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GROUPS WITH MANY SMALL SUBGROUPS, II

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As usual, $\mathbb{Z}$ and $\mathbb{Q}$ denote the groups of integer numbers and rational numbers respectively, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. We use $\mathbb{P}$ to denote the set of prime numbers.

Let G be a group. For subsets A, B of G, we let $AB = \{ab : a \in A, b \in B\}$ and $A^{-1} = \{a^{-1} : a \in A\}$. When G is abelian, we use the additive notation $A + B$ instead of AB and $-A$ instead of A^{-1}. For a subset A of G, we denote by $\langle A \rangle$ the smallest subgroup of G containing A. To simplify the notation, we write $\langle x \rangle$ instead of $\langle \{x\} \rangle$ for $x \in G$.

Let G be an abelian group. Following [6], we denote by $r_0(G)$ the free rank of G, by $r_p(G)$ the p-rank of G, and we let

$$r(G) = \max \left\{ r_0(G),\ \sum \{r_p(G) : p \in \mathbb{P}\} \right\}.$$

Following [3, Definition 7.2], we call the cardinal

$$r_d(G) = \min\{r(nG) : n \in \mathbb{N}^+\} \tag{1}$$

the *divisible rank* of G. The notion of the divisible rank was defined, under the name of *final rank*, by Szele [15] for p-groups.

All topological groups in this paper are assumed to be Hausdorff.

1. Minimally almost periodic and SSGP groups

A topological group is *minimally almost periodic* if every continuous homomorphism from it to a compact group is trivial. This class of topological groups was introduced by von Neumann and Wigner [12] in 1940, as a means of expanding the theory of almost periodic functions [11]. Examples of minimally almost periodic groups are notoriously difficult to construct. We refer the reader to [8, 1, 4, 5] for a historical overview of these examples.

Answering a long-standing question of Comfort and Protasov, Dikranjan and the first author gave a complete characterization of abelian groups which admit an introduction of a minimally almost periodic group topology [5]. In 1978, Dierolf and Warken [2] proved that every topological group is embedded in a topological group which is minimally almost periodic. The small subgroup generating property (SSGP) was defined by Gould in [8] as a generalization of a stronger property used by Dierolf and Warken in the proof of their result.

Following [4], we define

$$\mathrm{Cyc}(A) = \{x \in G : \langle x \rangle \subseteq A\} \quad \text{for every } A \subseteq G. \tag{2}$$

Definition 1.1. A topological group G has the *small subgroup generating property* (abbreviated to *SSGP*) if and only if $\langle \mathrm{Cyc}(U) \rangle$ is dense in G for every neighbourhood U of the identity of G. We shall say that a topological group G is *SSGP* if G satisfies the small subgroup generating property.

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This definition uses a convenient reformulation of the original definition of Gould given in [4]. The SSGP property was studied extensively by Comfort and Gould [8, 9, 1].

Comfort and Gould [1] asked the following question.

Question 1.2. [1, Question 5.2] What are the (abelian) groups which admit an SSGP group topology?

In the abelian case, Dikranjan and the first author completely resolved Question 1.2 for abelian groups of infinite divisible rank.

Theorem 1.3. [4, Theorem 3.2] *Every abelian group G satisfying $r_d(G) \geq \omega$ admits an* SSGP *group topology.*

An abelian group G satisfies $r_d(G) = 0$ if and only if G is a bounded torsion group; that is, if $nG = \{0\}$ for some $n \in \mathbb{N}^+$. Therefore, the following theorem is a reformulation of [4, Corollary 1.7] which itself is a combination of results of Gabriyelyan [7] and Comfort and Gould [1]:

Theorem 1.4. *A non-trivial abelian group G satisfying $r_d(G) = 0$ admits an* SSGP *group topology if and only if all leading Ulm-Kaplanski invariants of G are infinite.*

In the remaining case $0 < r_d(G) < \omega$, Dikranjan and the first author found a necessary condition on G in order to admit an SSGP topology, and they asked whether said condition was also sufficient. This question was reduced by the same authors to the following problem:

Question 1.5. [4, Question 13.1] Let $m \in \mathbb{N}^+$ and

$$G = G_0 \times \left(\bigoplus_{i=1}^{k} \mathbb{Z}(p_i^\infty) \right) \times F,$$

where F is a finite group, $k \in \mathbb{N}$, $p_1, p_2, \ldots, p_k$ are (not necessarily distinct) prime numbers, and G_0 is a subgroup of $\mathbb{Q}^m$ containing $\mathbb{Z}^m$ such that $G_0 \not\subseteq \mathbb{Q}_\pi^m$ for every finite set π of prime numbers. Is it true that G admits an SSGP group topology?

The notation $\mathbb{Q}_\pi$ appearing in the above question is given in the next definition.

Definition 1.6. For a set π of prime numbers, we use $\mathbb{Q}_\pi$ to denote the set of all rational numbers q whose irreducible representation $q = z/n$ with $z \in \mathbb{Z}$ and $n \in \mathbb{N}^+$ is such that all prime divisors of n belong to π.

Dikranjan and the first author proved a *provisional* theorem completely characterizing abelian groups G which admit an SSGP group topology in the remaining open case $0 < r_d(G) < \omega$ *provided that* the answer to Question 1.5 is positive [4, Theorem 13.2].

In [13], the authors gave a positive answer to a more general version of Question 1.5:

Theorem 1.7. [13, Theorem 2.10] *Suppose that $m \in \mathbb{N}^+$ and G_0 is a subgroup of $\mathbb{Q}^m$ containing $\mathbb{Z}^m$ such that $G_0 \not\subseteq \mathbb{Q}_\pi^m$ for every finite set π of prime numbers. Then for each at most countable abelian group H, the product $G = G_0 \times H$ admits a (separable) metric SSGP group topology.*

Thereby, the previously provisional result (of Dikranjan and the first author) for groups of finite divisible rank became complete and may be stated as follows:

Theorem 1.8. [13, Theorem 2.9] *For an abelian group G satisfying $1 \leq r_d(G) < \infty$, the following conditions are equivalent:*

(i) *G admits an* SSGP *group topology;*
(ii) *the quotient $H = G/t(G)$ of G with respect to its torsion part $t(G)$ has finite rank $r_0(H)$ and $r(H/A) = \omega$ for some (equivalently, every) free subgroup A of H such that H/A is torsion.*

Here $t(G) = \{x \in G : nx = 0 \text{ for some } n \in \mathbb{N}^+\}$.

2. THE ALGEBRAIC SMALL SUBGROUP GENERATING PROPERTY

As noted in [8], the notion of an SSGP group was introduced as a generalization of the stronger property which appeared (without a name) in Dierolf and Warken [2]. Below we give a name to this property and state it using the same notations adopted in Definition 1.1.

Definition 2.1. We say that a topological group G has the *algebraic small subgroup generating* property (ASSGP) if and only if $G = \langle \mathrm{Cyc}(U) \rangle$ for every neighbourhood U of the identity of G. We shall say that a topological group G is ASSGP if G satisfies the ASSGP property.

In contrast to Definition 1.1, the subgroup $\langle \mathrm{Cyc}(U) \rangle$ in the above definition is required to algebraically generate the whole group, rather than be only dense in it.

The following implications always hold:

$$\text{ASSGP} \to \text{SSGP} \to \text{ minimally almost periodic.} \tag{3}$$

The second arrow in (3) is not reversible [1, 8]. One of the goals of this paper is to show that the first arrow in (3) is not reversible either, even in the class of torsion abelian topological groups.

Before we state the original result of Dierolf and Warken, we recall the basics of Hartman-Mycielski construction from [10].

Let G be a group and denote the unit interval $[0, 1]$ by I. We denote by G^I the set of all functions from I to G, which is a group under the coordinate-wise operations. Given $g \in G$ and $t \in (0, 1]$ we define the function $g^t \in G^I$ such that

$$g_t(x) = \begin{cases} g \text{ if } x < t \\ e_G \text{ if } x \geq t, \end{cases}$$

where e_G is the identity element of G. It is known and easy to check that $G_t = \{g_t : g \in G\}$ is a subgroup of G^I that is isomorphic to G for every $t \in (0, 1]$. If G is abelian, then the sum

$$\mathrm{HM}(G) = \bigoplus_{t\in(0,1]} G_t$$

is direct [5, 3, 4]. If μ is the standard probability measure on I the *Hartman–Mycielski* topology on the group $\mathrm{HM}(G)$ is the topology generated by the family of all sets of the form

$$O(U, \epsilon) = \{g \in G^I : \mu(\{t \in I : g(t) \notin U\}) < \epsilon\}$$

where U is an open neighbourhood of e_G in G and $\epsilon > 0$, forms the base of the identity function of $\mathrm{HM}(G)$. This topology is known to be pathwise connected and locally pathwise connected [10].

The Hartman-Mycielski construction has been an invaluable tool for solving embedding problems such as the ones seen in [3, 4].

Following the construction by Dierolf and Warken in [2] (along with additional comments in [8] and [1]), we obtain the following:

Theorem 2.2. ([2, Theorem 1.1], [1, Theorem 3.2]) *Let G be a topological group. Then:*

(a) *G is closed in* $\mathrm{HM}(G)$*; and*
(b) $\mathrm{HM}(G)$ *is* ASSGP.

In particular, $\mathrm{HM}(G)$ *is* SSGP *and therefore minimally almost periodic.*

This theorem not only provides our first example of an ASSGP group but also establishes the fact that *every* topological group can be embedded as a closed subgroup in an ASSGP group.

3. Some of our results for ASSGP groups

To begin, the following basic properties hold:

Proposition 3.1. [14] *The class of* ASSGP *topological groups is closed under the following operations:*

(i) *taking continuous surjective homomorphisms,*
(ii) *taking topological products,*
(iii) *taking direct sums,*
(iv) *taking topological quotients.*

In the rest of this section we focus on the class of torsion groups.

Definition 3.2. A group G is:

(i) *torsion* if for every $g \in G$ there exists $n \in \mathbb{N}$ such that $g^n = e_G$ (the minimal n with this property is called the *order* of g).
(ii) *bounded* if there exists $n \in \mathbb{N}$ such that $G^n = \{e_g\}$.

Definition 3.3. A subgroup H of a group G is *essential* (in G) if $\langle g\rangle \cap H \neq \{e_G\}$ for every $g \in G \setminus \{e_G\}$.

Theorem 3.4. *Let G and H be torsion groups such that G is non-trivial and contains a finite essential subgroup E. Suppose that the orders of arbitrary $g \in G \setminus \{e_G\}$ and $h \in H \setminus \{e_H\}$ are relatively prime (i.e., do not contain common divisors). Then the direct sum $G \oplus H$ does not admit an* ASSGP *group topology.*

Proof. Consider any open neighbourhood V of e in an arbitrary group topology on $G \oplus H$. Since E is finite, the set $U = V \setminus (E \setminus \{e\})$ is also an open neighbourhood of e in this topology.

Claim 1. The inclusion $\langle Cyc(U)\rangle \subseteq H$ holds.

Proof. Assume that $g \in G$, $h \in H$ and $g + h \in \mathrm{Cyc}(U)$. Then

$$\langle g + h\rangle \subseteq U \tag{4}$$

by (2) .

Suppose that $g \neq e_G$. Let $n, m \in \mathbb{N}$ be orders of g and h. Then $mh = e_H$ by Definition 3.2(i), so

$$g^m = g^m + h^m = (g + h)^m \in \langle g + h\rangle, \tag{5}$$

as g and h commute. Since $\langle g + h\rangle$ is a group, from (4) and (5) we get

$$\langle g^m\rangle \subseteq U. \tag{6}$$

Now, since n and m are relatively prime and $g \neq e_G$, we have that $g^m \neq e_G$. By hypothesis, E is an essential subgroup of G, so $\langle g^m \rangle \cap E \neq \{e_G\}$ by Definition 3.3. Together with (6), this implies $U \cap E \neq \{e_G\}$ which contradicts the definition of U. The obtained contradiction means that $g = e_G$.

We have proved that $g + h = h \in H$. Since $g + h$ was an arbitrary element of $\mathrm{Cyc}(U)$, this establishes the inclusion $\mathrm{Cyc}(U) \subseteq H$. Since H is a group, we get $\langle Cyc(U) \rangle \subseteq H$. □

By the previous claim, we have that $\langle \mathrm{Cyc}(U) \rangle \subseteq H$. Since G is non-trivial, H is a proper subgroup of $G \oplus H$. Therefore, $\langle \mathrm{Cyc}(U) \rangle \neq G \oplus H$. By Definition 2.1, $G \oplus H$ does not have the ASSGP property. □

Theorem 3.4 allows us to show that the class of ASSGP groups is a proper subclass of SSGP groups, even in the class of torsion abelian groups.

Example 3.5. Let $P \subseteq \mathbb{P}$ be an infinite set of primes. The following (torsion abelian) groups admit an SSGP group topology but do not admit any ASSGP group topology:

(i) A direct sum $G = \bigoplus_{p \in P} \mathbb{Z}(p)$ of cyclic groups $\mathbb{Z}(p)$ of order p.
(ii) A direct sum $G = \bigoplus_{p \in P} \mathbb{Z}(p^\infty)$ of p-Prüfer groups $\mathbb{Z}(p^\infty)$.

Indeed, in either case the divisible rank $r_d(G)$ of G satisfies that $r_d(G) = \omega$. By [4, Theorem 3.2], G admits an SSGP group topology. If $q \in P$, then either

$$G = \mathbb{Z}(q) \oplus \left(\bigoplus_{p \in P \setminus \{q\}} \mathbb{Z}(p) \right) \quad \text{or} \quad G = \mathbb{Z}(q^\infty) \oplus \left(\bigoplus_{p \in P \setminus \{q\}} \mathbb{Z}(p^\infty) \right),$$

respectively. Since $\mathbb{Z}(q)$ is a finite essential subgroup of both $\mathbb{Z}(q)$ and $\mathbb{Z}(q^\infty)$, from Theorem 3.4 we conclude that G does not admit an ASSGP topology.

This example shows that the first arrow in (3) is not reversible.

Example 3.5 is best possible in a sense that torsion groups witnessing the non-reversibility of the first arrow in (3) cannot be made bounded.

Proposition 3.6. *Every bounded torsion* SSGP *group satisfies the* ASSGP *property.*

Proof. Let $n \in \mathbb{N}$ be the order of G and $U \subseteq G$ be any open neighbourhood of the identity e. Let V be an open neighborhood of e such that $V^n \subseteq U$.

Let $x \in V$. Then $\langle x \rangle = \{x^m : m = 1, \ldots, n\} \subseteq V^n \subseteq U$, so $x \in \mathrm{Cyc}(U)$. This shows that $V \subseteq \mathrm{Cyc}(U)$. Clearly, $\mathrm{Cyc}(U) \subseteq \langle \mathrm{Cyc}(U) \rangle$. Since the subgroup $\langle \mathrm{Cyc}(U) \rangle$ contains the non-empty open set V, it is clopen in G. Since it is also dense in G by the SSGP property of G and Definition 1.1, $\langle \mathrm{Cyc}(U) \rangle$ must coincide with G.

We have established the equation $G = \langle \mathrm{Cyc}(U) \rangle$ for an arbitrary open neighbourhood U of e in G. By Definition 2.1, G has the ASSGP property. □

As a corollary, we obtain the following:

Corollary 3.7. *Every minimally almost periodic bounded torsion abelian group is* ASSGP.

Proof. Let G be a minimally almost periodic bounded torsion abelian group. By [1, Corollary 3.28], every bounded minimally almost periodic group has the SSGP property, and so G is SSGP. By Proposition 3.6, G has the ASSGP property. □

4. The rationals and the ASSGP property

The group of rationals $\mathbb{Q}$ plays a fundamental role in the theory of SSGP groups, as evidenced in Question 1.5 and Theorem 1.7. In this section, we show that Theorem 1.7 cannot be strengthened to produce ASSGP groups.

Theorem 4.1. [14] *Suppose that an abelian* ASSGP *group G admits an algebraic decomposition $G = H \oplus T$ into a direct sum of a group H and a torsion group T. Assume that B is a finite subgroup of G such that $E = \mathbb{Z}^n + B$ is an essential subgroup of G and $t(G) \cap E = B$. Then:*

(i) *E is precompact in the subgroup topology inherited from G;*
(ii) *E is not dense in G.*

From this, we can obtain the following corollary:

Corollary 4.2. [14] *Let G be an abelian* ASSGP *group.*

(i) *If G has a dense and finitely generated essential subgroup, then G is the trivial group.*
(ii) *G cannot have a dense and essential subgroup isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{N}^+$.*
(iii) *If G is finitely generated, then it is trivial.*

As a consequence, the following was obtained:

Theorem 4.3. [14] *Let $n \in \mathbb{N}^+$ be arbitrary. Suppose G is a subgroup of $\mathbb{Q}^n$ such that $\mathbb{Z}^n \subseteq G$. If $P \subseteq \mathbb{P}$ is* any *subset of primes, then the group $H = G \oplus \bigoplus_{p \in P} \mathbb{Z}(p^\infty)^{m_p}$, where $m_p \in \mathbb{N}$ for all $p \in \mathbb{P}$, does not admit an* ASSGP *group topology.*

We highlight the importance of the word "any" in the previous theorem, as the set P can be taken as empty. As a particular case, we have the following:

Corollary 4.4. [14] *$\mathbb{Q}^n$ does not admit an* ASSGP *group topology for every $n \in \mathbb{N}^+$.*

Following [13], we call a subgroup G of $\mathbb{Q}^m$ *wide* if $\mathbb{Z}^m \subseteq G$ and $G \not\subseteq \mathbb{Q}^m_\pi$ for every finite set π of prime numbers. The following result is a re-statement of Theorem 1.7.

Theorem 4.5. *Let $n \in \mathbb{N}^+$ be arbitrary. A non-trivial subgroup $G \subseteq \mathbb{Q}^n$ admits an* SSGP *group topology if and only if G is a wide subgroup of $\mathbb{Q}^n$.*

Since all wide subgroups of $\mathbb{Q}^n$ satisfy the hypotheses of Theorem 4.3, we can combine it with Theorem 4.5 to obtain the following:

Corollary 4.6. *For every $n \in \mathbb{N}^+$, the following statements hold:*

(i) *The group $\mathbb{Q}^n$ contains no non-trivial* ASSGP *subgroups.*
(ii) *Every wide subgroup of $\mathbb{Q}^n$ admits an* SSGP *group topology but does not admit an* ASSGP *group topology.*

This corollary provides many additional examples of topological groups showing that the first arrow in (3) is not reversible.

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