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Asymptotic limit of strong stratification for the 3D inviscid Boussinesq system

Ryo Takada
Faculty of Mathematics, Kyushu University

1 Introduction

This note is the survey of our paper [20]. We consider the initial value problem for the 3D inviscid Boussinesq equations in the whole space $\mathbb{R}^3$:

$$\begin{cases} \partial_t v + (v\cdot\nabla)v = -\nabla q + \eta e_3 & t>0,\ x\in\mathbb{R}^3, \\ \partial_t \eta + (v\cdot\nabla)\eta = 0 & t>0,\ x\in\mathbb{R}^3, \\ \nabla\cdot v = 0 & t\geqslant 0,\ x\in\mathbb{R}^3, \\ v(0,x)=v_0(x), \quad \eta(0,x)=\eta_0(x) & x\in\mathbb{R}^3. \end{cases} \tag{1.1}$$

The unknown functions $v=(v_1(t,x),v_2(t,x),v_3(t,x))^T$, $\eta=\eta(t,x)$ and $q=q(t,x)$ represent the velocity field, the temperature and the scalar pressure of the fluids, respectively, while $v_0=(v_{0,1}(x),v_{0,2}(x),v_{0,3}(x))^T$ is the given initial velocity field satisfying the compatibility condition $\nabla\cdot v_0=0$ and $\eta_0=\eta_0(x)$ is the given initial temperature. The vertical unit vector is denoted by $e_3=(0,0,1)^T$.

It is known that the system (1.1) has an elementary explicit stationary solution (v_s,η_s,q_s) of the form

$$v_s\equiv 0, \quad \eta_s(x_3)=ax_3, \quad q_s(x_3)=\frac{a}{2}x_3^2 \qquad (a\in\mathbb{R}) \tag{1.2}$$

satisfying the hydrostatic balance $\dfrac{dq_s}{dx_3}=\eta_s$. Throughout this paper, we focus on the case of stable stratification:

$$a=\frac{d\eta_s}{dx_3}>0,$$

that is, the stable situation in which the temperature increases with height and warmer fluid is above colder one. We set $N=\sqrt{a}$, which is called the buoyancy or the Brunt-Väisälä frequency and represents the strength of stable stratification. Let us set

$$\theta(t,x)=\eta(t,x)-\eta_s(x_3), \quad q_\eta(t,x)=q(t,x)-q_s(x_3), \tag{1.3}$$

where η_s and q_s are given by (1.2) with $a = N^2 > 0$. Substituting (1.3) into (1.1) gives that (v, θ, q_η) solves

$$\begin{cases} \partial_t v + (v \cdot \nabla) v = -\nabla q_\eta + \theta e_3, \\ \partial_t \theta + (v \cdot \nabla)\theta = -N^2 v_3, \\ \nabla \cdot v = 0, \\ v(0,x) = v_0(x), \quad \theta(0,x) = \theta_0(x) = \eta_0(x) - N^2 x_3, \end{cases} \tag{1.4}$$

where θ_0 denotes the initial thermal disturbance. The above system (1.4) is called the inviscid Boussinesq equations for a stably stratified fluid.

For the original inviscid Boussinesq equations (1.1), it is known that for initial data $(v_0, \eta_0) \in H^s(\mathbb{R}^3)$ with $\nabla \cdot v_0 = 0$ and $s > 5/2$ there exists a $T_0 = T_0(s, \|(v_0, \eta_0)\|_{H^s}) > 0$ such that (1.1) possesses a unique classical solution (v, η) in the class $C([0, T_0); H^s(\mathbb{R}^3))$. Also, the local in time solution (v, η) in the class $C([0, T_0); H^s(\mathbb{R}^3))$ can be extended beyond $t = T_0$ provided that

$$\int_0^{T_0} \|\nabla v(t)\|_{L^\infty}\, dt < \infty \quad \text{or} \quad \int_0^{T_0} (\|\nabla \times v(t)\|_{L^\infty} + \|\nabla \eta(t)\|_{L^\infty})\, dt < \infty.$$

See [4–6, 9, 19] for the local existence theory of (1.1) in function spaces embedded in C^1 class such as the Hölder spaces, the Sobolev spaces and the Besov spaces, and the blow-up criteria of local solutions including the 2D cases.

For a stably stratified fluid, the system (1.4) exhibits a dispersive nature due to the presence of the stable stratification $(\theta e_3, -N^2 v_3)^T$. This phenomenon is closely related to the dispersive estimates for the propagator $e^{\pm iNt|D_h|/|D|}$ defined by the Fourier integral

$$e^{\pm iNt\frac{|D_h|}{|D|}} f(x) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm iNt\frac{|\xi_h|}{|\xi|}} \widehat{f}(\xi)\, d\xi, \qquad (t,x) \in \mathbb{R}^{1+3}.$$

Here, $\xi_h = (\xi_1, \xi_2) \in \mathbb{R}^2$ so that $|\xi_h| = \sqrt{\xi_1^2 + \xi_2^2}$ and $\widehat{f}$ denotes the Fourier transform of f. The sharp dispersive estimate for $e^{\pm iNt|D_h|/|D|}$ was established in [18]. Widmayer [21] proved the local well-posedness of (1.4) in $H^s(\mathbb{R}^3)$ with $s \geqslant 3$ for all $N \geqslant 0$. Furthermore, it is shown in [21] that for initial data $(v_0, \theta_0) \in H^{s+3}(\mathbb{R}^3) \cap W^{5,1}(\mathbb{R}^3)$ with $s \geqslant 3$, the local solution (v^N, θ^N) to (1.4) on $[0, T_0]$ can be decomposed into two parts as

$$(v^N, \theta^N/N) = (w^N, 0, 0) + (u^N, \rho^N), \qquad w^N = (w_1^N, w_2^N), \quad u^N = (u_1^N, u_2^N, u_3^N),$$

and there holds for every $0 < t \leqslant T_0$

$$\|(u^N, \rho^N)(t)\|_{W^{1,\infty}(\mathbb{R}^3)} \to 0, \quad \|w^N(t) - \overline{w}(t)\|_{L^2(\mathbb{R}^3)} \to 0$$

as $N \to \infty$, where $\overline{w} = (\overline{w}_1(t,x), \overline{w}_2(t,x))$ solves the 2D incompressible Euler equations (see (1.8) below). For the related singular limit problems to the rotating Navier-Stokes equations and the viscous rotating Boussinesq equations, we refer to [1–3, 7, 8].

In this manuscript, we prove the long time existence of classical solutions to (1.4) when the buoyancy frequency N is sufficiently high. More precisely, we shall show that for given initial disturbance $\phi = (v_0, \theta/N) \in H^{s+4}(\mathbb{R}^3)$ with $s \geqslant 3$ and for given finite time T, there exists a positive parameter $N_{\phi,T}$ such that the 3D inviscid stratified Boussinesq system (1.4) admits a unique classical solution $(v^N, \theta^N/N)$ on the time interval $[0, T]$ provided $N \geqslant N_{\phi,T}$. Furthermore, we consider the singular limit of the strong stratification as $N \to \infty$, and show that the long time classical solution v^N to (1.4) strongly converges to that of the 2D incompressible Euler equations in the space-time norm $L^q(0, T; W^{1,\infty}(\mathbb{R}^3))$ with the convergence rate $O(N^{-\frac{1}{q}})$ for $4 \leqslant q < \infty$.

To state our result more precisely, we first rewrite the sytem (1.4). Let us combine the velocity field with the rescaled thermal disturbance into the new unknown function

$$u := \left(v, \frac{\theta}{N}\right)^T = \left(v_1, v_2, v_3, \frac{\theta}{N}\right)^T.$$

Put

$$J := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \widetilde{\nabla} := (\nabla, 0)^T.$$

Then, the perturbed system (1.4) can be written as

$$\begin{cases} \partial_t u + NJu + (u \cdot \widetilde{\nabla})u + \widetilde{\nabla} q_\eta = 0, \quad \widetilde{\nabla} \cdot u = 0, \\ u(0, x) = \phi(x), \end{cases} \tag{1.5}$$

where $\phi := (v_0, \theta_0/N)^T$. Next, let $\mathbb{P}$ be the Helmholtz projection of the velocity v onto the divergence-free vector fields which is defined by

$$\mathbb{P} := \left(\begin{array}{c|c} (\delta_{jk} + R_j R_k)_{1 \leqslant j,k \leqslant 3} & 0 \\ \hline 0 & 1 \end{array}\right).$$

Here $\{R_j\}_{1 \leqslant j \leqslant 3}$ denote the Riesz transforms on $\mathbb{R}^3$. Applying the Helmholtz projection $\mathbb{P}$ to (1.5) gives the following evolution equation:

$$\begin{cases} \partial_t u + N\mathbb{P}J\mathbb{P}u + \mathbb{P}(u \cdot \widetilde{\nabla})u = 0, \quad \widetilde{\nabla} \cdot u = 0, \\ u(0, x) = \phi(x). \end{cases} \tag{1.6}$$

Here, we have used the facts that $\mathbb{P}\widetilde{\nabla} q_\eta = 0$ and $\mathbb{P}u = u$ since $\widetilde{\nabla} \cdot u = 0$.

The main result of the paper [20] reads as follows:

Theorem 1.1. *Let $s \in \mathbb{N}$ satisfy $s \geqslant 3$, and let $4 \leqslant q < \infty$. Then, for every $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \in H^{s+4}(\mathbb{R}^3)$ satisfying $\widetilde{\nabla} \cdot \phi = 0$ and for every $0 < T < \infty$, there exists a positive constant $N_{\phi,T}$ depending on s, q, T and $\|\phi\|_{H^{s+4}}$ such that if $N \geqslant N_{\phi,T}$ then (1.6) possesses a unique classical solution u^N in the class*

$$u^N \in C([0,T]; H^{s+4}(\mathbb{R}^3)) \cap C^1([0,T]; H^{s+3}(\mathbb{R}^3)).$$

Furthermore, there exists a positive constant $C = C(s, q, T, \|\phi\|_{H^{s+4}})$ such that

$$\|u^N - u^0\|_{L^q(0,T;W^{1,\infty})} \leqslant CN^{-\frac{1}{q}} \tag{1.7}$$

for all $N \geqslant N_{\phi,T}$, where $u^0 = (w, 0, 0)^T$ and $w = (w_1(t,x), w_2(t,x))^T$ is the classical solution of the two dimensional Euler equations

$$\begin{cases} \partial_t w + \mathbb{P}_h(w \cdot \nabla_h) w = 0 & t > 0,\ x \in \mathbb{R}^3, \\ \nabla_h \cdot w = 0 & t \geqslant 0,\ x \in \mathbb{R}^3, \\ w(0,x) = \mathbb{P}_h \phi_h(x) & x \in \mathbb{R}^3, \\ w \in C([0,T]; H^{s+4}(\mathbb{R}^3)) \cap C^1([0,T]; H^{s+3}(\mathbb{R}^3)). \end{cases} \tag{1.8}$$

Here, $\phi_h = (\phi_1, \phi_2)^T$, $\nabla_h = (\partial_1,\, \partial_2)^T$ and $\mathbb{P}_h = (\delta_{jk} + \partial_j \partial_k (-\Delta_h)^{-1})_{1 \leqslant j,k \leqslant 2}$ denotes the two dimensional Helmholtz projection.

This paper is organized as follows. In Section 2, we derive the explicit formula of linear solutions $e^{-tN\mathbb{P}J\mathbb{P}}\phi$, and establish the space-time estimates for the linear propagator $e^{\pm iNt|D_h|/|D|}$. In Section 3, we state the result on the global regularity of the limit system (1.8). In Section 4, we introduce the modified linear dispersive systems. In Section 5, we present the sketch of the proof of Theorem 1.1.

2 Linear solutions

In this section, we derive the explicit representation for the time evolution semigroup generated by the linear operator $-N\mathbb{P}J\mathbb{P}$, and establish the homogeneous and inhomogeneous space-time estimates for the linear propagator $e^{\pm iNt|D_h|/|D|}$.

We follow the argument in [18, Section 2]. Let us consider the linear equation of (1.6):

$$\begin{cases} \partial_t u + N\mathbb{P}J\mathbb{P}u = 0, \quad \widetilde{\nabla} \cdot u = 0, \\ u(0,x) = \phi(x). \end{cases} \tag{2.1}$$

Applying the Fourier transform to (2.1), we have

$$\begin{cases} \partial_t \widehat{u} + NP(\xi)JP(\xi)\widehat{u} = 0, \quad (\xi, 0)^T \cdot \widehat{u} = 0, \\ \widehat{u}(0,\xi) = \widehat{\phi}(\xi). \end{cases} \tag{2.2}$$

Here, $P(\xi)$ is the multiplier matrix of the projection $\mathbb{P}$ defined by $\widehat{\mathbb{P}u}(\xi) = P(\xi)\widehat{u}(\xi)$, which is given explicitly by

$$P(\xi) := \left(\begin{array}{c|c} \left(\delta_{jk} - \dfrac{\xi_j\xi_k}{|\xi|^2}\right)_{1\leqslant j,k\leqslant 3} & 0 \\ \hline 0 & 1 \end{array}\right).$$

Set $S(\xi) := -P(\xi)JP(\xi)$. Then, direct calculation yields

$$S(\xi) = \frac{1}{|\xi|^2}\begin{pmatrix} 0 & 0 & 0 & -\xi_1\xi_3 \\ 0 & 0 & 0 & -\xi_2\xi_3 \\ 0 & 0 & 0 & \xi_1^2+\xi_2^2 \\ \xi_1\xi_3 & \xi_2\xi_3 & -(\xi_1^2+\xi_2^2) & 0 \end{pmatrix},$$

and then

$$\det\{\lambda I - S(\xi)\} = \lambda^2\left(\lambda^2 + \frac{\xi_1^2+\xi_2^2}{|\xi|^2}\right).$$

Thus, the eigenvalues of $S(\xi)$ are $\left\{\pm i\frac{|\xi_h|}{|\xi|},\, 0,\, 0\right\}$, where $\xi_h = (\xi_1, \xi_2)$ and $|\xi_h| = \sqrt{\xi_1^2+\xi_2^2}$. Moreover, the corresponding eigenvectors are given by

$$a_\pm(\xi) = \frac{1}{\sqrt{2}|\xi_h||\xi|}\begin{pmatrix} \pm i\xi_1\xi_3 \\ \pm i\xi_2\xi_3 \\ \mp i|\xi_h|^2 \\ |\xi_h||\xi| \end{pmatrix}, \quad a_0(\xi) = \frac{1}{|\xi_h|}\begin{pmatrix} -\xi_2 \\ \xi_1 \\ 0 \\ 0 \end{pmatrix}, \quad b_0(\xi) = \frac{1}{|\xi|}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ 0 \end{pmatrix}. \tag{2.3}$$

We see that $\{a_+(\xi), a_-(\xi), a_0(\xi), b_0(\xi)\}$ is an orthonormal basis in $\mathbb{C}^4$ and satisfies

$$S(\xi)a_\pm(\xi) = \pm i\frac{|\xi_h|}{|\xi|}a_\pm(\xi), \quad S(\xi)a_0(\xi) = S(\xi)b_0(\xi) = 0.$$

Hence the solution to (2.2) can be written as

$$\widehat{u}(t,\xi) = e^{NtS(\xi)}\widehat{\phi}(\xi) = \sum_{\sigma\in\{\pm,0\}} e^{\sigma iNt\frac{|\xi_h|}{|\xi|}}\langle\widehat{\phi}(\xi), a_\sigma(\xi)\rangle_{\mathbb{C}^4}a_\sigma(\xi)$$

Here, we remark that $\langle\widehat{\phi}(\xi), b_0(\xi)\rangle_{\mathbb{C}^4} = 0$ by the divergence-free condition $\widetilde{\nabla}\cdot\phi = 0$. Let us set

$$P_j\phi := \mathcal{F}^{-1}[\langle\widehat{\phi}(\xi), a_j(\xi)\rangle_{\mathbb{C}^4}a_j(\xi)] \tag{2.4}$$

for $j = \pm, 0$, and define

$$e^{\pm iNtp(D)}f(x) := \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{ix\cdot\xi\pm iNtp(\xi)}\widehat{f}(\xi)\,d\xi, \qquad p(\xi) := \frac{|\xi_h|}{|\xi|}. \tag{2.5}$$

Then, the solution to (2.1) is explicitly given in terms of the evolution semigroup, and we obtain the following proposition.

Proposition 2.1. *For every* $N \geqslant 0$ *and for every* $\phi \in L^2(\mathbb{R}^3)$ *with* $\widetilde{\nabla} \cdot \phi = 0$, *there exists a unique solution* u *to* (2.1), *which is given explicitly by*

$$\begin{aligned} u(t,x) &= e^{-tN\mathbb{P}J\mathbb{P}}\phi(x) \\ &= e^{iNtp(D)}P_+\phi(x) + e^{-iNtp(D)}P_-\phi(x) + P_0\phi(x). \end{aligned}$$

Next, we shall prove the homogeneous and inhomogeneous space-time estimates for the linear propagator $e^{\pm iNt|D_h|/|D|}$ defined by (2.5). Since the phase $p(\xi) = |\xi_h|/|\xi|$ is homogeneous of degree 0, by the Littlewood-Paley decomposition and scaling, the matter is reduced to the frequency localized case. Also, the sign $\pm$ does not have any role. Hence we consider the operators

$$U_N(t)f(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi + iNtp(\xi)}\psi(\xi)^2 \widehat{f}(\xi)\, d\xi, \quad (t,x) \in \mathbb{R}^{1+3},$$

where ψ is a real-valued function in $\mathscr{S}(\mathbb{R}^3)$ satisfying $\operatorname{supp}\psi \subset \{2^{-2} \leqslant |\xi| \leqslant 2^2\}$ and $\psi(\xi) = 1$ on $\{2^{-1} \leqslant |\xi| \leqslant 2\}$. The sharp dispersive estimate for $U_N(t)$ is obtained in [18].

Lemma 2.2 ([18, Theorem 1.1])**.** *There exists a positive constant* $C = C(\psi) > 0$ *such that*

$$\|U_N(t)f\|_{L^\infty} \leqslant C(1 + N|t|)^{-\frac{1}{2}}\|f\|_{L^1}$$

for all $t \in \mathbb{R}$ *and* $f \in L^1(\mathbb{R}^3)$. *Also, the decay rate* $1/2$ *cannot be improved to a larger one.*

Now we investigate the boundedness of $U_N(t)$. We use the notation for the space-time norm

$$\|f\|_{L^q_t L^r_x} := \|f\|_{L^q(\mathbb{R};L^r(\mathbb{R}^3))}.$$

The following results are the homogeneous and inhomogeneous space-time estimates for the linear operator $U_N(t)$.

Lemma 2.3. *Let the exponents* $q, \tilde{q}, r, \tilde{r}$ *satisfy*

$$\frac{2}{q} + \frac{1}{r} \leqslant \frac{1}{2}, \quad \frac{2}{\tilde{q}} + \frac{1}{\tilde{r}} \leqslant \frac{1}{2}, \quad 4 \leqslant q, \tilde{q} \leqslant \infty, \quad 2 \leqslant r, \tilde{r} \leqslant \infty. \tag{2.6}$$

Then, there exist positive constants $C_1 = C_1(\psi, q, r)$ *and* $C_2 = C_2(\psi, q, \tilde{q}, r, \tilde{r})$ *such that*

$$\|U_N(t)f\|_{L^q_t L^r_x} \leqslant C_1 N^{-\frac{1}{q}}\|f\|_{L^2}, \tag{2.7}$$

$$\left\| \int_{-\infty}^t U_N(t-s)F(s)\, ds \right\|_{L^q_t L^r_x} \leqslant C_2 N^{-\frac{1}{q}-\frac{1}{\tilde{q}}}\|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} \tag{2.8}$$

for $f \in L^2(\mathbb{R}^3)$ *and* $F \in L^{\tilde{q}'}(\mathbb{R}; L^{\tilde{r}'}(\mathbb{R}^3))$, *where* $1/\tilde{r} + 1/\tilde{r}' = 1$ *and* $1/\tilde{q} + 1/\tilde{q}' = 1$.

Proof. We remark that the L^1-L^∞ decay rate of $U_N(t)$ is $-1/2$ and the admissible range (2.6) does not include the endpoint $q = 2$. Hence the proof follows from the standard TT^* argument and the interpolation (See [10, 15, 16]). For details, we refer to [20]. $\square$

From (2.7), (2.8), the Littlewood-Paley theory and scaling, we can show the space-time Strichartz estimates for the original propagator $e^{\pm iNt|D_h|/|D|}$ as a corollary of Lemma 2.3. Let φ_0 be a function in $\mathscr{S}(\mathbb{R}^3)$ satisfying

$$0 \leqslant \varphi_0(\xi) \leqslant 1 \quad \text{for all } \xi \in \mathbb{R}^3, \quad \operatorname{supp}\varphi_0 \subset \left\{\xi \in \mathbb{R}^3 \;\middle|\; 2^{-1} \leqslant |\xi| \leqslant 2\right\}$$

and

$$\sum_{j\in\mathbb{Z}} \varphi_j(\xi) = 1 \quad \text{for every } \xi \in \mathbb{R}^3 \setminus \{0\},$$

where $\varphi_j(\xi) := \varphi_0(2^{-j}\xi)$. We set $\Delta_j f := \mathcal{F}^{-1}[\varphi_j] * f$ for $j \in \mathbb{Z}$. Then, for $s \in \mathbb{R}$ and $1 \leqslant r, \sigma \leqslant \infty$, we define the semi-norm of the homogeneous Besov spaces $\dot{B}^s_{r,\sigma}(\mathbb{R}^3)$ as

$$\|f\|_{\dot{B}^s_{r,\sigma}} := \left\| \left\{2^{sj} \|\Delta_j f\|_{L^r}\right\}_{j\in\mathbb{Z}} \right\|_{\ell^\sigma(\mathbb{Z})}.$$

Also, we define the following space-time norm for $1 \leqslant q \leqslant \infty$:

$$\|F\|_{\widetilde{L^q_t}\dot{B}^s_{r,\sigma}} := \left\| \left\{2^{sj} \|\Delta_j F\|_{L^q_t L^r_x}\right\}_{j\in\mathbb{Z}} \right\|_{\ell^\sigma(\mathbb{Z})}.$$

Lemma 2.4 ([20])**.** *Let the exponents $q, \tilde{q}, r, \tilde{r}$ satisfy*

$$\frac{2}{q} + \frac{1}{r} \leqslant \frac{1}{2}, \quad \frac{2}{\tilde{q}} + \frac{1}{\tilde{r}} \leqslant \frac{1}{2}, \quad 4 \leqslant q, \tilde{q} \leqslant \infty, \quad 2 \leqslant r, \tilde{r} \leqslant \infty.$$

Then, there exist positive constants $C_1 = C_1(q, r)$ and $C_2 = C_2(q, \tilde{q}, r, \tilde{r})$ such that

$$\|e^{\pm iNtp(D)} f\|_{\widetilde{L^q_t}\dot{B}^0_{r,\sigma}} \leqslant C_1 N^{-\frac{1}{q}} \|f\|_{\dot{B}^{3(\frac{1}{2}-\frac{1}{r})}_{2,\sigma}}, \tag{2.9}$$

$$\left\| \int_{-\infty}^t e^{\pm iN(t-s)p(D)} F(s)\, ds \right\|_{\widetilde{L^q_t}\dot{B}^0_{r,\sigma}} \leqslant C_2 N^{-\frac{1}{q}-\frac{1}{\tilde{q}}} \|F\|_{\widetilde{L^{\tilde{q}'}_t}\dot{B}^{3(1-\frac{1}{r}-\frac{1}{\tilde{r}})}_{\tilde{r}',\sigma}} \tag{2.10}$$

for all $1 \leqslant \sigma \leqslant \infty$, $f \in \dot{B}^{3(\frac{1}{2}-\frac{1}{r})}_{2,\sigma}(\mathbb{R}^3)$ and $F \in \widetilde{L^{\tilde{q}'}}(\mathbb{R}; \dot{B}^{3(1-\frac{1}{r}-\frac{1}{\tilde{r}})}_{\tilde{r}',\sigma}(\mathbb{R}^3))$.

3 Global regularity of the limit system

In this section ,we shall state the result on the global regularity of the limit system (1.8), and give the global a priori $H^{s+3}(\mathbb{R}^3)$-estimate for the solution to (1.8). For the detailed proof, see [20]. We remark that the projection P_0 onto the stationary mode of the linear solution to (2.1) defined in (2.3) and (2.4) is also written as

$$\widehat{P_0\phi}(\xi) = \left(\begin{array}{c|c} \left(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi_h|^2}\right)_{1\leqslant j,k\leqslant 2} & 0 \\ \hline 0 & 0 \end{array}\right) \widehat{\phi}(\xi).$$

Hence we see that P_0 corresponds to the two dimensional Helmholtz projection

$$\mathbb{P}_h = \left(\delta_{jk} + \partial_j\partial_k(-\Delta_h)^{-1}\right)_{1\leqslant j,k\leqslant 2}, \quad P_0 = \left(\begin{array}{c|c} \mathbb{P}_h & 0 \\ \hline 0 & 0 \end{array}\right). \tag{3.1}$$

Now, let us consider the limit system of (1.6):

$$\begin{cases} \partial_t w + \mathbb{P}_h(w\cdot\nabla_h)w = 0 & t>0,\ x\in\mathbb{R}^3, \\ \nabla_h\cdot w = 0 & t\geqslant 0,\ x\in\mathbb{R}^3, \\ w(0,x) = \mathbb{P}_h\phi_h(x) & x\in\mathbb{R}^3, \end{cases} \tag{3.2}$$

where $w = (w_1(t,x), w_2(t,x))^T$, $\phi_h = (\phi_1(x),\phi_2(x))^T$ and $\nabla_h = (\partial_1,\partial_2)^T$. Note that for fixed $x_3\in\mathbb{R}$ the system (3.2) for $w = w(\cdot, x_3)$ corresponds to the two dimensional incompressible Euler equations (see [11, 13]).

The global regularity result for (3.2) reads as follows:

Theorem 3.1 ([20]). *Let $s\in\mathbb{N}$ satisfy $s\geqslant 3$. Then, for every $\phi_h\in H^{s+3}(\mathbb{R}^3)$ and for every $0<T<\infty$, there exists a unique classical solution w to (3.2) in the class*

$$w\in C([0,T];H^{s+3}(\mathbb{R}^3))\cap C^1([0,T];H^{s+2}(\mathbb{R}^3)).$$

Moreover, there exists a positive constant $C_L = C_L(s,T,\|\phi_h\|_{H^{s+3}})$ such that

$$\sup_{0\leqslant t\leqslant T}\|w(t)\|_{H^{s+3}} \leqslant C_L(s,T,\|\phi_h\|_{H^{s+3}}). \tag{3.3}$$

4 Linear Dispersive Solutions

In this section, we adapt the idea in [7] and introduce the modified linear dispersive equations. Making use of Lemma 2.4, we shall establish the global space-time estimates for the solutions to those systems.

Let $s\in\mathbb{N}$ satisfy $s\geqslant 3$, and let $0<T<\infty$. Then, for the initial data $\phi = (\phi_h,\phi_3,\phi_4)^T \in H^{s+4}(\mathbb{R}^3)$ with $\widetilde{\nabla}\cdot\phi = 0$, let $w = (w_1,w_2)\in C([0,T];H^{s+4}(\mathbb{R}^3))\cap C^1([0,T];H^{s+3}(\mathbb{R}^3))$ be the classical solution to (3.2) with $w(0,x) = \mathbb{P}_h\phi_h(x)$ constructed in Theorem 3.1 satisfying the H^{s+4}-estimate

$$\sup_{0\leqslant t\leqslant T}\|w(t)\|_{H^{s+4}} \leqslant C_L(s,T,\|\phi_h\|_{H^{s+4}}). \tag{4.1}$$

Now, we put $u^0 = (w, 0, 0)^T$, and consider the solution to the following linear systems with the external forces $P_\pm(u^0 \cdot \widetilde{\nabla})u^0$:

$$\begin{cases} \partial_t u^\pm \mp iNp(D)u^\pm + P_\pm(u^0 \cdot \widetilde{\nabla})u^0 = 0 & t > 0,\ x \in \mathbb{R}^3, \\ \widetilde{\nabla} \cdot u^\pm = 0 & t \geqslant 0,\ x \in \mathbb{R}^3, \\ u^\pm(0, x) = P_\pm \phi(x) & x \in \mathbb{R}^3, \end{cases} \tag{4.2}$$

where $p(D) = |D_h|/|D|$ is the Fourier multiplier, and the projections $P_\pm$ are defined in (2.3) and (2.4). By the Duhamel principle, the solutions to (4.2) are given by

$$u^\pm(t) = e^{\pm iNtp(D)}P_\pm\phi - \int_0^t e^{\pm iN(t-\tau)p(D)}P_\pm(u^0(\tau) \cdot \widetilde{\nabla})u^0(\tau)\, d\tau. \tag{4.3}$$

Lemma 4.1. *Let $s \in \mathbb{N}$ satisfy $s \geqslant 3$, and let $0 < T < \infty$. Then, for every $\phi \in H^{s+4}(\mathbb{R}^3)$ satisfying $\widetilde{\nabla} \cdot \phi = 0$, there exists a unique classical solution $u^\pm$ to (4.2) in the class*

$$u^\pm \in C([0, T]; H^{s+3}(\mathbb{R}^3)) \cap C^1([0, T]; H^{s+2}(\mathbb{R}^3)).$$

Moreover, there exists a positive constant $C = C(s, T, \|\phi\|_{H^{s+4}})$ such that

$$\sup_{0 \leqslant t \leqslant T} \|u^\pm(t)\|_{H^{s+3}} \leqslant \|\phi\|_{H^{s+3}} + C(s, T, \|\phi\|_{H^{s+4}}). \tag{4.4}$$

Also, for $4 \leqslant q < \infty$ there exist positive constants $C_q = C(q)$ and $C = C(s, q, T, \|\phi\|_{H^{s+4}})$ such that

$$\|\nabla^l u^\pm\|_{L^q(0,T;L^\infty)} \leqslant C_q N^{-\frac{1}{q}} \left(\|\phi\|_{H^{2+l}} + C(s, q, \|\phi\|_{H^{s+4}})\right) \tag{4.5}$$

for $l = 0, 1, 2, \ldots, s + 1$.

Proof. We shall give the sketch of proof for the space-time estimate (4.5). For the homogeneous term in (4.3), by the continuous embedding $\dot{B}^0_{\infty,1}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, the Minkowski inequality and (2.9) in Lemma 2.4, we have for $l = 0, 1, 2, \ldots, s + 1$

$$\begin{aligned} \left\|\nabla^l e^{\pm iNtp(D)}P_\pm\phi\right\|_{L^q(0,T;L^\infty)} &\leqslant C \left\|\nabla^l e^{\pm iNtp(D)}P_\pm\phi\right\|_{L^q(0,T;\dot{B}^0_{\infty,1})} \\ &\leqslant C\|\nabla^l e^{\pm iNtp(D)}P_\pm\phi\|_{\widetilde{L^q}(0,T;\dot{B}^0_{\infty,1})} \\ &\leqslant CN^{-\frac{1}{q}}\|\nabla^l P_\pm\phi\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \leqslant CN^{-\frac{1}{q}}\|\phi\|_{H^{2+l}}. \end{aligned} \tag{4.6}$$

For the inhomogeneous term in (4.3), similarly to (4.6), it follows from (2.10) in Lemma 2.4 with $(\tilde{q}, \tilde{r}) = (\infty, 2)$ that

$$\begin{aligned} &\left\|\nabla^l \int_0^t e^{\pm iN(t-\tau)p(D)}P_\pm(u^0(\tau) \cdot \widetilde{\nabla})u^0(\tau)\, d\tau\right\|_{L^q(0,T;L^\infty)} \\ &\quad\leqslant CN^{-\frac{1}{q}} \left\|\nabla^l P_\pm(u^0 \cdot \widetilde{\nabla})u^0\right\|_{\widetilde{L^1}(0,T;\dot{B}^{\frac{3}{2}}_{2,1})}. \end{aligned} \tag{4.7}$$

Here, we have by the H^{s+4}-estimates (4.1) for $w(t)$

$$\begin{aligned}\left\|\nabla^l P_\pm(u^0\cdot\widetilde{\nabla})u^0\right\|_{\widetilde{L^1}(0,T;\dot{B}^{\frac{3}{2}}_{2,1})} &= \int_0^T \left\|\nabla^l P_\pm(u^0(t)\cdot\widetilde{\nabla})u^0(t)\right\|_{\dot{B}^{\frac{3}{2}}_{2,1}} dt \\ &\leqslant C\int_0^T \left\|(u^0(t)\cdot\widetilde{\nabla})u^0(t)\right\|_{H^{2+l}} dt \leqslant C\int_0^T \|w(t)\|^2_{H^{3+l}}\,dt \\ &\leqslant C\int_0^T \|w(t)\|^2_{H^{s+4}}\,dt \leqslant C(s,T,\|\phi_h\|_{H^{s+4}}). \end{aligned} \tag{4.8}$$

Combining (4.6), (4.7) and (4.8) yields the desired estimate (4.5). □

5 Proof of Main Theorem

We are now ready to present the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $s\in\mathbb{N}$ with $s\geqslant 3$, and let $\phi=(\phi_h,\phi_3,\phi_4)^T\in H^{s+4}(\mathbb{R}^3)$ satisfying $\widetilde{\nabla}\cdot\phi=0$. Since $\mathbb{P}J\mathbb{P}$ is skew-symmetric and then $\langle \mathbb{P}J\mathbb{P}u,\,u\rangle_{H^s}=0$, it follows from the standard local well-posedness theory for the 3D Euler equations in $H^s(\mathbb{R}^3)$ by [12, 14, 17] that there exists a local time $T_0=T_0(s,\|\phi\|_{H^s})>0$ such that (1.6) possesses a unique classical solution u^N for all $N\geqslant 0$ in the class

$$u^N\in C([0,T_0];H^s(\mathbb{R}^3))\cap C^1([0,T_0];H^{s-1}(\mathbb{R}^3)). \tag{5.1}$$

In particular, there exist positive constants $C_0=C_0(s)$ and $C_1=C_1(s)$ such that

$$T_0\geqslant \frac{C_0}{\|\phi\|_{H^s}},\qquad \sup_{0\leqslant t\leqslant T_0}\|u^N(t)\|_{H^s}\leqslant C_1\|\phi\|_{H^s}. \tag{5.2}$$

Let $0<T<\infty$. We shall first show that the local solution u^N in the class (5.1) can be extended to the arbitrary finite time interval $[0,T]$ provided that the buoyancy frequency N is sufficiently high.

Let $w=(w_1,w_2)\in C([0,T];H^{s+4}(\mathbb{R}^3))\cap C^1([0,T];H^{s+3}(\mathbb{R}^3))$ be the classical solution to the limit system (3.2) with $w(0,x)=\mathbb{P}_h\phi_h(x)$ constructed in Theorem 3.1. We put $u^0=(w,0,0)^T$. Then, by (3.1), we see that u^0 is the classical solution to the system

$$\begin{cases}\partial_t u^0+P_0(u^0\cdot\widetilde{\nabla})u^0=0, & \widetilde{\nabla}\cdot u^0=0,\\ u^0(0,x)=P_0\phi.\end{cases}$$

Also, let $u^\pm\in C([0,T];H^{s+3}(\mathbb{R}^3))\cap C^1([0,T];H^{s+2}(\mathbb{R}^3))$ be the classical solutions to the linear systems (4.2) constructed in Lemma 4.1 satisfying (4.4) and (4.5).

Now we set

$$v^N:=u^N-u^+-u^--u^0.$$

Then, since there hold $\phi = \mathbb{P}\phi = P_+\phi + P_-\phi + P_0\phi$ and $P_j u^j = u^j$ for $j \in \{0, \pm\}$, the perturbation v^N should solve

$$\begin{cases} \partial_t v^N + N\mathbb{P}J\mathbb{P}v^N + \mathbb{P}(u^N \cdot \widetilde{\nabla})v^N + \sum_{j=0,\pm} \mathbb{P}(v^N \cdot \widetilde{\nabla})u^j + \sum_{\substack{j,k=0,\pm \\ (j,k)\neq(0,0)}} \mathbb{P}(u^j \cdot \widetilde{\nabla})u^k = 0, \\ \widetilde{\nabla} \cdot v^N = 0, \\ v^N(0,x) = 0 \end{cases} \tag{5.3}$$

on the local time interval $[0, T_0]$. Let us derive the H^s-estimate for $v^N(t)$. Taking the H^s inner product of (5.3) with v^N gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v^N(t)\|_{H^s}^2 + \langle (u^N(t) \cdot \widetilde{\nabla})v^N(t),\, v^N(t)\rangle_{H^s} \\ &\quad + \sum_{j=0,\pm} \langle (v^N(t) \cdot \widetilde{\nabla})u^j(t),\, v^N(t)\rangle_{H^s} + \sum_{\substack{j,k=0,\pm \\ (j,k)\neq(0,0)}} \langle (u^j(t) \cdot \widetilde{\nabla})u^k(t),\, v^N(t)\rangle_{H^s} = 0. \end{aligned} \tag{5.4}$$

Since it holds

$$\int_{\mathbb{R}^3} (u^N \cdot \widetilde{\nabla})\partial^\alpha v^N \cdot \partial^\alpha v^N \, dx = 0$$

for $\alpha \in (\mathbb{N} \cup \{0\})^3$ with $|\alpha| \leqslant s$ by the divergence-free condition, we have

$$\begin{aligned} \left| \langle (u^N \cdot \widetilde{\nabla})v^N,\, v^N \rangle_{H^s} \right| &\leqslant \sum_{|\alpha|\leqslant s} \sum_{0<\beta\leqslant\alpha} C_{\alpha,\beta} \left\| (\partial^\beta u^N \cdot \widetilde{\nabla})\partial^{\alpha-\beta} v^N \right\|_{L^2} \left\| \partial^\alpha v^N \right\|_{L^2} \\ &\leqslant C\|u^N\|_{H^s}\|v^N\|_{H^s}^2. \end{aligned} \tag{5.5}$$

Here, we have used the estimates (see [12, Lemma in page 302])

$$\left\| (\partial^\beta u^N \cdot \widetilde{\nabla})\partial^{\alpha-\beta} v^N \right\|_{L^2} \leqslant \begin{cases} C\|u^N\|_{H^3}\|v^N\|_{H^{|\alpha|}} & 0 < \beta \leqslant \alpha,\ |\beta| = 1, 2, \\ C\|u^N\|_{H^{|\beta|}}\|v^N\|_{H^{|\alpha|-|\beta|+3}} & 0 < \beta \leqslant \alpha,\ |\beta| \geqslant 3. \end{cases}$$

For the third term in the left hand side of (5.4), since $s \geqslant 3$ and $H^s(\mathbb{R}^3)$ is a Banach algebra, we see that

$$\begin{aligned} \left| \sum_{j=0,\pm} \langle (v^N \cdot \widetilde{\nabla})u^j,\, v^N \rangle_{H^s} \right| &\leqslant \sum_{j=0,\pm} \left\| (v^N \cdot \widetilde{\nabla})u^j \right\|_{H^s} \left\| v^N \right\|_{H^s} \\ &\leqslant C \sum_{j=0,\pm} \|u^j\|_{H^{s+1}}\|v^N\|_{H^s}^2. \end{aligned} \tag{5.6}$$

For the fourth term in the left hand side of (5.4), the Schwartz inequality gives

$$\left| \sum_{\substack{j,k=0,\pm \\ (j,k)\neq(0,0)}} \langle (u^j \cdot \widetilde{\nabla})u^k,\, v^N \rangle_{H^s} \right| \leqslant \sum_{\substack{j,k=0,\pm \\ (j,k)\neq(0,0)}} \|(u^j \cdot \widetilde{\nabla})u^k\|_{H^s}\|v^N\|_{H^s}. \tag{5.7}$$

Let us derive the estimates for $\|(u^j\cdot\widetilde{\nabla})u^k\|_{H^s}$. It follows from the the Leibniz rule that

$$\|(u^j\cdot\widetilde{\nabla})u^k\|_{H^s}^2=\sum_{|\alpha|\leqslant s}\sum_{\beta\leqslant\alpha}\sum_{\gamma\leqslant\alpha}C_{\alpha,\beta,\gamma}\int_{\mathbb{R}^3}(\partial^\beta u^j\cdot\widetilde{\nabla})\partial^{\alpha-\beta}u^k\cdot(\partial^\gamma u^j\cdot\widetilde{\nabla})\partial^{\alpha-\gamma}u^k\,dx. \tag{5.8}$$

For $(j,k)=(\pm,\pm)$, we have by the Hölder inequality

$$\begin{aligned}&\sum_{|\alpha|\leqslant s}\sum_{\beta\leqslant\alpha}\sum_{\gamma\leqslant\alpha}C_{\alpha,\beta,\gamma}\int_{\mathbb{R}^3}(\partial^\beta u^\pm\cdot\widetilde{\nabla})\partial^{\alpha-\beta}u^\pm\cdot(\partial^\gamma u^\pm\cdot\widetilde{\nabla})\partial^{\alpha-\gamma}u^\pm\,dx\\&\quad\leqslant\sum_{|\alpha|\leqslant s}\sum_{\beta\leqslant\alpha}\sum_{\gamma\leqslant\alpha}C_{\alpha,\beta,\gamma}\|\partial^\beta u^\pm\|_{L^\infty}\|\partial^\gamma u^\pm\|_{L^\infty}\|\nabla\partial^{\alpha-\beta}u^\pm\|_{L^2}\|\nabla\partial^{\alpha-\gamma}u^\pm\|_{L^2}\\&\quad\leqslant C\|u^\pm\|_{H^{s+1}}^2\left(\sum_{l=0}^{s}\|\nabla^l u^\pm\|_{L^\infty}\right)^2.\end{aligned}\tag{5.9}$$

Similarly to (5.9), we see that for $(j,k)=(\pm,\mp),(\pm,0),(0,\pm)$

$$\left|\int_{\mathbb{R}^3}(\partial^\beta u^\pm\cdot\widetilde{\nabla})\partial^{\alpha-\beta}u^\mp\cdot(\partial^\gamma u^\pm\cdot\widetilde{\nabla})\partial^{\alpha-\gamma}u^\mp\,dx\right|\leqslant C\|u^\mp\|_{H^{s+1}}^2\left(\sum_{l=0}^{s}\|\nabla^l u^\pm\|_{L^\infty}\right)^2,\tag{5.10}$$

$$\left|\int_{\mathbb{R}^3}(\partial^\beta u^\pm\cdot\widetilde{\nabla})\partial^{\alpha-\beta}u^0\cdot(\partial^\gamma u^\pm\cdot\widetilde{\nabla})\partial^{\alpha-\gamma}u^0\,dx\right|\leqslant C\|u^0\|_{H^{s+1}}^2\left(\sum_{l=0}^{s}\|\nabla^l u^\pm\|_{L^\infty}\right)^2,\tag{5.11}$$

$$\left|\int_{\mathbb{R}^3}(\partial^\beta u^0\cdot\widetilde{\nabla})\partial^{\alpha-\beta}u^\pm\cdot(\partial^\gamma u^0\cdot\widetilde{\nabla})\partial^{\alpha-\gamma}u^\pm\,dx\right|\leqslant C\|u^0\|_{H^s}^2\left(\sum_{l=0}^{s+1}\|\nabla^l u^\pm\|_{L^\infty}\right)^2.\tag{5.12}$$

Combining (5.7)–(5.12), we obtain

$$\left|\sum_{\substack{j,k=0,\pm\\(j,k)\neq(0,0)}}\langle(u^j\cdot\widetilde{\nabla})u^k,\,v^N\rangle_{H^s}\right|\leqslant C\sum_{j=0,\pm}\|u^j\|_{H^{s+1}}\sum_{l=0}^{s+1}\left(\|\nabla^l u^+\|_{L^\infty}+\|\nabla^l u^-\|_{L^\infty}\right)\|v^N\|_{H^s}.\tag{5.13}$$

Substituting (5.5), (5.6) and (5.13) into (5.4), we have

$$\begin{aligned}\frac{d}{dt}\|v^N(t)\|_{H^s}&\leqslant C\left(\|u^N\|_{H^s}+\sum_{j=0,\pm}\|u^j\|_{H^{s+1}}\right)\|v^N\|_{H^s}\\&\quad+C\sum_{j=0,\pm}\|u^j\|_{H^{s+1}}\sum_{l=0}^{s+1}\left(\|\nabla^l u^+\|_{L^\infty}+\|\nabla^l u^-\|_{L^\infty}\right).\end{aligned}\tag{5.14}$$

Here, it follows from the uniform H^{s+3} estimates (3.3), (4.4) and (5.2) that there exists a positive constant $C=C(s,T,\|\phi\|_{H^{s+4}})$ such that

$$\begin{aligned}\|u^N(t)\|_{H^s}+\sum_{j=0,\pm}\|u^j(t)\|_{H^{s+1}}&\leqslant\sup_{0\leqslant t\leqslant T_0}\|u^N(t)\|_{H^s}+\sum_{j=0,\pm}\sup_{0\leqslant t\leqslant T}\|u^j(t)\|_{H^{s+3}}\\&\leqslant C(s,T,\|\phi\|_{H^{s+4}})\end{aligned}\tag{5.15}$$

for $0 \leqslant t \leqslant T_0$. Then, by (5.14), (5.15) and $v^N(0) = 0$, we have

$$\|v^N(t)\|_{H^s} \leqslant C(s,T,\|\phi\|_{H^{s+4}}) \sum_{l=0}^{s+1} \int_0^t \left(\|\nabla^l u^+(\tau)\|_{L^\infty} + \|\nabla^l u^-(\tau)\|_{L^\infty}\right) d\tau$$
$$+ C(s,T,\|\phi\|_{H^{s+4}}) \int_0^t \|v^N(\tau)\|_{H^s}\, d\tau. \tag{5.16}$$

Here, it follows from the Hölder inequality and the space-time estimates (4.5) in Lemma 4.1 that for $4 \leqslant q < \infty$

$$\sum_{l=0}^{s+1} \int_0^t \|\nabla^l u^\pm(\tau)\|_{L^\infty}\, d\tau \leqslant T^{1-\frac{1}{q}} \sum_{l=0}^{s+1} \|\nabla^l u^\pm\|_{L^q(0,T;L^\infty)} \leqslant C(s,q,T,\|\phi\|_{H^{s+4}}) N^{-\frac{1}{q}} \tag{5.17}$$

for $0 \leqslant t \leqslant T_0 < T$. Hence we have by (5.16), (5.17) and the Gronwall inequality

$$\sup_{0\leqslant t\leqslant T_0} \|v^N(t)\|_{H^s} \leqslant C(s,q,T,\|\phi\|_{H^{s+4}}) N^{-\frac{1}{q}} e^{C(s,T,\|\phi\|_{H^{s+4}})T}. \tag{5.18}$$

Therefore, there exists a positive constant $N_0 = N_0(s,q,T,\|\phi\|_{H^{s+4}}) > 0$ such that there holds

$$\sup_{0\leqslant t\leqslant T_0} \|v^N(t)\|_{H^s} \leqslant 1 \tag{5.19}$$

for all $N \geqslant N_0$. Then, since $v^N = u^N - u^0 - u^+ - u^-$, it follows from (3.3), (4.4) and (5.19) that there exists a positive constant $C_* = C_*(s,T,\|\phi\|_{H^{s+4}})$ such that

$$\|u^N(T_0)\|_{H^s} \leqslant \|v^N(T_0)\|_{H^s} + \sum_{j=0,\pm} \|u^j(T_0)\|_{H^s}$$
$$\leqslant \sup_{0\leqslant t\leqslant T_0} \|v^N(t)\|_{H^s} + \sum_{j=0,\pm} \sup_{0\leqslant t\leqslant T} \|u^j(t)\|_{H^{s+3}}$$
$$\leqslant 1 + C_*(s,T,\|\phi\|_{H^{s+4}}). \tag{5.20}$$

Note that the constant $C_*(s,T,\|\phi\|_{H^{s+4}})$ is independent of the local time T_0. Therefore, the local solution u^N can be extended to $[T_0,T_1]$, where

$$T_1 - T_0 \geqslant \frac{C_0}{1 + C_*(s,T,\|\phi\|_{H^{s+4}})}, \tag{5.21}$$

and there holds

$$\sup_{T_0\leqslant t\leqslant T_1} \|u^N(t)\|_{H^s} \leqslant C_1\left(1 + C_*(s,T,\|\phi\|_{H^{s+4}})\right). \tag{5.22}$$

We repeat the same procedure as (5.4)–(5.18) on the time interval $[T_0,T_1]$. Since we have the global estimates for u^j $(j = 0,\pm)$ on $[0,T]$, it suffices to modify the above argument for the initial data $\|v(T_0)\|_{H^s}$ and the H^s estimates for u^N as in (5.2) and (5.22). Then, similarly to (5.18), we have

$$\sup_{T_0\leqslant t\leqslant T_1} \|v^N(t)\|_{H^s} \leqslant \tilde{C}(s,q,T,\|\phi\|_{H^{s+4}}) N^{-\frac{1}{q}} e^{\tilde{C}(s,T,\|\phi\|_{H^{s+4}})T}$$

for $N \geqslant N_0$. Hence one can take $N_1 = N_1(s, q, T, \|\phi\|_{H^{s+4}}) \geqslant N_0$ so that there holds

$$\sup_{T_0 \leqslant t \leqslant T_1} \|v^N(t)\|_{H^s} \leqslant 1 \tag{5.23}$$

for all $N \geqslant N_1$. Then, we have by (3.3), (4.4) and (5.23)

$$\begin{aligned}
\|u^N(T_1)\|_{H^s} &\leqslant \|v^N(T_1)\|_{H^s} + \sum_{j=0,\pm} \|u^j(T_1)\|_{H^s} \\
&\leqslant \sup_{T_0 \leqslant t \leqslant T_1} \|v^N(t)\|_{H^s} + \sum_{j=0,\pm} \sup_{0 \leqslant t \leqslant T} \|u^j(t)\|_{H^{s+3}} \\
&\leqslant 1 + C_*(s, T, \|\phi\|_{H^{s+4}})
\end{aligned} \tag{5.24}$$

for all $N \geqslant N_1$. Note that the above bound (5.24) is exactly same as (5.20). Hence the local solution u^N can be uniquely extended to the solution of (1.6) on the time interval $[T_1, T_1 + (T_1 - T_0)]$ (defined in (5.21)) for $N \geqslant N_1$ and satisfies

$$\sup_{T_1 \leqslant t \leqslant 2T_1 - T_0} \|u^N(t)\|_{H^s} \leqslant C_1 \left(1 + C_*(s, T, \|\phi\|_{H^{s+4}})\right). \tag{5.25}$$

Also note that the bound (5.25) is exactly same as (5.22). Since T is arbitrary *finite* time, we repeat a finite number of the same procedures in the above, and continue the local solution u^N to the given time interval $[0, T]$ in the class $C([0, T]; H^s(\mathbb{R}^3)) \cap C^1([0, T]; H^{s-1}(\mathbb{R}^3))$ for $N \geqslant N_{\phi,T}$, where $N_{\phi,T} = N(s, q, T, \|\phi\|_{H^{s+4}})$ is some large positive constant. Also, since we have the $H^s(\mathbb{R}^3)$-estimate, it is easy to see that the solution u^N belongs to the class $C([0, T]; H^{s+4}(\mathbb{R}^3)) \cap C^1([0, T]; H^{s+3}(\mathbb{R}^3))$ by the standard extension criterion. Note that it follows from the above procedure on the extension of solutions that the long time solution u^N on $[0, T]$ satisfies the uniform H^s estimate as

$$\sup_{0 \leqslant t \leqslant T} \|u^N(t)\|_{H^s} \leqslant C(s, q, T, \|\phi\|_{H^{s+4}}) \tag{5.26}$$

with some positive constant $C(s, q, T, \|\phi\|_{H^{s+4}})$ for $N \geqslant N_{\phi,T}$.

It remains to prove the convergence result (1.7). Let $N \geqslant N_{\phi,T}$. Since there holds the uniform H^s estimate (5.26) for $u^N(t)$, we have similarly to (5.18)

$$\sup_{0 \leqslant t \leqslant T} \|v^N(t)\|_{H^s} \leqslant C(s, q, T, \|\phi\|_{H^{s+4}}) N^{-\frac{1}{q}} e^{C(s, T, \|\phi\|_{H^{s+4}})T}. \tag{5.27}$$

Recall that $v^N = u^N - u^0 - u^+ - u^-$. Therefore, by (4.5), (5.27) and the continuous embedding $H^s(\mathbb{R}^3) \hookrightarrow W^{1,\infty}(\mathbb{R}^3)$, we obtain for $4 \leqslant q < \infty$

$$\begin{aligned}
\|u^N - u^0\|_{L^q(0,T;W^{1,\infty})} &\leqslant \|v^N\|_{L^q(0,T;W^{1,\infty})} + \sum_{j=\pm} \|u^j\|_{L^q(0,T;W^{1,\infty})} \\
&\leqslant T^{\frac{1}{q}} \sup_{0 \leqslant t \leqslant T} \|v^N(t)\|_{H^s} + \sum_{j=\pm} \|u^j\|_{L^q(0,T;W^{1,\infty})} \\
&\leqslant C(s, q, T, \|\phi\|_{H^{s+4}}) N^{-\frac{1}{q}}
\end{aligned}$$

for all $N \geqslant N_{\phi,T}$. This completes the proof of Theorem 1.1. □

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Ryo Takada
Faculty of Mathematics
Kyushu University
Fukuoka 819-0395
JAPAN
E-mail address: takada@math.kyushu-u.ac.jp