On the phase shift of line solitary waves for the KP-II equation

Tetsu Mizumachi (Hiroshima University)

1 Introduction

The KP-II equation

(1.1) $$\partial_x(\partial_t u+\partial_x^3 u+3\partial_x(u^2))+3\partial_y^2 u=0 \quad \text{for } t>0 \text{ and } (x,y)\in\mathbb{R}^2,$$

is a generalization to two spatial dimensions of the KdV equation

(1.2) $$\partial_t u+\partial_x^3 u+3\partial_x(u^2)=0\,,$$

and has been derived as a model in the study of the transverse stability of solitary wave solutions to the KdV equation with respect to two dimensional perturbation when the surface tension is weak or absent (see [13]).

The global well-posedness of (1.1) in $H^s(\mathbb{R}^2)$ ($s\ge 0$) on the background of line solitons has been studied by Molinet, Saut and Tzvetkov [22] whose proof is based on the work of Bourgain [4]. For the other contributions on the Cauchy problem of the KP-II equation, see e.g. [9, 10, 11, 12, 28, 29, 30, 31] and the references therein.

Let

$$\varphi_c(x)\equiv c\cosh^{-2}\left(\sqrt{\frac{c}{2}}\,x\right),\quad c>0.$$

Then φ_c is a solution of

(1.3) $$\varphi_c''-2c\varphi_c+3\varphi_c^2=0\,,$$

and $\varphi_c(x-2ct)$ is a solitary wave solution of the KdV equation (1.2) and a line soliton solution of (1.1) as well. In this article, we report my recent result on the phase shift of modulating line solitons.

Let us briefly explain known results on stability of 1-solitons for the KdV equation first. Stability of the 1-soliton $\varphi_c(x-2ct)$ of (1.2) was proved by [2, 3, 32] using the fact that φ_c is a minimizer of the Hamiltonian on the manifold $\{u\in H^1(\mathbb{R})\mid \|u\|_{L^2(\mathbb{R})}=\|\varphi_c\|_{L^2(\mathbb{R})}\}$.

Solitary waves of the KdV equation travel at speeds faster than the maximum group velocity of linear waves and the larger solitary wave moves faster to the right. Using this property, Pego and Weinstein [24] prove asymptotic stability of solitary wave solutions of (1.2) in an exponentially weighted space. Later, Martel and Merle established the Liouville theorem for the generalized KdV equations by using a virial type identity and prove the asymptotic stability of solitary waves in $H^1_{loc}(\mathbb{R})$ (see e.g. [16]).

For the KP-II equation, its Hamiltonian is infinitely indefinite and the variational characterization of line soliton is not useful. So it is natural to study stability of line solitons using strong linear stability of line solitons as in [24]. Spectral transverse stability of line solitons of (1.1) has been studied by [1, 5]. Alexander *et al.* [1] proved that the spectrum of the linearized operator in $L^2(\mathbb{R}^2)$ consists of the entire imaginary axis. On the other hand, in an exponentially weighted space where the size of perturbations are biased in the direction of motion, the spectrum of the linearized operator consists of a curve of resonant continuous eigenvalues which goes through 0 and the set of continuous spectrum which locates in the stable half plane and is away from the imaginary axis (see [5, 17]). The former one appears because line solitons are not localized in the transversal direction and 0, which is related to the symmetry of line solitons, cannot be an isolated eigenvalue of the linearized operator. Such a situation is common with planer traveling wave solutions for the heat equation. See e.g. [14, 15, 33].

Transverse stabiliity of line solitons for the KP-II equation has been proved for localized perturbations as well as for perturbations which have 0-mean along all the lines parallel to the x-axis ([17, 18]).

Theorem 1.1. ([18, Theorem 1.1]) *Let $c_0 > 0$ and $u(t,x,y)$ be a solution of* (1.1) *satisfying $u(0,x,y) = \varphi_{c_0}(x) + v_0(x,y)$. There exist positive constants ε_0 and C satisfying the following: if $v_0 \in \partial_x L^2(\mathbb{R}^2)$ and $\|v_0\|_{L^2(\mathbb{R}^2)} + \||D_x|^{1/2}v_0\|_{L^2(\mathbb{R}^2)} + \||D_x|^{-1/2}|D_y|^{1/2}v_0\|_{L^2(\mathbb{R}^2)} < \varepsilon_0$ then there exist C^1-functions $c(t,y)$ and $x(t,y)$ such that for every $t \geq 0$ and $k \geq 0$,*

$$\|u(t,x,y) - \varphi_{c(t,y)}(x - x(t,y))\|_{L^2(\mathbb{R}^2)} \leq C\|v_0\|_{L^2}\,, \tag{1.4}$$

$$\|c(t,\cdot) - c_0\|_{H^k(\mathbb{R})} + \|\partial_y x(t,\cdot)\|_{H^k(\mathbb{R})} + \|x_t(t,\cdot) - 2c(t,\cdot)\|_{H^k(\mathbb{R})} \leq C\|v_0\|_{L^2}\,, \tag{1.5}$$

$$\lim_{t\to\infty}\left(\|\partial_y c(t,\cdot)\|_{H^k(\mathbb{R})} + \left\|\partial_y^2 x(t,\cdot)\right\|_{H^k(\mathbb{R})}\right) = 0\,, \tag{1.6}$$

and for any $R > 0$,

$$\lim_{t\to\infty}\left\|u(t, x + x(t,y), y) - \varphi_{c(t,y)}(x)\right\|_{L^2((x>-R)\times\mathbb{R}_y)} = 0\,. \tag{1.7}$$

Let $\langle x\rangle = \sqrt{1+x^2}$ for $x \in \mathbb{R}$.

Theorem 1.2. ([18, Theorem 1.2]) *Let $c_0 > 0$ and $s > 1$. Suppose that u is a solutions of* (1.1) *satisfying $u(0,x,y) = \varphi_{c_0}(x) + v_0(x,y)$. Then there exist positive constants ε_0 and C such that if $\|\langle x\rangle^s v_0\|_{H^1(\mathbb{R}^2)} < \varepsilon_0$, there exist $c(t,y)$ and $x(t,y)$ satisfying* (1.6), (1.7) *and*

$$\|u(t,x,y) - \varphi_{c(t,y)}(x - x(t,y))\|_{L^2(\mathbb{R}^2)} \leq C\|\langle x\rangle^s v_0\|_{H^1(\mathbb{R}^2)}\,, \tag{1.8}$$

$$\|c(t,\cdot) - c_0\|_{H^k(\mathbb{R})} + \|\partial_y x(t,\cdot)\|_{H^k(\mathbb{R})} + \|x_t(t,\cdot) - 2c(t,\cdot)\|_{H^k(\mathbb{R})} \leq C\|\langle x\rangle^s v_0\|_{H^1(\mathbb{R}^2)} \tag{1.9}$$

for every $t \geq 0$ and $k \geq 0$.

Remark 1.1. The parameters $c(t_0, y_0)$ and $x(t_0, y_0)$ represent the local amplitude and the local phase shift of the modulating line soliton $\varphi_{c(t,y)}(x - x(t,y))$ at time t_0 along the line $y = y_0$ and that $x_y(t,y)$ represents the local orientation of the crest of the line soliton.

Remark 1.2. In view of Theorem 1.1,

$$\lim_{t\to\infty}\sup_{y\in\mathbb{R}}(|c(t,y) - c_0| + |x_y(t,y)|) = 0\,,$$

and as $t \to \infty$, the modulating line soliton $\varphi_{c(t,y)}(x - x(t,y))$ converges to a y-independent modulating line soliton $\varphi_{c_0}(x - x(t,0))$ in $L^2(\mathbb{R}_x \times (|y| \leq R))$ for any $R > 0$.

For the KdV equation as well as for the KP-II equation posed on $L^2(\mathbb{R}_x \times \mathbb{T}_y)$, the dynamics of a modulating soliton $\varphi_{c(t)}(x - x(t))$ is described by a system of ODEs

$$\dot{c} \simeq 0\,, \quad \dot{x} \simeq 2c\,.$$

See [24] for the KdV equation and [20] for the KP-II equation with the y-periodic boundary condition. However, to analyze transverse stability of line solitons for localized perturbation in $\mathbb{R}^2$, we need to study a system of PDEs for $c(t, y)$ and $x(t, y)$ in [17, 18] as is the case with the planar traveling waves for the heat equations (e.g. [14, 15, 33]) and planar kinks for the ϕ^4-model ([6]).

In [17, Theorems 1.4 and 1.5], we find that the phase shift $x(t, y)$ in (1.4) and (1.7) is not uniform in y because of the diffraction of modulating line solitons around $y = \pm 2\sqrt{2c_0}t + O(\sqrt{t})$ and that the set of exact 1-line solitons

$$\mathcal{K} = \{\varphi_c(x + ky - (2c + 3k^2)t + \gamma) \mid c > 0\,,\ k\,, \gamma \in \mathbb{R}\}$$

is not stable in $L^2(\mathbb{R}^2)$.

Theorem 1.3. *Let* $c_0 > 0$. *Then for any* $\varepsilon > 0$, *there exists a solution of* (1.1) *such that* $\|\langle x\rangle(\langle x\rangle + \langle y\rangle)\,\{u(0, x, y) - \varphi_{c_0}(x)\}\|_{H^1(\mathbb{R}^2)} < \varepsilon$ *and* $\liminf_{t\to\infty} t^{-1/2} \inf_{v\in\mathcal{A}} \|u(t, \cdot) - v\|_{L^2(\mathbb{R}^2)} > 0$.

Theorem 1.4. *Let* $c_0 = 2$ *and* $u(t)$ *be as in Theorem 1.2. There exist positive constants* ε_0 *and* C *such that if* $\varepsilon := \|\langle x\rangle(\langle x\rangle + \langle y\rangle)v_0\|_{H^1(\mathbb{R}^2)} < \varepsilon_0$, *then there exist* C^1*-functions* $c(t, y)$ *and* $x(t, y)$ *satisfying* (1.4)–(1.7) *and*

$$\left\| \begin{pmatrix} c(t,\cdot) - 2 \\ x_y(t,\cdot) \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u_B^+(t, y + 4t) \\ u_B^-(t, y - 4t) \end{pmatrix} \right\|_{L^2(\mathbb{R})} = o(\varepsilon t^{-1/4}) \tag{1.10}$$

as $t \to \infty$, *where* $u_B^\pm$ *are self similar solutions of the Burgers equation*

$$\partial_t u = 2\partial_y^2 u \pm 4\partial_y(u^2)$$

such that

$$u_B^\pm(t, y) = \frac{\pm m_\pm H_{2t}(y)}{2\left(1 + m_\pm \int_0^y H_{2t}(y_1)\, dy_1\right)}\,, \quad H_t(y) = (4\pi t)^{-1/2} e^{-y^2/4t}\,,$$

and that $m_\pm$ *are constants satisfying*

$$\int_{\mathbb{R}} u_B^\pm(t, y)\, dy = \frac{1}{4} \int_{\mathbb{R}} (c(0, y) - 2)\ dy + O(\varepsilon^2)\,.$$

Remark 1.3. Since (1.1) is invariant under the scaling $u \mapsto \lambda^2 u(\lambda^3 t, \lambda x, \lambda^2 y)$, we may assume that $c_0 = 2$ without loss of generality.

Remark 1.4. The linearized operator around the line soliton solution has resonant continuous eigenvalues near $\lambda = 0$ whose corresponding eigenmodes grow exponentially as $x \to -\infty$. See (**??**)–(**??**). The diffraction of the line soliton around $y = \pm 4t$ can be thought as a mechanism to emit energy from those resonant continuous eigenmodes.

Theorems 1.3 and 1.4 are improvement of [17, Theorems 1.4 and 1.5].

If we disregard damping effect and nonlinearity of waves propagating along the crest of line solitons, then time evolution of the phase shift is expected to be described by the 1-dimensional wave equation

$$x_{tt} = 8c_0 x_{yy}\,.$$

So it seems natural to expect that $\sup_{t,y\in\mathbb{R}} |x(t,y) - 2c_0t|$ remains small for localized perturbations even if the $L^2(\mathbb{R}_y)$ norm of $x(t,y) - 2c_0t$ grows as $t \to \infty$. We have the following result for the phase shift of modulating 1-line solitons.

Theorem 1.5. *Let $u(t,x,y)$ and $x(t,y)$ be as in Theorem 1.2. There exist positive constants ε_0 and C such that if $\varepsilon := \|\langle x\rangle(\langle x\rangle + \langle y\rangle)v_0\|_{H^1(\mathbb{R}^2)} < \varepsilon_0$, then $\sup_{t\ge 0,\, y\in\mathbb{R}} |x(t,y) - 2c_0t| \le C\varepsilon$.*

Moreover, there exists an $h \in \mathbb{R}$ such that for any $\delta > 0$,

$$\begin{cases} \lim_{t\to\infty} \|x(t,\cdot) - 2c_0t - h\|_{L^\infty(|y|\le(\sqrt{8c_0}-\delta)t} = 0\,, \\ \lim_{t\to\infty} \|x(t,\cdot) - 2c_0t\|_{L^\infty(|y|\ge(\sqrt{8c_0}+\delta)t)} = 0\,. \end{cases} \tag{1.11}$$

In the case where the surface tension is weak, we find in [19] that time evolution of resonant continuous eigenmondes for the linearized Benney-Luke equation around line solitary waves is similar to (1.11). We except that (1.11) is true for modulating line solitary waves of the 2D Benney-Luke equation.

References

[1] J. C. Alexander, R. L. Pego and R. L. Sachs, *On the transverse instability of solitary waves in the Kadomtsev-Petviashvili equation*, Phys. Lett. A **226** (1997), 187–192.

[2] T. Benjamin, *The stability of solitary waves*, Proc. R. Soc. Lond. A **328** (1972), 153-183.

[3] J. L. Bona, *The stability of solitary waves*, Proc. R. Soc. Lond. A **344** (1975), 363-374.

[4] J. Bourgain, *On the Cauchy problem for the Kadomtsev-Petviashvili equation*, GAFA **3** (1993), 315-341.

[5] S. P. Burtsev, *Damping of soliton oscillations in media with a negative dispersion law*, Sov. Phys. JETP **61** (1985).

[6] S. Cuccagna, *On asymptotic stability in 3D of kinks for the ϕ^4 model*, Trans. Amer. Math. Soc. **360** (2008), 2581-2614.

[7] A. de Bouard and Y. Martel, *Non existence of L^2-compact solutions of the Kadomtsev-Petviashvili II equation*, Math. Ann. **328** (2004) 525-544.

[8] A. Bouard and J. C. Saut, *Remarks on the stability of generalized KP solitary waves*, Mathematical problems in the theory of water waves, 75–84, Contemp. Math. **200**, Amer. Math. Soc., Providence, RI, 1996.

[9] A. Grünrock, M. Panthee and J. Drumond Silva, *On KP-II equations on cylinders*, Ann. IHP Analyse non linéaire **26** (2009), 2335-2358.

[10] M. Hadac, *Well-posedness of the KP-II equation and generalizations*, Trans. Amer. Math. Soc. **360** (2008), 6555-6572.

[11] M. Hadac, S. Herr and H. Koch, *Well-posedness and scattering for the KP-II equation in a critical space*, Ann. IHP Analyse non linéaire **26** (2009), 917-941.

[12] P. Isaza and J. Mejia, *Local and global Cauchy problems for the Kadomtsev-Petviashvili (KP-II) equation in Sobolev spaces of negative indices*, Comm. Partial Differential Equations **26** (2001), 1027-1057.

[13] B. B. Kadomtsev and V. I. Petviashvili, *On the stability of solitary waves in weakly dispersive media*, Sov. Phys. Dokl. **15** (1970), 539-541.

[14] T. Kapitula, *Multidimensional stability of planar traveling waves*, Trans. Amer. Math. Soc. **349** (1997), 257–269.

[15] C. D. Levermore and J. X. Xin, *Multidimensional stability of traveling waves in a bistable reaction-diffusion equation, II.* Comm. Partial Differential Equations **17** (1992), 1901–1924.

[16] Y. Martel and F. Merle, *A Liouville theorem for the critical generalized Korteweg-de Vries equation*, J. Math. Pures Appl. **79** (2000), 339-425.

[17] T. Mizumachi, *Stability of line solitons for the KP-II equation in* $\mathbb{R}^2$, Mem. of AMS **238** (2015), 1125.

[18] T. Mizumachi, *Stability of line solitons for the KP-II equation in* $\mathbb{R}^2$*. II*, Proc. Roy. Soc. Edinburgh Sect. A. **148** (2018), 149–198.

[19] T. Mizumachi and Y. Shimabukuro, *Asymptotic linear stability of Benney-Luke line solitary waves in 2D*, Nonlinearity **30** (2017), 3419–3465.

[20] T. Mizumachi and N. Tzvetkov, *Stability of the line soliton of the KP-II equation under periodic transverse perturbations*, Mathematische Annalen **352** (2012), 659–690.

[21] L. Molinet, J.-C. Saut and N. Tzvetkov, *Remarks on the mass constraint for KP-type equations*, SIAM J. Math. Anal. **39** (2007), 627-641.

[22] L. Molinet, J. C. Saut and N. Tzvetkov, *Global well-posedness for the KP-II equation on the background of a non-localized solution*, Ann. Inst. H. Poincaré Anal. Non Linéaire **28** (2011), 653–676.

[23] G. Pedersen, *Nonlinear modulations of solitary waves*, J. Fluid Mech. **267** (1994), 83–108.

[24] R. L. Pego and M. I. Weinstein, *Asymptotic stability of solitary waves*, Comm. Math. Phys. **164** (1994), 305–349.

[25] F. Rousset and N. Tzvetkov, *Transverse nonlinear instability for two-dimensional dispersive models* , Ann. IHP, Analyse Non Linéaire **26** (2009), 477-496.

[26] F. Rousset and N. Tzvetkov, *Transverse nonlinear instability for some Hamiltonian PDE's*, J. Math. Pures Appl. **90** (2008), 550-590.

[27] F. Rousset and N. Tzvetkov, *Stability and instability of the KDV solitary wave under the KP-I flow*, Commun. Math. Phys. **313** (2012), 155–173.

[28] H. Takaoka, *Global well-posedness for the Kadomtsev-Petviashvili II equation*, Discrete Contin. Dynam. Systems **6** (2000), 483-499.

[29] H. Takaoka and N. Tzvetkov, *On the local regularity of Kadomtsev-Petviashvili-II equation*, IMRN **8** (2001), 77-114.

[30] N. Tzvetkov, *Global low regularity solutions for Kadomtsev-Petviashvili equation*, Diff. Int. Eq. **13** (2000), 1289-1320.

[31] S. Ukai, *Local solutions of the Kadomtsev-Petviashvili equation*, J. Fac. Sc. Univ. Tokyo Sect. IA Math. **36** (1989), 193–209.

[32] M. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math. **39** (1986), 51–68.

[33] J. X. Xin, *Multidimensional stability of traveling waves in a bistable reaction-diffusion equation, I.* Comm. Partial Differential Equations **17** (1992), 1889–1899.

[34] V. Zakharov, *Instability and nonlinear oscillations of solitons*, JEPT Lett. **22**(1975), 172-173.