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On the Mathematical Analysis of a Leapfrogging Pair of Coaxial Vortex Rings

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1 Introduction

We are interested in the mathematical modelling and analysis of the interaction of two vortex rings sharing a common axis of symmetry (coaxial vortex rings) in an incompressible and inviscid fluid. A vortex ring is a thin torus-shaped region in the fluid in which the vorticity of the fluid is concentrated. The study of such interaction dates back to 1858, where in his seminal paper Helmholtz [1] observed that a pair of vortex rings may exhibit what is now known as "leapfrogging". Leapfrogging is a motion pattern where two vortex rings pass through each other repeatedly due to the induced flow of the rings acting on each other. Under the classical definition of leapfrogging motion, the pair as a whole also moves in one direction along the common axis of symmetry. This is the case we focus on in this paper. Dyson [2, 3] also considered the motion of coaxial vortex rings and proposed a system of ordinary differential equations describing such motion. Based on this model system, Dyson also observed that leapfrogging may occur. The complex, yet tangible nature of the leapfrogging phenomenon fascinated many researchers and since the observation by Helmholtz, leapfrogging of a pair of coaxial vortex rings as well as the interaction of coaxial vortex rings in general are well studied theoretically, numerically, and experimentally. Notably, although the leapfrogging phenomenon was theoretically observed for a long time, the first experiment which successfully provided photographic proof of the leapfrogging phenomenon in a laboratory setting was the one conducted by Yamada and Matsui [16] in 1978. They used vortex rings made of air and used smoke for visualization and successfully created a leapfrogging pair of rings.

In more recent years, Borisov, Kilin, and Mamaev [20] gave a thorough description of the possible motion patterns of two interacting vortex rings moving under Dyson's model. Hence, much is already known for Dyson's model, but the model has one drawback in that it is derived as a system of ordinary differential equations for the radius and the displacement along the common axis of the rings. It is observed by Maxworthy [31], Widnall and Tsai [32], Widnall and Sullivan [33], and Fukumoto and Hattori [34], that even a small perturbation which destroys the axisymmetry of a vortex ring can grow and eventually cause instability (this kind of instability is called the curvature instability by Fukumoto and Hattori). This suggests that when considering the motion of vortex rings, it is important to model the motion within a framework which can incorporate the effects of these kind of perturbations in order to further understand the behavior of a pair of coaxial vortex rings, but this is not possible under Dyson's model.

Given these situations, we propose a new model describing the interaction of coaxial vortex rings, in particular, we propose a system of partial differential equations so that the model can incorporate the effects of non-symmetric perturbations.

The rest of the paper is organized as follows. In Section 2, we derive the model system via the localized induction approximation. We also give some exact solutions of the obtained system to show that the model is capable of describing well known motions of straight vortex filaments which are parallel to each other. In Section 3, we consider the case when the two filaments are circular with a common axis of symmetry and the vorticity strengths have the same sign. We show that the problem can be reduced to a two-dimensional Hamiltonian system. From here, we give a condition for leapfrogging to occur, and prove that the condition is necessary and sufficient. The precise statement will be given in the beginning of Section 3.

2 Interaction of Two Vortex Filaments

We consider the interaction of two vortex filaments and derive a system of nonlinear partial differential equations which describe their motion. The obtained model admits solutions which correspond to well known motions of point vortices when the two filaments are straight parallel lines, and also gives a clear view of the dynamics when the filaments are arranged as coaxial circles.

2.1 Derivation of the Model System

Following the work of Arms and Hama [30], we apply the localized induction approximation to the Biot–Savart law to obtain a system of partial differential equations approximating the motion of two interacting vortex filaments. The velocity $\boldsymbol{v}(\boldsymbol{x})$ at some point $\boldsymbol{x} \in \mathbf{R}^3$ of an infinite body of incompressible and inviscid fluid induced by a pair of vortex filaments whose positions are parametrized by $\xi \in J$ at time $t \geq 0$ as $\boldsymbol{X}(\xi, t)$ and $\boldsymbol{Y}(\xi, t)$ is given by

$$\boldsymbol{v}(\boldsymbol{x}) = \frac{\Gamma_1}{4\pi} \int_J \frac{\boldsymbol{X}_\xi(r,t) \times (\boldsymbol{x} - \boldsymbol{X}(r,t))}{|\boldsymbol{x} - \boldsymbol{X}(r,t)|^3} \, \mathrm{d}r + \frac{\Gamma_2}{4\pi} \int_J \frac{\boldsymbol{Y}_\xi(r,t) \times (\boldsymbol{x} - \boldsymbol{Y}(r,t))}{|\boldsymbol{x} - \boldsymbol{Y}(r,t)|^3} \, \mathrm{d}r \tag{2.1}$$

where $\times$ is the exterior product in the three-dimensional Euclidean space, Γ_1 is the vorticity strength of the filament $\boldsymbol{X}$, Γ_2 is the vorticity strength of the filament $\boldsymbol{Y}$, $J = \mathbf{R}$ or $\mathbf{R}/2\pi\mathbf{Z}$, and subscripts denote the partial differentiation with the respective variables. The above equation is the Biot–Savart law when the vorticity is concentrated on two vortex filaments. The case $J = \mathbf{R}$ corresponds to when $\boldsymbol{X}$ and $\boldsymbol{Y}$ are infinitely long filaments, and the case $J = \mathbf{R}/2\pi\mathbf{Z}$ corresponds to when $\boldsymbol{X}$ and $\boldsymbol{Y}$ are closed filaments. To determine the velocity of a point on one of the filaments (say $\boldsymbol{X}(\xi,t)$), one would like to substitute $\boldsymbol{x} = \boldsymbol{X}(\xi,t)$ in equation (2.1), but this would result in the divergence of the first integral on the right-hand side. Hence we apply the localized induction approximation to approximate the the velocity at $\boldsymbol{X}(\xi,t)$ by the following equation.

$$\begin{aligned}\boldsymbol{v}(\boldsymbol{X}(\xi,t)) &= \frac{\Gamma_1}{4\pi}\int_{\varepsilon\le|\xi-r|\le L}\frac{\boldsymbol{X}_\xi(r,t)\times(\boldsymbol{X}(\xi,t)-\boldsymbol{X}(r,t))}{|\boldsymbol{X}(\xi,t)-\boldsymbol{X}(r,t)|^3}\,\mathrm{d}r\\ &\quad+\frac{\Gamma_2}{4\pi}\int_{|\xi-r|\le\delta}\frac{\boldsymbol{Y}_\xi(r,t)\times(\boldsymbol{X}(\xi,t)-\boldsymbol{Y}(r,t))}{|\boldsymbol{X}(\xi,t)-\boldsymbol{Y}(r,t)|^3}\,\mathrm{d}r\\ &=: I_1 + I_2.\end{aligned}$$

Here, $\varepsilon > 0$ and $\delta > 0$ are small parameters, and $L > 0$ is a cut-off parameter. I_1 is the effect of self-induction, and I_2 is the effect of interaction. The approximation applied in I_1 is the well known localized induction approximation. To obtain I_2, we have further assumed that the filaments $\boldsymbol{X}$ and $\boldsymbol{Y}$ are positioned in a way that $\boldsymbol{Y}(\xi,t)$ is the closest point to $\boldsymbol{X}(\xi,t)$ and the contributions from points far away from $\boldsymbol{Y}(\xi,t)$ can be ignored. This kind of geometric assumption is true for the situations that we treat in this paper, but does not hold, for example, when the filaments are knotted together. By the calculations in Arms and Hama [30], it is known that I_1 can be expanded in terms of small ε as follows.

$$I_1 = -\frac{\Gamma_1}{4\pi}\log\left(\frac{L}{\varepsilon}\right)\frac{\boldsymbol{X}_\xi\times\boldsymbol{X}_{\xi\xi}}{|\boldsymbol{X}_\xi|^3} + O(1).$$

The above is obtained by substituting the Taylor expansion of $\boldsymbol{X}(r,t)$ and $\boldsymbol{X}_\xi(r,t)$ with respect to r around ξ into the integrand. We further substitute

$$\begin{aligned}\boldsymbol{Y}(r,t) &= \boldsymbol{Y}(\xi,t)+\boldsymbol{Y}_\xi(\xi,t)(r-\xi)+O((r-\xi)^2),\\ \boldsymbol{Y}_\xi(r,t) &= \boldsymbol{Y}_\xi(\xi,t)+\boldsymbol{Y}_{\xi\xi}(\xi,t)(r-\xi)+O((r-\xi)^2),\end{aligned}$$

into I_2 to obtain

$$I_2 = \frac{\delta\Gamma_2}{2\pi}\frac{\boldsymbol{Y}_\xi\times(\boldsymbol{X}-\boldsymbol{Y})}{|\boldsymbol{X}-\boldsymbol{Y}|^3} + O(\delta^2).$$

Hence, after fixing L and taking sufficiently small ε and δ, the leading order terms of I_1 and I_2 yield

$$\boldsymbol{X}_t = -\frac{\Gamma_1}{4\pi}\log(\frac{L}{\varepsilon})\frac{\boldsymbol{X}_\xi \times \boldsymbol{X}_{\xi\xi}}{|\boldsymbol{X}_\xi|^3} + \frac{\delta\Gamma_2}{2\pi}\frac{\boldsymbol{Y}_\xi \times (\boldsymbol{X}-\boldsymbol{Y})}{|\boldsymbol{X}-\boldsymbol{Y}|^3},$$

where we also used the fact that $\boldsymbol{v}(\boldsymbol{X}(\xi,t)) = \boldsymbol{X}_t(\xi,t)$ by the definition of velocity. By rescaling time by a factor of $-\log(\frac{L}{\varepsilon})/4\pi$, we obtain

$$\boldsymbol{X}_t = \Gamma_1\frac{\boldsymbol{X}_\xi \times \boldsymbol{X}_{\xi\xi}}{|\boldsymbol{X}_\xi|^3} - \alpha\Gamma_2\frac{\boldsymbol{Y}_\xi \times (\boldsymbol{X}-\boldsymbol{Y})}{|\boldsymbol{X}-\boldsymbol{Y}|^3},$$

where $\alpha = 2\delta/\log(\frac{L}{\varepsilon}) > 0$. The calculations for the velocity at points on $\boldsymbol{Y}$ are the same and hence we arrive at the following system.

$$\begin{cases} \boldsymbol{X}_t = \Gamma_1\dfrac{\boldsymbol{X}_\xi \times \boldsymbol{X}_{\xi\xi}}{|\boldsymbol{X}_\xi|^3} - \alpha\Gamma_2\dfrac{\boldsymbol{Y}_\xi \times (\boldsymbol{X}-\boldsymbol{Y})}{|\boldsymbol{X}-\boldsymbol{Y}|^3}, \\ \boldsymbol{Y}_t = \Gamma_2\dfrac{\boldsymbol{Y}_\xi \times \boldsymbol{Y}_{\xi\xi}}{|\boldsymbol{Y}_\xi|^3} - \alpha\Gamma_1\dfrac{\boldsymbol{X}_\xi \times (\boldsymbol{Y}-\boldsymbol{X})}{|\boldsymbol{X}-\boldsymbol{Y}|^3}. \end{cases} \tag{2.2}$$

All the analysis that follows will be based on the above system (2.2).

2.2 Dynamics of Two Parallel Lines

As a preliminary analysis, we show that for a pair of infinitely long, straight, and parallel vortex filaments, the dynamics of the filaments according to equation (2.2) are the same as that of two point vortices moving in a plane. Suppose the two filaments are initially parametrized as

$$\boldsymbol{X}_0(\xi) = {}^t(l,0,\xi), \qquad \boldsymbol{Y}_0(\xi) = {}^t(-l,0,\xi),$$

where $l > 0$ is arbitrary. In this situation, it is expected that the motions of the filaments become two-dimensional and resemble that of two point vortices. Indeed, if we make the ansatz

$$\boldsymbol{X}(\xi,t) = {}^t(x_1(t),x_2(t),\xi), \qquad \boldsymbol{Y}(\xi,t) = {}^t(y_1(t),y_2(t),\xi),$$

and substitute it into (2.2), we obtain

$$\begin{cases} \dot{x_1} = \dfrac{\alpha\Gamma_2(x_2 - y_2)}{\left((x_1 - x_1)^2 + (x_2 - y_2)^2\right)^{3/2}}, \\ \dot{x_2} = -\dfrac{\alpha\Gamma_2(x_1 - y_1)}{\left((x_1 - y_1)^2 + (x_2 - y_2)^2\right)^{3/2}}, \\ \dot{y_1} = \dfrac{\alpha\Gamma_1(y_2 - x_2)}{\left((x_1 - x_1)^2 + (x_2 - y_2)^2\right)^{3/2}}, \\ \dot{y_2} = -\dfrac{\alpha\Gamma_1(y_1 - x_1)}{\left((x_1 - x_1)^2 + (x_2 - y_2)^2\right)^{3/2}}, \end{cases}$$

where a dot over a variable denotes the derivative with respect to time. Further setting $z_1 = x_1 + ix_2$ and $z_2 = y_1 + iy_2$, where i is the imaginary unit, we have

$$\begin{cases} \dot{z_1} = -i\alpha\Gamma_2 \dfrac{z_1 - z_2}{|z_1 - z_2|^{3/2}}, \\ \dot{z_2} = -i\alpha\Gamma_1 \dfrac{z_2 - z_1}{|z_1 - z_2|^{3/2}}. \end{cases}$$

We see from direct calculation that when $\Gamma_1 + \Gamma_2 \neq 0$,

$$C = \frac{\Gamma_1 z_1 + \Gamma_2 z_2}{\Gamma_1 + \Gamma_2}, \qquad D = |z_1 - z_2|,$$

are conserved quantities. C is known as the center of vorticity. Utilizing these quantities, the equations can be decoupled to obtain

$$\begin{pmatrix} \dot{z_1} \\ \dot{z_2} \end{pmatrix} = -\frac{i\alpha(\Gamma_1 + \Gamma_2)}{D^{3/2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 - C \\ z_2 - C \end{pmatrix}.$$

The above equation can be solved explicitly and we have

$$z_j(t) = (z_j(0) - C)e^{i\omega t} + C$$

for $j = 1, 2$, where $\omega = -\alpha(\Gamma_1 + \Gamma_2)/D^{3/2}$. This shows that the two filaments rotate in a two-dimensional circular pattern and the center and radius of rotation is determined by the center of vorticity. When $\Gamma_1 + \Gamma_2 = 0$, we see that $z_1 - z_2$ is conserved and hence we have

$$\dot{z_j} = -\frac{i\alpha\Gamma_2}{D^{3/2}} w_0 = \text{const.},$$

for $j = 1, 2$ with $w_0 = z_1(0) - z_2(0)$. This shows that the two filaments travel in a straight line at a constant speed while keeping their parallel configuration. These dynamics of the filaments directly correspond to the motion of two point vortices moving in a plane, which is well known in the literature such as Newton [39]. Hence, we see that system (2.2) is capable of describing the motion of two parallel lines in the expected manner.

3 Leapfrogging for a Pair of Filaments with Vorticity Strengths of the Same Sign

We consider the case when the two filaments are arranged as coaxial circles and $\Gamma_1, \Gamma_2 > 0$. Rescaling the time variable by a factor of Γ_2 in (2.2) yields

$$\begin{cases} \boldsymbol{X}_t = \beta \dfrac{\boldsymbol{X}_\xi \times \boldsymbol{X}_{\xi\xi}}{|\boldsymbol{X}_\xi|^3} - \alpha \dfrac{\boldsymbol{Y}_\xi \times (\boldsymbol{X} - \boldsymbol{Y})}{|\boldsymbol{X} - \boldsymbol{Y}|^3}, \\[2ex] \boldsymbol{Y}_t = \dfrac{\boldsymbol{Y}_\xi \times \boldsymbol{Y}_{\xi\xi}}{|\boldsymbol{Y}_\xi|^3} - \alpha\beta \dfrac{\boldsymbol{X}_\xi \times (\boldsymbol{Y} - \boldsymbol{X})}{|\boldsymbol{X} - \boldsymbol{Y}|^3}, \end{cases} \tag{3.1}$$

where $\beta = \Gamma_1/\Gamma_2$. We assume without loss of generality that $\beta \geq 1$, since the case $\beta < 1$ is reduced to the case $\beta > 1$ by renaming the filaments.

Suppose that for some $R_{1,0}, R_{2,0} > 0$ and $z_{1,0}, z_{2,0} \in \mathbf{R}$, the initial filaments $\boldsymbol{X}_0$ and $\boldsymbol{Y}_0$ are parametrized by $\xi \in [0, 2\pi)$ as follows.

$$\boldsymbol{X}_0(\xi) = {}^t(R_{1,0}\cos(\xi), R_{1,0}\sin(\xi), z_{1,0}), \quad \boldsymbol{Y}_0(\xi) = {}^t(R_{2,0}\cos(\xi), R_{2,0}\sin(\xi), z_{2,0}),$$

where we assume that $(R_{1,0} - R_{2,0})^2 + (z_{1,0} - z_{2,0})^2 > 0$, which means that the two circles are not overlapping. Now, we make the ansatz

$$\boldsymbol{X}(\xi, t) = {}^t(R_1(t)\cos(\xi), R_1(t)\sin(\xi), z_1(t)), \quad \boldsymbol{Y}(\xi, t) = {}^t(R_2(t)\cos(\xi), R_2(t)\sin(\xi), z_2(t)),$$

and substitute it into (3.1). From the equation for $\boldsymbol{X}$ we have

$$\dot{R}_1\cos(\xi) = -\frac{\alpha R_2(z_1 - z_2)\cos(\xi)}{\left((R_1 - R_2)^2 + (z_1 - z_2)^2\right)^{3/2}},$$

$$\dot{R}_1\sin(\xi) = -\frac{\alpha R_2(z_1 - z_2)\sin(\xi)}{\left((R_1 - R_2)^2 + (z_1 - z_2)^2\right)^{3/2}},$$

$$\dot{z}_1 = \frac{\beta}{R_1} + \frac{\alpha R_2(R_1 - R_2)}{\left((R_1 - R_2)^2 + (z_1 - z_2)^2\right)^{3/2}}.$$

The dependence of the system on ξ is eliminated by multiplying the first two equations by $\cos(\xi)$ and $\sin(\xi)$, respectively, and adding. The equations for $\boldsymbol{Y}$ are calculated in the same way and we arrive at

$$\begin{cases} \dot{R}_1 = -\dfrac{\alpha R_2(z_1 - z_2)}{\big((R_1 - R_2)^2 + (z_1 - z_2)^2\big)^{3/2}}, \\ \dot{z}_1 = \dfrac{\beta}{R_1} + \dfrac{\alpha R_2(R_1 - R_2)}{\big((R_1 - R_2)^2 + (z_1 - z_2)^2\big)^{3/2}}, \\ \dot{R}_2 = \dfrac{\alpha\beta R_1(z_1 - z_2)}{\big((R_1 - R_2)^2 + (z_1 - z_2)^2\big)^{3/2}}, \\ \dot{z}_2 = \dfrac{1}{R_2} - \dfrac{\alpha\beta R_1(R_1 - R_2)}{\big((R_1 - R_2)^2 + (z_1 - z_2)^2\big)^{3/2}}, \\ (R_1(0), z_1(0), R_2(0), z_2(0)) = (R_{1,0}, z_{1,0}, R_{2,0}, z_{2,0}). \end{cases} \tag{3.2}$$

We note here that a system similar to (3.2) was derived independently by Munakata [40] by directly approximating the induced velocities of vortex rings. First, we observe that z_1 and z_2 can be reduced to one variable, namely $W = z_1 - z_2$. Furthermore, we see by direct calculation that $\beta R_1^2 + R_2^2$ is a conserved quantity. Hence, setting $d^2 = \beta R_{1,0}^2 + R_{2,0}^2$ with $d > 0$, we make the change of variables

$$R_1(t) = \frac{d}{\beta^{1/2}} \cos(\theta(t)), \quad R_2(t) = d \sin(\theta(t))$$

to further reduce the system. We then arrive at

$$\begin{cases} \dot{\theta} = \dfrac{\alpha \beta^{1/2} W}{\big(\frac{d^2}{\beta}(\beta^{1/2}\sin\theta - \cos\theta)^2 + W^2\big)^{3/2}} =: F_1(\theta, W), \\ \dot{W} = \dfrac{\beta^{3/2}\sin\theta - \cos\theta}{d \sin\theta\cos\theta} - \dfrac{\alpha d^2(\sin\theta + \beta^{1/2}\cos\theta)(\beta^{1/2}\sin\theta - \cos\theta)}{\beta^{1/2}\big(\frac{d^2}{\beta}(\beta^{1/2}\sin\theta - \cos\theta)^2 + W^2\big)^{3/2}} =: F_2(\theta, W), \end{cases} \tag{3.3}$$

with initial data (θ_0, W_0). Here, $W_0 = z_{1,0} - z_{2,0}$ and θ_0 is determined uniquely from the relation

$$R_{1,0} = \frac{d}{\beta^{1/2}} \cos\theta_0, \quad R_{2,0} = d \sin\theta_0.$$

Note that from our problem setting, (θ_0, W_0) is contained in the open set $\Omega_\beta \subset \mathbf{R}^2$ given by

$$\Omega_\beta = \big\{(\theta, W) \in \mathbf{R}^2 \mid 0 < \theta < \frac{\pi}{2}, W \in \mathbf{R}, (\theta, W) \neq (\theta_\beta, 0)\big\},$$

where θ_β is the unique solution of

$$\beta^{1/2} \sin\theta_\beta - \cos\theta_\beta = 0,$$

which is given explicitly by $\theta_\beta = \arctan(1/\beta^{1/2})$. The excluded point in the above definition corresponds to the two filaments overlapping. Since we can reconstruct the solution of (3.2) from the solution $(\theta(t), W(t))$ of (3.3) by

$$R_1(t) = \frac{d}{\beta^{1/2}} \cos(\theta(t)), \quad R_2(t) = d\sin(\theta(t)),$$

$$z_1(t) = \int_0^t \frac{\beta}{R_1(\tau)} + \frac{\alpha R_2(\tau)(R_1(\tau) - R_2(\tau))}{\big((R_1(\tau) - R_2(\tau))^2 + W(\tau)^2\big)^{3/2}} \mathrm{d}\tau,$$

$$z_2(t) = \int_0^t \frac{1}{R_2(\tau)} - \frac{\alpha\beta R_2(\tau)(R_1(\tau) - R_2(\tau))}{\big((R_1(\tau) - R_2(\tau))^2 + W(\tau)^2\big)^{3/2}} \mathrm{d}\tau,$$

we focus on the solvability and behavior of the solution to system (3.3). It can be checked by direct calculation that the system (3.3) is a Hamiltonian system and the Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{H}(\theta, W) = \frac{1}{2d} \log\left(\frac{(1 - \sin\theta)^{\beta^{3/2}}(1 - \cos\theta)}{(1 + \sin\theta)^{\beta^{3/2}}(1 + \cos\theta)}\right) - \frac{\alpha\beta^{1/2}}{\big(\frac{d^2}{\beta}(\beta^{1/2}\sin\theta - \cos\theta)^2 + W^2\big)^{1/2}}. \tag{3.4}$$

In other words, $F_1 = \frac{\partial\mathcal{H}}{\partial W}$ and $F_2 = -\frac{\partial\mathcal{H}}{\partial\theta}$. Of course, the Hamiltonian is a conserved quantity of motion. In this formulation, closed orbits revolving around the point $(\theta_\beta, 0)$ correspond to leapfrogging. From here, we treat (3.3) as a two-dimensional dynamical system in Ω_β with parameters d,β, and α, and make use of many tools known for two-dimensional dynamical systems and Hamiltonian systems, for example in Hirsch and Smale [41], to determine the dynamics of the filaments.

We state our main theorems.

Theorem 3.1 *For any $\alpha, d > 0$, $\beta \geq 1$, and $(\theta_0, W_0) \in \Omega_\beta$, there exists a unique time-global solution $(\theta, W) \in C^1(\mathbf{R}) \times C^1(\mathbf{R})$ of* (3.3).

Theorem 3.2 *In addition to the assumptions of Theorem* 3.1, *if we assume* $0 < \alpha < 1/3$, *then system* (3.3) *has two equilibrium points* $(\theta_*, 0)$ *and* $(\theta_{**}, 0)$ *with* $\theta_* \in (0, \theta_\beta)$ *and* $\theta_{**} \in (\theta_\beta, \pi/2)$*, and the following two statements are equivalent.*

(i) *The solution with initial data* (θ_0, W_0) *is a leapfrogging solution. In other words, the solution curve is a closed orbit revolving around the point* $(\theta_\beta, 0)$*.*

(ii) $\theta_0 \in (\theta_*, \theta_{**})$ *and* $\mathcal{H}(\theta_0, W_0) < \min\{\mathcal{H}(\theta_*, 0), \mathcal{H}(\theta_{**}, 0)\}$*.*

Remark 3.3 (Note on the assumption for α in Theorem 3.2) *Recall that* $\alpha > 0$ *was given by* $\alpha = 2\delta / \log(\frac{L}{\varepsilon})$*, where* $\delta, \varepsilon > 0$ *were small parameters with* $L > 0$ *fixed. These parameters were introduced in the course of the derivation of the model system* (2.2)*. Hence, it is natural to assume that* α *is small and also important that the smallness assumption for* α *in Theorem* 3.2 *is independent of the parameters* d *and* β*.*

The rest of the section is devoted to the proof of the above two theorems.

Proof of Theorem 3.1. Since F_1 and F_2 are smooth in Ω_β, the time-local unique solvability is known. Suppose the maximum existence time $T > 0$ is finite. From the standard theory of dynamical systems, for any compact set $K \subset \Omega_\beta$, there exists $t' \in [0, T)$ such that $(\theta(t'), W(t')) \notin K$. On the other hand, since the Hamiltonian is conserved, there exists $\eta > 0$ and $r > 0$ such that for all $t \in [0, T)$,

$$(\theta(t), W(t)) \in \big([\eta, \frac{\pi}{2} - \eta] \times \mathbf{R}\big) \setminus B_r(\theta_\beta, 0),$$

where $B_r(\theta_\beta, 0)$ is the open ball in $\mathbf{R}^2$ with center $(\theta_\beta, 0)$ and radius r. This follows from the fact that the Hamiltonian diverges to $-\infty$ at $\theta = 0, \pi/2$ uniformly with respect to W and at the point $(\theta_\beta, 0)$. In particular, since the solution curve is uniformly separated from the point $(\theta_\beta, 0)$, there exists $c_0 > 0$ such that

$$\frac{d^2}{\beta}(\beta^{1/2}\sin\theta(t) - \cos\theta(t))^2 + W(t)^2 \geq c_0$$

for all $t \in [0, T)$. Hence from the second equation in (3.3), we have

$$|\dot{W}| \leq \frac{\beta^{3/2} + 1}{d \sin\eta \cos(\pi/2 - \eta)} + \frac{\alpha d^2 (\beta^{1/2} + 1)^2}{\beta^{1/2} c_0^{3/2}} =: M,$$

which yields

$$|W(t)| \leq |W(0)| + Mt \leq |W_0| + MT$$

for all $t \in [0,T)$. Finally, this shows that for all $t \in [0,T)$, $(\theta(t), W(t))$ is contained in the compact set K' given by

$$K' = \Big([\eta, \frac{\pi}{2} - \eta] \times [-|W_0| - MT, |W_0| + MT]\Big) \setminus B_r(\theta_\beta, 0),$$

which is a contradiction. The same argument holds for $t < 0$ and hence, the solution exists globally in time and is defined for all $t \in \mathbf{R}$. □

Proof of Theorem 3.2. We divide the proof of Theorem 3.2 into subsections. First we prove that system (3.3) has exactly two equilibriums as stated in Theorem 3.2.

3.1 Equilibriums of System (3.3)

From the form of F_1, we see that an equilibrium can only exist on the line segment $(0, \pi/2) \times \{0\}$, and thus, we set $f(\theta) := F_2(\theta, 0)$ and investigate the zeroes of f. First we consider the zeroes in the interval $(0, \theta_\beta)$. Keeping in mind that $\beta^{1/2}\sin\theta - \cos\theta < 0$ in $(0, \theta_\beta)$, by a change of variable $\theta = \arctan x$ we have

$$f(\arctan x) = \frac{(1+x^2)^{1/2} g_\alpha(x)}{dx(\beta^{1/2}x - 1)^2},$$

where g_α is given by

$$g_\alpha(x) = \beta^{5/3}x^3 - \beta(2\beta+1)x^2 + \beta^{1/2}(\beta+2)x - 1 + \alpha\beta(x^2 + \beta^{1/2}x)$$

for $x \in (0, 1/\beta^{1/2})$. We further make the change of variable $y = \beta^{1/2}x$ for simplification and investigate the zeroes of the function h_α given by

$$h_\alpha(y) = \beta y^3 - (2\beta+1)y^2 + (\beta+2)y - 1 + \alpha(y^2 + \beta y)$$

in the interval $I_1 = (0,1)$. We treat h_α as a perturbation of h_0 given by

$$h_0(y) = \beta y^3 - (2\beta+1)y^2 + (\beta+2)y - 1,$$

which is h_α with $\alpha = 0$ and prove that h_α has exactly one zero in I_1. We see from direct calculation that h_0 has one local maximum and one local minimum at

$$y_1 = \frac{\beta+2}{3\beta}, \quad y_2 = 1,$$

respectively, and

$$h_0(y_1) = \frac{4}{27\beta^2}(\beta-1)^3 > 0, \quad h_0(y_2) = 0.$$

Since the zero at y_2 is singular, we cannot directly apply the method of perturbation to h_α. Instead, we analyze the positions of the local extrema for $0 < \alpha < 1/3$ to determine the number of zeroes of h_α. First, we observe that the discriminant Δ of the quadratic equation $h'_\alpha(y) = 0$ is given by

$$\Delta = 4[(1-3\alpha)\beta^2 - 2(1+2\alpha)\beta + (\alpha - 1)] =: 4\phi(\beta).$$

$\phi(\beta) = 0$ has two roots $\beta_\pm$ given by

$$\beta_- = \frac{1+2\alpha - \sqrt{3\alpha(3-\alpha-\alpha^2)}}{1-3\alpha}, \quad \beta_+ = \frac{1+2\alpha + \sqrt{3\alpha(3-\alpha-\alpha^2)}}{1-3\alpha}$$

and under the assumption $0 < \alpha < 1/3$, we see that

$$\phi(\beta) < 0 \text{ for } 1 \leq \beta < \beta_+, \quad \phi(\beta) \geq 0 \text{ for } \beta_+ \leq \beta,$$

where we also used the fact that $\phi(1) = -\alpha(9-\alpha) < 0$. This shows that when $1 \leq \beta < \beta_+$, $\Delta < 0$ which implies $h'_\alpha > 0$ for $y \in (0,1)$. Since, $h_\alpha(0) = -1$ and $h_\alpha(1) = \alpha(1+\beta) > 0$, there is exactly one zero in I_1.

When $\beta_+ \leq \beta$, the roots $y_\pm$ of $h'_\alpha(y) = 0$ are given by

$$y_\pm = \frac{2\beta + 1 - \alpha \pm \sqrt{\phi(\beta)}}{3\beta},$$

where y_- is the local maximum and y_+ is the local minimum. Since h_α is a third order polynomial, it is sufficient to prove that $h_\alpha(y_+) > 0$ to prove that h_α has exactly one root. We have

$$\begin{aligned} y_+ \geq \frac{1}{3\beta}(2\beta + 1 - \alpha) &\geq \frac{1}{3\beta}(\beta + 2 + (\beta_+ - 1) - \alpha) \\ &= \frac{1}{3\beta}\left\{\beta + 2 + \frac{\alpha^{1/2}}{1-3\alpha}[\left((3(3-\alpha-\alpha^2)\right)^{1/2} + 5\alpha^{1/2} - (1-3\alpha)\alpha^{1/2}]\right\} \\ &\geq \frac{\beta+2}{3\beta}, \end{aligned}$$

which implies $h_0(y_+) \geq 0$. Finally, we have

$$h_\alpha(y_+) = h_0(y_+) + \alpha(y_+^2 + \beta y_+) > 0$$

which shows that h_α also has exactly one root when $\beta_+ \leq \beta$. Hence we have proven that for any $\beta \geq 1$ and $0 < \alpha < 1/3$, h_α has exactly one zero y_* in I_1 and $h'_\alpha(y_*) > 0$. Hence, $\theta_* = \arctan(y_*/\beta^{1/2})$ is the desired zero of $f(\theta)$ in the interval $(0, \theta_\beta)$ and we see that $f'(\theta_*) > 0$. By a similar argument, we see that there exists a unique $\theta_{**} \in (\theta_\beta, \pi/2)$ such that $f(\theta_{**}) = 0$ and $f'(\theta_{**}) > 0$. We note here that because θ_* and θ_{**} are the only zeroes in the interval $(0, \theta_\beta)$ and $(\theta_\beta, \pi/2)$ respectively, and $f'(\theta_*), f'(\theta_{**}) > 0$, we have the following property for $f(\theta)$.

$$\begin{aligned} &f(\theta) < 0, \quad \text{for } \theta \in (0, \theta_*) \cup (\theta_{**}, \pi/2), \\ &f(\theta) > 0, \quad \text{for } \theta \in (\theta_*, \theta_\beta) \cup (\theta_\beta, \theta_{**}). \end{aligned} \tag{3.5}$$

3.2 Analysis for Solutions with Initial Data of the Form $(\theta_0, 0)$

Since a leapfrogging solution corresponds to a closed orbit revolving around the point $(\theta_\beta, 0)$ in Ω_β, a leapfrogging solution always crosses the lines $(0, \theta_\beta) \times \{0\}$ and $(\theta_\beta, \pi/2) \times \{0\}$ in Ω_β. To this end, we first characterize the solutions with initial data of the form $(\theta_0, 0)$, and prove that the condition given in Theorem 3.2 is necessary and sufficient for leapfrogging to occur.

First we prove that (ii) implies (i). Set $H_* := \min\{\mathcal{H}(\theta_*, 0), \mathcal{H}(\theta_{**}, 0)\}$. Let $\theta_0 \in (\theta_*, \theta_{**})$ satisfy $\mathcal{H}(\theta_0, 0) < H_*$. To make the situation more concrete, we further assume that $\mathcal{H}(\theta_*, 0) > \mathcal{H}(\theta_{**}, 0)$ and make a remark on the case $\mathcal{H}(\theta_*, 0) \le \mathcal{H}(\theta_{**}, 0)$ at the end. From (3.5) and the fact that $\frac{\partial \mathcal{H}}{\partial \theta}(\theta, 0) = -f(\theta)$, we have

$$\begin{aligned} &\frac{\partial \mathcal{H}}{\partial \theta}(\theta, 0) > 0, \quad \text{for } \theta \in (0, \theta_*) \cup (\theta_{**}, \pi/2), \\ &\frac{\partial \mathcal{H}}{\partial \theta}(\theta, 0) < 0, \quad \text{for } \theta \in (\theta_*, \theta_\beta) \cup (\theta_\beta, \theta_{**}). \end{aligned} \tag{3.6}$$

Moreover, since $\mathcal{H}(\theta_*, 0) > \mathcal{H}(\theta_{**}, 0)$, and $\mathcal{H}(\theta, 0) \to -\infty$ monotonically as $\theta \to \theta_\beta-$, there exists a unique $\tilde{\theta} \in (\theta_*, \theta_\beta)$ such that $\mathcal{H}(\tilde{\theta}, 0) = H_*$. This implies that $\theta_0 \in (\tilde{\theta}, \theta_{**}) \setminus \{\theta_\beta\}$.

We assume that $\theta_0 \in (\tilde{\theta}, \theta_\beta)$ since the arguments for the case $\theta_0 \in (\theta_\beta, \theta_{**})$ is the same. We prove that the unique time-global solution $(\theta(t), W(t))$ starting from $(\theta_0, 0)$ obtained in Theorem 3.1, which is defined for $t \in \mathbf{R}$, is a closed orbit revolving around $(\theta_\beta, 0)$. First, we show that the solution is bounded. We observe that as a function of W, the Hamiltonian achieves a minimum at $W = 0$ for each fixed θ. Hence for all $W \in \mathbf{R}$, we have

$$\begin{aligned} \mathcal{H}(\tilde{\theta}, W) &\ge \mathcal{H}(\tilde{\theta}, 0) = H_* > \mathcal{H}(\theta_0, 0), \\ \mathcal{H}(\theta_{**}, W) &\ge \mathcal{H}(\theta_{**}, 0) = H_* > \mathcal{H}(\theta_0, 0). \end{aligned}$$

The above and from the conservation and continuity of the Hamiltonian, there exists

$\eta > 0$ and $r > 0$ such that

$$(\theta(t), W(t)) \in \big([\tilde{\theta} + \eta, \theta_{**} - \eta] \times \mathbf{R}\big) \setminus B_r(\theta_\beta, 0),$$

for all $t \in \mathbf{R}$. Furthermore, if we set

$$\phi(\theta) := \frac{1}{2d} \log \left(\frac{(1 - \sin\theta)^{\beta^{3/2}}(1 - \cos\theta)}{(1 + \sin\theta)^{\beta^{3/2}}(1 + \cos\theta)} \right),$$

we see that as a function of θ, $\mathcal{H}(\theta, W)$ converges to ϕ uniformly as $W \to \infty$. Since we have

$$\phi'(\theta) = -\frac{(\beta^{3/2}\sin\theta - \cos\theta)}{d\cos\theta\sin\theta},$$

we see that ϕ achieves a maximum at $\theta = \arctan(1/\beta^{3/2}) =: \theta_c$ with $0 < \theta_c < \theta_\beta$ and ϕ is monotone in the intervals $(0, \theta_c)$ and $(\theta_c, \pi/2)$. If $0 < \theta_c \leq \tilde{\theta}$, for $\varepsilon_1 > 0$ given by

$$\varepsilon_1 = \frac{\alpha\beta^{1/2}}{2\big\{\frac{d^2}{\beta}\big(\beta^{1/2}\sin\theta_{**} - \cos\theta_{**}\big)^2\big\}^{1/2}},$$

there exists $W_1 > 0$ such that for all $\theta \in (\tilde{\theta}, \theta_{**})$, and $W > W_1$ we have

$$\mathcal{H}(\theta, W) > \phi(\theta) - \varepsilon_1 > \phi(\theta_{**}) - 2\varepsilon_1 = \mathcal{H}(\theta_{**}, 0) = H_* > \mathcal{H}(\theta_0, 0).$$

If $\tilde{\theta} < \theta_c < \theta_\beta$, choose $\theta' \in \{\tilde{\theta}, \theta_{**}\}$ so that $\phi(\theta') = \min\{\phi(\tilde{\theta}), \phi(\theta_{**})\}$. Then for $\varepsilon_2 > 0$ given by

$$\varepsilon_2 = \frac{\alpha\beta^{1/2}}{2\big\{\frac{d^2}{\beta}\big(\beta^{1/2}\sin\theta' - \cos\theta'\big)^2\big\}^{1/2}},$$

there exists $W_2 > 0$ such that for all $\theta \in (\tilde{\theta}, \theta_{**})$ and $W > W_2$, we have

$$\mathcal{H}(\theta, W) > \phi(\theta) - \varepsilon_2 > \phi(\theta') - 2\varepsilon_2 = \mathcal{H}(\theta', 0) = H_* > \mathcal{H}(\theta_0, 0).$$

In either case, we see that the value of the Hamiltonian on the segment $[\tilde{\theta}, \theta_{**}] \times \{W_*\}$, where $W_* = \max\{W_1, W_2\}$, is strictly greater than $\mathcal{H}(\theta_0, 0)$ and hence the solution curve cannot cross this segment. Since the Hamiltonian is symmetric with respect to $W = 0$, we finally see that

$$(\theta(t), W(t)) \in \big([\tilde{\theta} + \eta, \theta_{**} - \eta] \times [-W_*, W_*]\big) \setminus B_r(\theta_\beta, 0) =: K_*,$$

for all $t \in \mathbf{R}$, and in particular, the solution is bounded.

Next we set

$$L_0 := \{(\theta, W) \in \Omega_\beta \mid \mathcal{H}(\theta, W) = \mathcal{H}(\theta_0, 0)\} \cap K_*.$$

As a closed subset of the compact set K_*, L_0 is a compact subset of Ω_β. From the conservation of the Hamiltonian and the way we chose η, r, and W_*, we see that L_0 is also an invariant set and hence we have

$$L_\omega(\theta_0, 0) \subset L_0,$$

where $L_\omega(\theta_0, 0)$ is the ω-limit set of $(\theta_0, 0)$. Since $(\theta(t), W(t))$ is bounded for $t > 0$, it converges along some series $\{t_n\}_{n=1}^\infty$ with $t_n \to \infty$ as $n \to \infty$, and in particular, $L_\omega(\theta_0, 0)$ is not empty. Since $L_\omega(\theta_0, 0)$ is a non-empty compact set and contains no equilibriums (recall that the equilibriums $(\theta_*, 0)$ and $(\theta_{**}, 0)$ are outside the set L_0), it is a closed orbit by the Poincaré–Bendixson Theorem. Moreover, the point $(\theta_\beta, 0)$ is in the interior of this closed orbit, because if it is not, then the closed orbit would enclose an open subset of Ω_β in which an equilibrium must exist, which leads to a contradiction. This proves that $L_\omega(\theta_0, 0)$ is a closed orbit revolving around $(\theta_\beta, 0)$. Since $L_\omega(\theta_0, 0) \subset L_0$, there exists $\theta_1 \in (\tilde{\theta} + \eta, \theta_\beta)$ and $\theta_2 \in (\theta_\beta, \theta_{**} - \eta)$ such that $(\theta_1, 0), (\theta_2, 0) \in L_\omega(\theta_0, 0)$. The values θ_1 and θ_2 satisfying this property are unique in their respective intervals because the Hamiltonian is monotone along the line segments $[\tilde{\theta} + \eta, \theta_\beta] \times \{0\}$ and $[\theta_\beta, \theta_{**} - \eta] \times \{0\}$. This uniqueness implies that $\theta_1 = \theta_0$, which proves that $L_\omega(\theta_0, 0)$ coincides with the orbit starting from $(\theta_0, 0)$.

In summary, we have proven that the orbit starting from $(\theta_0, 0)$ is a closed orbit revolving around $(\theta_\beta, 0)$ corresponding to a leapfrogging solution. We further have the characterization

$$L_\omega(\theta_0, 0) = L_0,$$

which we prove by contradiction. Suppose there exists $(\overline{\theta}, \overline{W}) \in L_0$ such that $(\overline{\theta}, \overline{W}) \notin L_\omega(\theta_0, 0)$. We first see that $\overline{W} \neq 0$, since $(\overline{\theta}, 0) \in L_0$ implies $\overline{\theta} = \theta_1$ or θ_2, which contradicts $(\overline{\theta}, 0) \notin L_\omega(\theta_0, 0)$. Henceforth, we assume $\overline{W} > 0$ since the proof for the other case is the same. Now, if $\overline{\theta} \in [\tilde{\theta} + \eta, \theta_1]$, we have

$$\mathcal{H}(\overline{\theta}, \overline{W}) > \mathcal{H}(\overline{\theta}, 0) \geq \mathcal{H}(\theta_1, 0) = \mathcal{H}(\theta_0, 0)$$

from the monotonicity of $\mathcal{H}$ along the line $\{\overline{\theta}\} \times \mathbf{R}$ and the monotonicity along the line segment $[\overline{\theta}, \theta_1] \times \{0\}$, and this contradicts $(\overline{\theta}, \overline{W}) \in L_0$. The case $\overline{\theta} \in [\theta_2, \theta_{**} - \eta]$ leads to a contradiction by the same argument. If $\overline{\theta} \in (\theta_1, \theta_2)$ and $(\overline{\theta}, \overline{W})$ is in the interior of the closed orbit $L_\omega(\theta_0,0)$, there exists $\tilde{W} > \overline{W}$ such that $(\overline{\theta},\tilde{W}) \in L_\omega(\theta_0,0)$. Then we have

$$\mathcal{H}(\overline{\theta},\overline{W}) < \mathcal{H}(\overline{\theta},\tilde{W}) = \mathcal{H}(\theta_0,0),$$

which contradicts $(\overline{\theta},\overline{W}) \in L_0$. Similarly, if $(\overline{\theta},\overline{W})$ is outside of the closed orbit, there exists $\tilde{W} < \overline{W}$ such that $(\overline{\theta},\tilde{W}) \in L_\omega(\theta_0,0)$. Again, this implies the estimate

$$\mathcal{H}(\overline{\theta},\overline{W}) > \mathcal{H}(\overline{\theta},\tilde{W}) = \mathcal{H}(\theta_0,0),$$

which contradicts $(\overline{\theta},\overline{W}) \in L_0$. Hence we have $L_\omega(\theta_0,0) = L_0$. We can express L_0 as

$$L_0 = \{(\theta,W) \in \Omega_\beta \mid \mathcal{H}(\theta,W) = \mathcal{H}(\theta_0,0)\} \cap M,$$

with $M = [\theta_*,\theta_{**}] \times \mathbf{R}$, because the value of the Hamiltonian on $M \setminus K_*$ is different from $\mathcal{H}(\theta_0,0)$, and thus, replacing K_* with M does not add any points. This expression will be utilized to derive the necessary and sufficient condition for leapfrogging to occur for solutions with general initial data.

Finally, we make some remarks on the case $\mathcal{H}(\theta_*,0) \not> \mathcal{H}(\theta_{**},0)$. When $\mathcal{H}(\theta_*,0) = \mathcal{H}(\theta_{**},0)$, the same proof holds with $\tilde{\theta} = \theta_*$. When $\mathcal{H}(\theta_*,0) < \mathcal{H}(\theta_{**},0)$, there is a unique $\hat{\theta} \in (\theta_\beta,\theta_{**})$ such that $\mathcal{H}(\hat{\theta},0) = H_*$. This $\hat{\theta}$ plays the same role as $\tilde{\theta}$, and the same arguments for the case $\mathcal{H}(\theta_*,0) < \mathcal{H}(\theta_{**},0)$ holds.

Next we prove that (i) implies (ii). Suppose that a solution starting from $(\theta_0,0)$ is a leapfrogging solution. Since $\mathcal{H}(\theta_*,0)$ and $\mathcal{H}(\theta_{**},0)$ are the maximum value of $\mathcal{H}(\theta,0)$ in their respective intervals $(0,\theta_\beta)$ and $(\theta_\beta,\pi/2)$, in order for a solution curve to cross over the segments $(0,\theta_\beta)\times\{0\}$ and $(\theta_\beta,\pi/2)\times\{0\}$, the value of the Hamiltonian on this solution curve must be less than or equal to the smaller of the two. In other words, $\mathcal{H}(\theta_0,0) \le H_*$ holds. If $\mathcal{H}(\theta_0,0) = H_*$ holds, the only possible points at which the solution curve can cross the segments $(0,\theta_\beta) \times \{0\}$ and $(\theta_\beta,\pi/2) \times \{0\}$ are at the equilibrium points. This would result in the solution converging to one of the equilibrium points, and is not a leapfrogging solution. Hence, for a leapfrogging solution, $\mathcal{H}(\theta_0,0) < H_*$ holds.

Furthermore, $(\theta_0,0)$ is not on the lines $\{\theta_*\} \times \mathbf{R}$ or $\{\theta_{**}\} \times \mathbf{R}$ since the value of the Hamiltonian is greater than or equal to H_* along these lines. Consequently, if $\theta_0 \in (0,\theta_*) \cup (\theta_{**},\pi/2)$, the solution curve cannot cross over from one side of these lines to the other, which means that the solution is not a leapfrogging solution. This implies that $\theta_0 \in (\theta_*,\theta_{**})$, and condition (ii) holds.

We summarize the conclusions of this subsection in the following lemma.

Lemma 3.4 *For initial data of the form $(\theta_0,0) \in \Omega_\beta$, we have the following.*

(i) *If $\theta_0 \in (\theta_*,\theta_{**})$ and $\mathcal{H}(\theta_0,0) < H_*$, then the solution starting from $(\theta_0,0)$ is a leapfrogging solution. Moreover, the closed orbit $L_\omega(\theta_0,0)$ can be expressed as*

$$L_\omega(\theta_0,0) = \{(\theta,W) \in \Omega_\beta \mid \mathcal{H}(\theta,W) = \mathcal{H}(\theta_0,0)\} \cap M,$$

where $M = [\theta_*, \theta_{**}] \times \mathbf{R}$.

(ii) *Otherwise, the solution is not a leapfrogging solution.*

3.3 Remarks on Solutions with General Initial Data

Let $(\theta_0, W_0) \in \Omega_\beta$ satisfy $\theta_0 \in (\theta_*, \theta_{**})$ and $\mathcal{H}(\theta_0, W_0) < H_*$. Since $\mathcal{H}(\theta, 0)$ takes all values between $-\infty$ and H_* on the set $(\theta_*, \theta_\beta) \cup (\theta_\beta, \theta_{**})$, there exists $\theta_{LF} \in (\theta_*, \theta_\beta) \cup (\theta_\beta, \theta_{**})$ such that $\mathcal{H}(\theta_{LF}, 0) = \mathcal{H}(\theta_0, W_0)$. Moreover, from Lemma 3.4, the orbit containing $(\theta_{LF}, 0)$ is a closed orbit corresponding to a leapfrogging solution. Since

$$(\theta_0, W_0) \in \{(\theta, W) \in \Omega_\beta \mid \mathcal{H}(\theta, W) = \mathcal{H}(\theta_{LF}, 0)\} \cap M,$$

Lemma 3.4 implies that (θ_0, W_0) is on the closed orbit containing $(\theta_{LF}, 0)$ and hence, the solution starting from (θ_0, W_0) is a leapfrogging solution.

On the other hand, suppose either $\mathcal{H}(\theta_0, W_0) \geq H_*$ or $\theta_0 \notin (\theta_*, \theta_{**})$ holds. We prove that solution curves starting from these initial data are not leapfrogging solutions. If $\mathcal{H}(\theta_0, W_0) \geq H_*$, then the solution starting from (θ_0, W_0) is not a leapfrogging solution since the value of the Hamiltonian of a leapfrogging solution is strictly less than H_* from Lemma 3.4. If $\theta_0 \notin (\theta_*, \theta_{**})$ holds, we only need to consider the case when $\mathcal{H}(\theta_0, W_0) < H_*$ also holds. Since $\mathcal{H}(\theta_0, W_0) < H_*$, $\theta_0 \in (0, \theta_*) \cup (\theta_{**}, \pi/2)$ because the value of the Hamiltonian on the lines $\{\theta_*\} \times \mathbf{R}$ and $\{\theta_{**}\} \times \mathbf{R}$ are greater than or equal to H_*. Furthermore, since the Hamiltonian is conserved, the solution curve starting from (θ_0, W_0) cannot cross over from one side of these lines to the other and hence, the solution is not a leapfrogging solution. This finishes the proof of Theorem 3.2. $\square$

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