge 0$, $r \ge 1$) denotes a smooth function $\mathcal{R} : B_R(0) \subseteq \chi(\varepsilon D)H^1(\mathbb{R}) \to H^s(\mathbb{R})$ which satisfies the estimates

$$\|\mathcal{R}(\rho)\|_s \lesssim \varepsilon^\gamma\|\rho\|_1^r, \quad \|\mathrm{d}\mathcal{R}[\rho]\|_{\mathcal{L}(H^1(\mathbb{R}),H^s(\mathbb{R}))} \lesssim \varepsilon^\gamma\|\rho\|_1^{r-1}$$

for each $\rho \in B_R(0)$.

Proposition 15 *One has that*

$$\left|\frac{\varepsilon^2}{\varepsilon^2+g(\varepsilon s)}-\frac{1}{1+(\beta-\frac{1}{3})s^2}\right| \lesssim \varepsilon^2$$

for all $|s|<\delta/\varepsilon$.

Proof. Obviously

$$\left|\frac{\varepsilon^2}{\varepsilon^2+g(\varepsilon s)}-\frac{1}{1+(\beta-\frac{1}{3})s^2}\right| = \frac{g(\varepsilon s)-(\beta-\frac{1}{3})s^2\varepsilon^2}{(\varepsilon^2+g(\varepsilon s))(1+(\beta-\frac{1}{3})s^2)};$$

furthermore

$$g(k)-(\beta-\tfrac{1}{3})k^2 \lesssim k^4, \qquad |k|\le\delta$$

and

$$g(k)\gtrsim k^2, \qquad k\in\mathbb{R}.$$

It follows that

$$\left|\frac{\varepsilon^2}{\varepsilon^2+g(\varepsilon s)}-\frac{1}{1+(\beta-\frac{1}{3})s^2}\right| \lesssim \frac{\varepsilon^2 s^4}{(1+s^2)^2}\le\varepsilon^2 \qquad |s|<\delta/\varepsilon$$

(because $s^4/(1+s^2)^2\le 1$ for all s). □

Using this proposition, one can write equation (23) as

$$\rho+\mathcal{G}_\varepsilon(\rho)=0, \tag{24}$$

where

$$\mathcal{G}_\varepsilon(\rho)=\tfrac{3}{2}\left(1-(\beta-\tfrac{1}{3})\partial_x^2\right)^{-1}\chi(\varepsilon D)\rho^2+\chi(\varepsilon D)\underline{\mathcal{O}}_0(\varepsilon^2\|\rho\|_1^2).$$

Finally, note that the solutions $\rho\in B_R(0)\subseteq\chi(\varepsilon D)H^1(\mathbb{R})$ of (24) coincide with the solutions $\rho\in B_R(0)\subseteq H^1(\mathbb{R})$ of

$$\rho+\mathcal{H}_\varepsilon(\rho)=0, \tag{25}$$

where

$$\mathcal{H}_\varepsilon(\rho)=\mathcal{G}_\varepsilon(\chi(\varepsilon D)\rho);$$

furthermore the entire reduction can be carried out in spaces of functions which are even in x (denoted by the subscript ‘e’).

Equation (25) is solved using the following version of the implicit-function theorem.

Theorem 16 *Let $\mathcal{Y}$ be a Banach space, Y_0 and Λ_0 be open neighbourhoods of respectively $y^\star$ in $\mathcal{Y}$ and the origin in $\mathbb{R}^n$ and $F : Y_0 \times \Lambda_0 \to \mathcal{Y}$ be a function which is differentiable with respect to $y \in Y_0$ for each $\lambda \in \Lambda_0$. Furthermore, suppose that $F(y^\star, 0) = 0$, $\mathrm{d}_1 F[y^\star, 0] : \mathcal{Y} \to \mathcal{Y}$ is an isomorphism, $\mathrm{d}_1 F[\cdot, 0]$ is continuous at the point $y^\star$ and*

$$\lim_{\lambda \to 0} F(y, \lambda) = F(y, 0), \quad \lim_{\lambda \to 0} \mathrm{d}_1 F[y, \lambda] = \mathrm{d}_1 F[y, 0]$$

uniformly over $y \in Y_0$.

There exist open neighbourhoods Y of $y^\star$ in $\mathcal{Y}$ and Λ of 0 in $\mathbb{R}^n$ (with $Y \subseteq Y_0$, $\Lambda \subseteq \Lambda_0$) and a uniquely determined mapping $h : \Lambda \to Y$ with the properties that

(i) *h is continuous at the origin (with $h(0) = y^\star$),*

(ii) *$F(h(\lambda), \lambda) = 0$ for all $\lambda \in \Lambda$,*

(iii) *$y = h(\lambda)$ whenever $(y, \lambda) \in Y \times \Lambda$ satisfies $F(y, \lambda) = 0$.*

Define $\mathcal{Y} = H^1_{\mathrm{e}}(\mathbb{R})$ and $F : B_R(0) \times [0, \varepsilon_0) \to H^1_{\mathrm{e}}(\mathbb{R})$ by

$$F(\rho, \varepsilon) := \rho + \mathcal{H}_\varepsilon(\rho).$$

Note that

$$\begin{aligned} &F(\rho, \varepsilon) - F(\rho, 0) \\ &\quad = \tfrac{3}{2}\left(1 - (\beta - \tfrac{1}{3})\partial_x^2\right)^{-1}[\chi(\varepsilon D)(\chi(\varepsilon D)\rho)^2 - \rho^2] + \chi(\varepsilon D)\underline{\mathcal{O}}_1(\varepsilon\|\chi(\varepsilon D)\rho\|_1^2) \end{aligned}$$

(because $\chi(\varepsilon D)\underline{\mathcal{O}}_1(\cdot) = \varepsilon^{-1}\chi(\varepsilon D)\underline{\mathcal{O}}_0(\cdot)$), so that

$$F(\rho, \varepsilon) - F(\rho, 0) \to 0, \quad \mathrm{d}_1 F[\rho, \varepsilon] - \mathrm{d}_1 F[\rho, 0] \to 0$$

uniformly over $\rho \in B_R(0)$. The equation

$$F(\rho, 0) = \rho + \tfrac{3}{2}\left(1 - (\beta - \tfrac{1}{3})\partial_x^2\right)^{-1}\rho^2 = 0$$

has the (unique) solution

$$\rho^\star(x) = -\operatorname{sech}^2\left(\frac{x}{2(\beta - \frac{1}{3})^{1/2}}\right)$$

in $H^1_{\mathrm{e}}(\mathbb{R})$ and

$$\mathrm{d}_1 F[\rho^\star, 0] = I + 3\left(1 - (\beta - \tfrac{1}{3})\partial_x^2\right)^{-1}(\rho^\star \cdot).$$

The existence proof is thus completed by the familiar result that the operator $I + 3\left(1 - (\beta - \frac{1}{3})\partial_x^2\right)^{-1}(\rho^\star \cdot)$ is an isomorphism $H^1_{\mathrm{e}}(\mathbb{R}) \to H^1_{\mathrm{e}}(\mathbb{R})$ (see Kirchgässner [5, Proposition 5.1] or Friesecke & Pego [4, §4]).

Theorem 17 *For each sufficiently small value of $\varepsilon > 0$ equation* (25) *has a unique small-amplitude solution $\rho = \rho(\varepsilon)$ in $H^1_{\mathrm{e}}(\mathbb{R})$ which satisfies $\rho \to \rho^\star$ as $\varepsilon \to 0$.*

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