Wave turbulence in a two-layer fluid system

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1   Introduction

Two-layer density stratified fluid systems can be observed in many natural and engineering situations, e.g., at estuaries where fresh water from rivers flowing onto brine, and the thin layers called pycnoclines observed in the ocean where the density changes rather rapidly.

In various wave phenomena in general, resonant interactions between waves are known to play crucial roles (Phillips 1960). The dispersion relation for the surface gravity waves in a single-layer fluid is of non-decay type, and the three-wave resonance is not possible for any combination of wavenumbers. For a two-layer fluid system, on the other hand, it has been known (Ball 1964, Alam 2012) that several types of three-wave resonances are possible. Among them there is a type of resonance called "Class 3" which involves two surface waves and one interfacial wave. The Class 3 resonance is especially interesting in the sense that it is the only type of resonance in which the constituent three waves all propagate in the same direction.

In our previous work (Tanaka & Wakayama 2015), we investigated numerically the energy transfer from the surface waves to the interfacial waves which occurs due to this Class 3 resonance. In that work we found an interesting phenomenon that a sharp peak appears in the surface wave spectrum around some wavenumber during the course of the spectral evolution, and that the peak grows with a much faster time-scale than that expected from the conventional wave turbulence theory. The aim of the present study is to clarify the origin of this peculiar phenomenon.
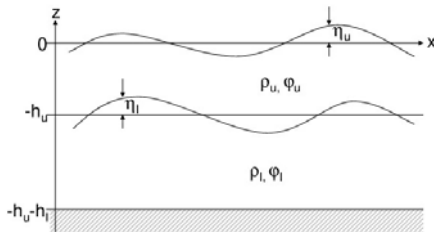

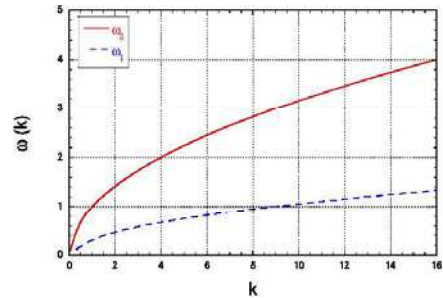Figure 1: Definition sketch of a two-layer fluid system.

Figure 2: The linear dispersion relation ($h_u = 1.0$, $h_l = 2.0$, $R = 0.80$)

2   Governing equations

We investigate wave motions in a two-layer fluid system as shown in Fig. 1. It is assumed that the water is incompressible, inviscid, and the velocity fields are irrotational, and the bottom is flat. It is also assumed that the motion of water is confined in the vertical xz plane, and that the waves propagate in one dimension along the x axis. Then the governing equations of the

problem are given as follows:

$$\nabla^2 \phi_u = 0, \quad -h_u + \eta_l < z < \eta_u, \tag{1a}$$

$$\nabla^2 \phi_l = 0, \quad -h_u - h_l < z < -h_u + \eta_l, \tag{1b}$$

$$\eta_{u,t} + \eta_{u,x}\phi_{u,x} - \phi_{u,z} = 0, \quad z = \eta_u, \tag{1c}$$

$$\phi_{u,t} + \frac{1}{2}\left(\phi_{u,x}^2 + \phi_{u,z}^2\right) + g\eta_u = 0, \quad z = \eta_u, \tag{1d}$$

$$\eta_{l,t} + \eta_{l,x}\phi_{u,x} - \phi_{u,z} = 0, \quad z = -h_u + \eta_l, \tag{1e}$$

$$\eta_{l,t} + \eta_{l,x}\phi_{l,x} - \phi_{l,z} = 0, \quad z = -h_u + \eta_l, \tag{1f}$$

$$\rho_u\left[\phi_{u,t} + \frac{1}{2}\left(\phi_{u,x}^2 + \phi_{u,z}^2\right) + g\eta_l\right]$$
$$- \rho_l\left[\phi_{l,t} + \frac{1}{2}\left(\phi_{l,x}^2 + \phi_{l,z}^2\right) + g\eta_l\right] = 0, \quad z = -h_u + \eta_l, \tag{1g}$$

$$\phi_{l,z} = 0, \quad z = -h_u - h_l. \tag{1h}$$

The system of equations (1) admits a linear solution corresponding to a monochromatic wave-train $\propto e^{\mathrm{i}(kx-\omega t)}$ with an infinitesimal amplitude. The frequency ω is related to the wavenumber k by the linear dispersion relation

$$(1 + R\,T_u\,T_l)\,\omega^4 - gk(T_u + T_l)\,\omega^2 + (1 - R)g^2k^2\,T_uT_l = 0, \tag{2}$$

where $T_u = \tanh kh_u$, $T_l = \tanh kh_l$, and $R = \rho_u/\rho_l$ is the density ratio. The dispersion relation (2) is bi-quadratic with respect to ω, and has four solutions $\pm\omega_s$, $\pm\omega_i$ for each k with $\omega_s > \omega_i$. In the mode corresponding to the larger (smaller) value of ω, the amplitude of the surface displacement is larger (smaller) than that of the interface displacement, and we call it the "surface (interfacial) wave". As an example of the dispersion relation, we show in Fig. 2 ω_s and ω_i as functions of k when $h_u = 1.0$. $h_l = 2.0$ and $R = 0.80$. In all the results shown here, the mass, the time, and the length are normalized so that $\rho_l = 1$, $h_u = 1$, and $g = 1$.

3 Numerical results

The direct numerical simulation (DNS) of the governing equation has been performed. We employ the numerical method based on the High-Order Spectral Method (HOSM) developed by Alam et al. (2009). The amplitude expansion which is included in HOSM is truncated at $O(\epsilon^2)$ ($M = 2$), corresponding to taking into account of up to three wave interactions. For time-integration, we use the 4th-order Runge-Kutta method with a fixed interval. For the detailed description of the numerical method and the settings of parameters such as the number of nodes, size of the timestep, etc., see Alam et al. (2009) and Tanaka & Wakayama (2015).

As an example of the results of DNS, here we show only those which are obtained when $h_l = 2$ and $R = 0.8$. Figure 3 shows the wavenumbers k_{S1}, k_{S2} of two surface waves which can constitute the Class 3 resonant triad with an interfacial wave whose wavenumber is given by $|k_1 - k_2|$. It should be noted that there is a minimum value $k_{\min}$ that the wavenumber of the surface wave can take in order to be a member of a Class 3 resonant triad. When $h_l = 2$, $R = 0.8$, $k_{\min} \approx 1.73$
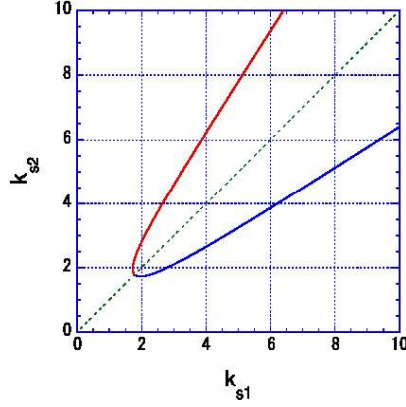
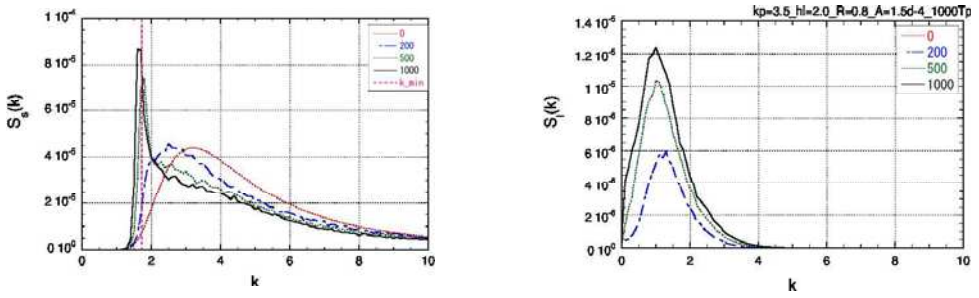Figure 3: Resonance curve ($h_l = 2$, $R = 0.8$)

Figure 4: Evolution of $S_s(k)$ (left) and $S_i(k)$ (right) ($h_l = 2.0$, $R = 0.80$)

Figure 4 shows the evolution of the energy spectra $S_s(k)$ and $S_i(k)$ of surface and interfacial waves when $h_l = 2$, $R = 0.8$. The initial shape of $S_i(k)$ and $S_i(k)$ are specified as follows:

$$S_s(k) = A \left(\frac{k}{k_p} \right)^{-3} \exp \left[-\frac{5}{4} \left(\frac{k}{k_p} \right)^{-2} \right] \quad (k > 0), \tag{3a}$$

$$S_s(k) = 0 \quad (k < 0), \tag{3b}$$

$$S_i(k) = 0 \quad (-\infty < k < \infty). \tag{3c}$$

Equation (3) indicates that the initial wave field consists only of the surface waves propagating in the positive x direction, and there is no interfacial waves. The parameter k_p controls the peak wavenumber of the spectrum and is chosen as $k_p = 3.5$ ($\lambda_p \approx 1.8 h_u$), while the parameter A controlling the energy density of the wave field is chosen as $A = 1.5 \times 10^{-4}$, corresponding to $ak \approx 0.08$.

As is observed in Fig. (4), the spectrum $S_s(k)$ of the surface waves shows a rather peculiar behavior. In the initial stage of the evolution it gradually downshifts to lower wavenumber. But the downshift stops at some time, then a sharp peak appears around $k_{\min}$ and grows very rapidly. On the other hand, the evolution of $S_i(k)$ seems to be smooth and monotonic throughout the spectral evolution.

4 Weak turbulence theory and the kinetic equation

By applying the standard procedure of weak turbulence theory to the present problem of two layer fluid system, we obtain the following kinetic equations which govern the spectral evolution: (Zakharov et al.1992, Nazarenko2011; Hasselmann1962, Janssen2003)

$$\frac{dn_k}{dt} = -2\pi \iint \bigg[|V_{k12}|^2 \{(n_k - n_1)N_2 + n_k n_1\}\delta(\omega_k - \omega_1 - \sigma_2)\,\delta(k - k_1 - k_2)dk_{1,2}$$

$$- |V_{1k2}|^2 \{(n_1 - n_k)N_2 + n_k n_1\}\delta(\omega_k - \omega_1 + \sigma_2)\,\delta(k - k_1 + k_2)dk_{1,2}\bigg], \qquad (4a)$$

$$\frac{dN_k}{dt} = 2\pi \iint |V_{12k}|^2 \{(n_1 - n_2)N_k + n_1 n_2\}\delta(\sigma_k - \omega_1 + \omega_2)\,\delta(k - k_1 + k_2)dk_{1,2}, \qquad (4b)$$

where n_k and N_k are the action density spectra of surface and interfacial waves, respectively, and are related to the energy spectra by $S_s(k) = \omega_k n_k$ and $S_i(k) = \sigma_k N_k$. The existence of two delta functions, one for k and another for ω, implies that exchange of energy is possible only among those three waves which satisfy the resonance condition. In (4) we have neglected all the contributions from resonant interactions other than Class 3.

Figure 5 shows the rates of change $dS_s(k)/dt$ and $dS_i(k)/dt$ evaluated at $t = 0$, i.e., the time when we know the spectra $S_s(k)$ and $S_i(k)$ exactly. The blue dashed line shows the result predicted by the kinetic equation (4), while the red solid line shows the results obtained by the DNS which is estimated from the difference of $S_s(k)$ and $S_i(k)$ from their respective initial shape. The results show that the spectra are actually evolving in time according to the kinetic equation (4) at least at around $t = 0$, and probably be so until the time when the sharp peak appears in $S_s(k)$ and starts to grow with a much faster timescale of $O(1/\epsilon)$ instead of $O(1/\epsilon^2)$, i.e., the timescale expected from the kinetic equation (4).
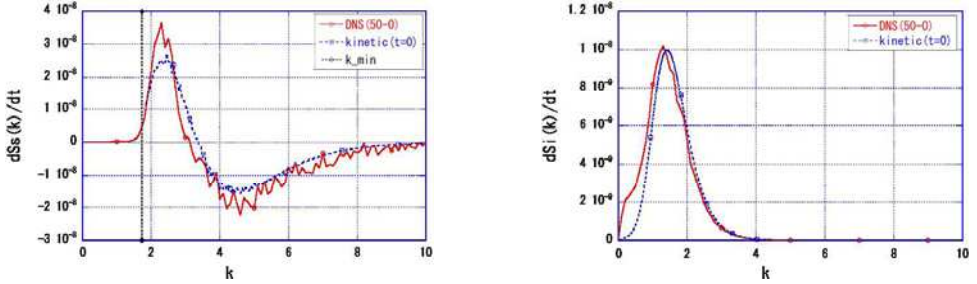

Figure 5: $\frac{dS_s(k)}{dt}$(left) and $\frac{dS_i(k)}{dt}$(right) evaluated at $t = 0$.

5 Breakdown of the kinetic equation

The resonance condition requires that

$$k_{s2} = k_{s1} + k_i, \quad \omega(k_{s2}) = \omega(k_{s1}) + \sigma(k_i) \quad (k_{s2} > k_{s1}). \qquad (5)$$

If we regard k_{s1} as a function of k_{s2}, $dk_{s1}/dk_{s2} = 0$ at the point A on the resonance curve where $k_{s1} = k_{\min}$. (See Fig. 3) Then the k_{s2}-derivative of the resonance condition gives

$$\frac{d\omega(k)}{dk}\bigg|_{k_{s2}} = \frac{d\omega(k)}{dk}\bigg|_{k_{s1}} \frac{dk_{s1}}{dk_{s2}}\bigg|_A + \frac{d\sigma(k)}{dk}\bigg|_{k_i}\left(1 - \frac{dk_{s1}}{dk_{s2}}\bigg|_A\right) = \frac{d\sigma(k)}{dk}\bigg|_{k_i} \qquad (6)$$

This implies that when the wavenumber of one of the two surface waves among the resonant triad of Class3 is equal to $k_{\min}$, the group velocity of the other surface wave and that of the interfacial wave are equal.

By removing the double integral in (4) by exploiting the two delta functions, the kinetic equation for $n_k \, (= S_s(k)/\omega(k))$ reads

$$\frac{dn_k}{dt} = -2\pi \iint \left[|V_{k12}|^2 \{(n_k - n_1)N_2 + n_k n_1\} \delta(\omega_k - \omega_1 - \sigma_2)\, \delta(k - k_1 - k_2) \right.$$

$$\left. - |V_{1k2}|^2 \{(n_1 - n_k)N_2 + n_k n_1\} \delta(\omega_k - \omega_1 + \sigma_2)\, \delta(k - k_1 + k_2) \right] dk_{1,2}$$

$$= -2\pi \frac{|V_{k12}|^2 \{(n_k - n_1)N_2 + n_k n_1\}}{\left| \frac{d\omega(k_1)}{dk} - \frac{d\sigma(k_2)}{dk} \right|} \qquad (\longleftarrow k^S = k_1^S + k_2^I; \ k^S > k_1^S)$$

$$+ 2\pi \frac{|V_{1k2}|^2 \{(n_1 - n_k)N_2 + n_k n_1\}}{\left| \frac{d\omega(k_1)}{dk} - \frac{d\sigma(k_2)}{dk} \right|}. \qquad (\longleftarrow k^S = k_1^S - k_2^I; \ k^S < k_1^S) \tag{7}$$

The denominator of the second term vanishes at $k = k_{\min}$. The divergence of dn_k/dt at $k = k_{\min}$ implies that the kinetic equation cannot express the behavior of $S_s(k)$ near $k = k_{\min}$ correctly. The divergence of dn_k/dt to ∞ also suggests that around $k_{\min}$ $S_s(k)$ changes with a much faster timescale than $O(1/\epsilon^2)$ expected from the kinetic equation. We actually observe in DNS that $S_s(k)$ changes with a timescale of $O(1/\epsilon)$ instead of $O(1/\epsilon^2)$ around $k_{\min}$.

The kinetic equation for the interfacial wave spectrum $S_i(k)$ can also be rewritten to a form which does not include double integral as follows:

$$\frac{dN_k}{dt} = 2\pi \iint |V_{12k}|^2 \{(n_1 - n_2)N_k + n_1 n_2\}$$

$$\delta(\sigma_k - \omega_1 + \omega_2)\, \delta(k - k_1 + k_2) dk_{1,2}$$

$$= 2\pi \frac{|V_{12k}|^2 \{(n_1 - n_2)N_k + n_1 n_2\}}{\left| \frac{d\omega(k_1)}{dk} - \frac{d\omega(k_2)}{dk} \right|} \qquad (\longleftarrow k^I = k_1^S - k_2^S) \tag{8}$$

The denominator does not vanish so long as $k_1^S \neq k_2^S$, hence the kinetic equation remains valid. (It vanishes at $k_i = 0$, but V_{12k} also vanishes then.) This is consistent with the result of DNS that the timescale of $S_i(k)$ remains $O(1/\epsilon^2)$ throughout the spectral evolution even after the sharp peak appears in $S_s(k)$ around $k_{\min}$ and the kinetic equation for $S_s(k)$ has broken down.

6   Discussion and conjecture

The possibility that the kinetic equation breaks down at the wavenumber where the group velocities of the two constituent waves among a resonant triad coincide was pointed out many years ago and is called the "double resonance" (Benney and Saffman 1966). In the present study it has become clear that the appearance of a sharp peak in $S_s(k)$ which we had observed and reported in our previous work is closely connected to this double resonance. Although the possibility of double resonance has been known for a long time, there are not many works which have investigated the effect of double resonance to a wave turbulence in some specific physical system. As far as we know, the work by Soomere (1993) in the context of Rossby wave turbulence is the only work of such kind. The present study has shown that the two-layer fluid

system provides another interesting example of a realistic physical system for which the double resonance is possible to occur.

An important question to be addressed here is if it ever be possible to describe the rapid growth of the sharp peak in $S_s(k)$ around $k_{\min}$, a phenomenon with the deterministic timescale of $O(1/\epsilon)$ instead of the spectral timescale of $O(1/\epsilon^2)$, within the framework of the weak turbulence theory. As we have seen above, the kinetic equation of Zakharov–Hasselmann type ("Hasselmann's equation" for short), which is the most standard type of kinetic equation given by the weak turbulence theory, breaks down at $k_{\min}$, and cannot describe the phenomenon properly. However we now have more general kinetic equations ("GKE" for short), as those proposed by Janssen (2003) and Annenkov-Shrira (2006).

Recently we have studied the weak turbulence in a model Hamiltonian system with a decay dispersion relation. (Note that this system has nothing to do with the two-layer fluid system which is studied here.) In that study we compared the rates of change of the spectrum $dS(k,t)/dt$ given by Hasselmann's equation, Janssen's GKE and the DNS. The model Hamiltonian is defined as follows:

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3, \quad \mathcal{H}_2 = \int \omega(\boldsymbol{k})|a(\boldsymbol{k})|^2 d\boldsymbol{k}, \tag{9}$$

$$\mathcal{H}_3 = \frac{1}{2} \int \left\{ V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) a^*(\boldsymbol{k}) a(\boldsymbol{k}_1) a(\boldsymbol{k}_2) + \text{c.c.} \right\}, \tag{10}$$

$$\omega(\boldsymbol{k}) = k^{3/2}, \quad V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) = (kk_1k_2)^{1/4}. \tag{11}$$

The corresponding dynamic equation is given by

$$\frac{da_{\boldsymbol{k}}}{dt} = -i\frac{\delta \mathcal{H}}{\delta a_{\boldsymbol{k}}^*} = -i\omega_{\boldsymbol{k}} a_{\boldsymbol{k}} - \frac{i}{2} \int V_{012} a_1 a_2 \delta_{0-1-2}^{\boldsymbol{k}} d\boldsymbol{k}_{12} - i \int V_{102}^* a_1 a_2^* \delta_{1-0-2}^{\boldsymbol{k}} d\boldsymbol{k}_{12} d\boldsymbol{k}_{12}, \tag{12}$$

from which we also obtain the equation for the spectrum as follows:

$$\frac{\partial n_{\boldsymbol{k}}}{\partial t} = \text{Im} \int \left[V_{012} \, J_{012} \, \delta_{0-1-2}^{\boldsymbol{k}} + 2V_{102}^* \, J_{102}^* \, \delta_{1-0-2}^{\boldsymbol{k}} \right] d\boldsymbol{k}_{12}, \tag{13}$$

where

$$\langle a_{\boldsymbol{k}} a_{\boldsymbol{k}'}^* \rangle = n_{\boldsymbol{k}} \, \delta(\boldsymbol{k} - \boldsymbol{k}'), \quad \langle a_0^* a_1 a_2 \rangle = J_{012} \, \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2). \tag{14}$$

Differentiating $\langle a_0^* a_1 a_2 \rangle$ with respect to t, using (12), and making use of the "random-phase approximation" which implies

$$\langle a_1^* a_2^* a_3 a_4 \rangle = n_1 n_2 \left[\delta_{1-3}^{\boldsymbol{k}} \, \delta_{2-4}^{\boldsymbol{k}} + \delta_{1-4}^{\boldsymbol{k}} \, \delta_{2-3}^{\boldsymbol{k}} \right], \tag{15}$$

we obtain the evolution equation for J_{012} as follows:

$$\left\{ \frac{\partial}{\partial t} - i \, \Delta\omega_{012} \right\} J_{012} = iV_{012}^* f_{012}, \qquad f_{012} := n_1 n_2 - n_0(n_1 + n_2), \tag{16}$$

where $\Delta\omega_{012}$ is the frequency mismatch $\Delta\omega_{012} = \omega(\boldsymbol{k}) - \omega(\boldsymbol{k}_1) - \omega(\boldsymbol{k}_2)$. By solving (16) for $J_{012}(t)$ and inserting it to (13), we get a kinetic equation which is closed with respect to $n_{\boldsymbol{k}}(t)$. The general solution of (16) is given by

$$J_{012}(t) = iV_{012}^* \int_0^t f_{012}(t') \, \mathrm{e}^{i\Delta\omega_{012}(t-t')} dt' + J_{012}(0)\mathrm{e}^{i\Delta\omega_{012}t}, \tag{17}$$

where $J_{012}(0) = 0$ in order to be consistent with the initial condition of DNS. Substituting this general solution for $J_{012}(t)$ into (13) gives the Annenkov-Shrira's GKE. If we assume that the change of $n_{\boldsymbol{k}}$ is much slower that $e^{i\Delta\omega_{012}t}$ and make the approximation as

$$J_{012}(t) \approx iV_{012}^* f_{012}(t) \int_0^t e^{i\Delta\omega_{012}(t-t')} dt' = V_{012}^* f_{012}(t) \frac{e^{i\Delta\omega_{012}t} - 1}{\Delta\omega_{012}}, \tag{18}$$

before substituting $J_{012}(t)$ into (13), we obtain Janssen's GKE. Furthermore if we take the limit of $t \to \infty$ in (18), and employ the resultant simple form for $J_{012}(t)$ given by

$$J_{012}(t) = V_{012}^* f_{012}(t) \left[-\frac{\mathcal{P}}{\Delta\omega_{012}} + \pi i\delta(\Delta\omega_{012}) \right], \tag{19}$$

we obtain Hasselmann's kinetic equation, the most standard type of kinetic equation given by the weak turbulence theory.

Figure 6 shows the rate of change of the energy spectrum $dE_{\boldsymbol{k}}/dt$ at $t = 0.3T_p$ and $t = 0.6T_p$ obtained for the model Hamiltonian system, with T_p being the period corresponding to the peak of the spectrum. The figure clearly shows that $dE_{\boldsymbol{k}}/dt$ changes very rapidly with a linear timescale of $O(1)$. The figure also shows that Janssen's GKE can keep up with the rapid change and gives $dE_{\boldsymbol{k}}/dt$ which is very close to that given by the DNS, while Hasselmann's equation cannot.
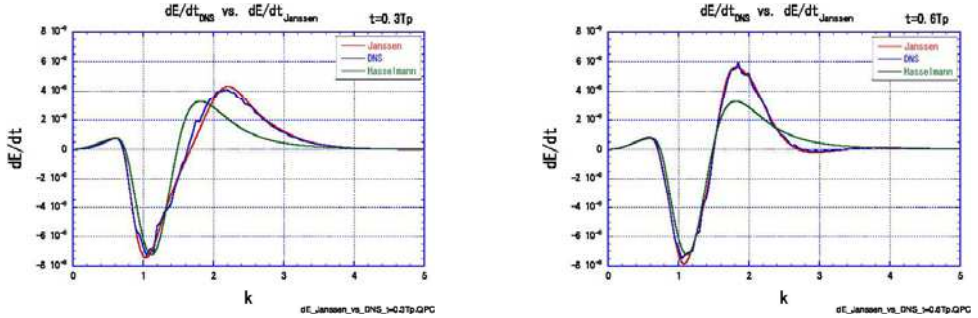

Figure 6: rapid change of $dE_{\boldsymbol{k}}/dt$ observed in the model Hamiltonian system.

However we cannot expect that Janssen's equation would equally be successful in the present case of the two-layer fluid system from the following reasons. As we explained above, Janssen's GKE assumes slow change of $n_{\boldsymbol{k}}(t)$ and takes $f_{012}(t)$ out of the integral with respect to t when evaluating J_{012} (cf. (18)). In the case of the model system, the change of $n_{\boldsymbol{k}}(t)$ was actually slow even though $dn_{\boldsymbol{k}}/dt$ changes rapidly. On the other hand, the sharp peak of $S_s(k)$ in the problem of the two-layer fluid system grows with the timescale of $(1/\epsilon)$, much faster than the spectral timescale of $(1/\epsilon^2)$. Furthermore, Janssen's GKE is known to become equivalent to Hasselmann's equation for $t \gg 1$, hence it would be destined to break down around $k_{\min}$ just like Hasselmann's equation does.

On the other hand, Annenkov & Shrira's GKE keeps $f_{012}(t)$ within the integral with respect to t in (17), and treats $f_{012}(t)$ (hence $n(k)$) in the same manner as the rapidly oscillating term $e^{i\Delta\omega t}$. Thus we conjecture that Annenkov & Shrira's GKE might be able to describe the whole spectral evolution of the two-layer fluid system, including the rapid growth of the sharp peak of $S_s(k)$ around $k_{\min}$. The project is still under way and further investigation is necessary for a definite conclusion.

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